Introduction

The present thesis, consisting of five chapters, is primarily confined to a study on Γ-semirings.

In chapter I, the existing literature relevant to the work of the thesis, basic definitions, preliminary results used elsewhere in the thesis are briefly presented.

The content of chapter II has been published in the journal “International journal of computational and mathematical sciences” under the title “Commuting regular Γ-semirings”. In chapter II, the concept of regular Γ-semirings is discussed. The notion of commuting regular Γ-semirings is introduced and some properties of commuting regular Γ-semirings are determined. A necessary and sufficient condition for Γ-semirings to possess commuting regularity is also obtained.

The content of chapter III has been published in the seminar “Proceedings of the national seminar on recent advancements in mathematics” under the title “On inverse Γ-semirings” and in the journal “Journal of ultra scientist of physical sciences” under the title “Homomorphisms of inverse Γ-semirings”. In chapter III, the concept of inverse Γ-semirings is introduced and some results of inverse Γ-semirings are discussed. Homomorphisms of inverse Γ-semirings are also determined.
The content of chapter IV has been published in the journal “Advances in algebra” under the title “Orthodox Γ-semirings”. In chapter IV, the notion of orthodox Γ-semirings is introduced and some characterizations of orthodox Γ-semirings are discussed. Homomorphisms on orthodox Γ-semirings are also determined.

The content of chapter V has been published in the journal “International journal of algebra” under the title “On pseudo symmetric ideals in Γ-semirings” and in the journal “International journal of mathematical sciences and engineering applications” under the title “Semipseudo symmetric Γ-semirings”. In chapter V, the notions of pseudo symmetric ideals in Γ-semirings and pseudo symmetric Γ-semirings are introduced. A class of Γ-semirings known as semipseudo symmetric Γ-semiring is investigated. Necessary conditions for semipseudo symmetric Γ-semirings to possess semisimple elements are obtained. Maximal ideals in Archimedean semipseudo symmetric Γ-semirings are characterized.

A brief survey of the Literature

It is the existence of an inverse for each element which distinguishes a group from a monoid and the existence of an identity which distinguishes a monoid from semigroup. A semigroup \((S, \circ)\) is a set \(S\) with an associative binary composition \(\circ\). These differences are attenuated by the possibility of defining an inverse of some elements of an arbitrary semigroup \(S\) by the following device. Elements \(a\) and \(b\) are inverses of each other if \(a = aba\) and \(b = bab\). The standard notation for the set of all inverses of \(a\) is \(V(a)\). Not all elements of \(S\) may have an inverse but those that do are called regular. The notion of the regularity in semigroups and rings was introduced by J.Von Neumann in [69], 1936 and his work initiated investigation of many other types of
the regularity. Left, right and complete regularity were introduced and studied by A.H.Clifford in [14], 1941 and R.Croisot in [18], 1953, intra regularity by R.Croisot, 1953 and J.Callais[11], 1961 defined and studied left and right quasi regularity. All types of the regularity of semigroups determined by equations of the form \( a = a^m xa^n \) with \( m, n \geq 0, \ m + n \geq 2 \) were studied by R.Croisot in [18], 1953 who proved that any of them is equivalent to the regularity, left, right or complete regularity. A similar problem concerning all types of the regularity determined by equations of the form \( a = a^p xa^q ya^r \) with \( p, q, r \geq 0 \) was treated by S.Lajos and G.Szazg in [38], 1975. A semigroup \( S \) is called regular if for each \( a \in S \), there exists \( x \in S \) such that \( axa = a \) i.e. a semigroup is called regular if all its elements are regular and \( S \) is called an inverse semigroup if for each \( a \in S \), there is a unique \( x \in S \) such that \( axa = a \) and \( xax = x \) i.e. \( S \) is an inverse semigroup if every element of \( S \) has a unique inverse. A band is a semigroup in which each element is idempotent and an orthodox semigroup is a regular semigroup in which the idempotents form a semigroup. Inverse semigroups were introduced by Preston[52] in 1954 and in 1968, Meakin generalized the same to orthodox semigroup. A different structure theorem for orthodox semigroups in terms of bands and inverse semigroups has already been given by Yamada in [71]. A Clifford semigroup \( S \) is a regular semigroup in which all idempotents are central. In A.H.Clifford’s paper[15], a diagram was given presenting the relationship between various classes of regular semigroups and certain minimum congruences. Green’s relations for semigroups first studied by J.A.Green[25] have played a fundamental role in the development of semigroup theory [30]. A conventional semigroup \( S \) is a regular semigroup in which \( E \), the set of idempotents, is self-conjugate i.e. \( cEc' \subseteq E \) for
each $c \in S$ and for each inverse $c'$ of $c$. Conventional semigroups were first developed by Masat in [43] and stemmed directly from generalizations of Meakin’s work in [44]. Viewed as classes, the following relationships hold with all the inclusions being proper: Inverse semigroups $\subset$ Orthodox semigroups $\subset$ Conventional semigroups $\subset$ Regular semigroups. Recently commuting regular rings and semigroups are studied in [6],[19],[20],[72]. A semigroup or a ring $A$ is called commuting regular if for any $x, y \in A$, there exists $z \in A$ such that $xy = yxzyx$. The notion of a $\Gamma$-semigroup was introduced by M.K.Sen[63] in 1981. $\Gamma$-semigroups generalize semigroups. Many classical notions of semigroups have been extended to $\Gamma$-semigroups, see [12],[13],[63],[64]. Since every semigroup is a $\Gamma$-semigroup, Sen and Saha[61] and Saha[55],[56],[57] have extended many results of semigroups to $\Gamma$-semigroups. The notion of orthodox $\Gamma$-semigroup was introduced by M.K.Sen and N.K.Saha[65] and they extended results of Hall[27] and Yamada[70] to $\Gamma$-semigroups.

During the past century a lot of literature has been devoted to investigations of semirings. Semiring is a well known universal algebra. In a ring, if we do away with the requirement of having additive inverse of each element, then the resulting algebraic structure becomes a semiring i.e. A semiring $(S, +, \circ)$ is a non-empty set $S$, endowed with associative operations of addition and multiplication such that the multiplicative semigroup $(S, \circ)$ distributes over the addition. The explicit notion of a semiring was introduced by H.S.Vandiver[66] in 1934. The additive identity(if it exists) of a semiring is called zero and denoted by 0. An additively commutative semiring $S$ with a zero satisfying $0 \circ x = x \circ 0 = 0$ for all $x \in S$, is called a hemiring. A halfring is an additively cancelative hemiring. An element $a$ in a semiring $S$ is called idempotent[central] if and only if it satisfies $a + a = aa = a[ax = xa]$ for all
x ∈ S. A semiring is called idempotent [commutative] if and only if all of its elements are idempotent [central]. A semiring (S, +, ◦) is additive regular if for every element a ∈ S, there exists an element x ∈ S such that a + x + a = a. Additive regular semirings were first studied by J. Zeleznekow [73] in 1981. We call a semiring (S, +, ◦) an additive inverse semiring if (S, +) is an inverse semigroup i.e. for each a ∈ S, there exists a unique element a′ ∈ S such that a + a′ + a = a and a′ + a + a′ = a′. Additive inverse semirings were first studied by Karvellas [34] in 1974. An additive inverse semiring S is called orthodox if (S, ◦) is an orthodox semigroup. Semirings constitute a fairly natural generalization of rings with broad applications in the mathematical foundations of computer science. It has also found applications in various branches of mathematics. Moreover, among semirings there are such combinatorially interesting systems as the Boolean algebra of subsets of a finite set (with addition being union and multiplication being intersection), non-negative integers and reals (with the usual arithmetic), etc. Matrix theory over semirings is an object of much study in the last decades, see for example [35]. In particular many authors have investigated various rank functions for matrices over semirings and their property, see [8], [9], [26], [32] and references therein. Semiring arises very naturally as the non-negative cone of a totally ordered ring. But the non-positive cone of a totally ordered ring does not form a semiring because multiplication is no longer a binary composition. However, non-positive cone of a totally ordered ring can provide an algebraic home which is nothing but a Γ-semiring on which the authors have made a study.

The notion of a Γ-ring was first introduced by N. Nobusawa [47] in 1964. The class of Γ-rings contains not only all rings but Hestenes ternary rings [29]. Later on, W.E. Barnes [7] weakened slightly the conditions in the definition of Γ-rings in the sense
of Nobusawa. Actually W.E.Barnes, J.Luh and S.Kyuno studied the structures of \( \Gamma \)-rings and obtained various generalizations analogous to the corresponding parts in ring theory. K.R.Goodeare investigated the structure of regular rings and developed their basic properties. P.Ribenboim studied the regular rings and obtained some of their characterizations. Also, S.Kyuno, N.Nobusawa and B.Smith defined a certain type of regular \( \Gamma \)-rings and they developed their various properties. Strongly prime rings were then introduced by Handelmann and Lawrence. Later in 1992, for a \( \Gamma \)-ring, G.L.Booth and N.J.Groenewald defined the left and right operator rings. They studied relationships between \( \Gamma \)-rings and their associated left and right operator rings in 1992, see [10].

The notion of \( \Gamma \)-semirings was introduced by M.Muralikrishna Rao in 1995 i.e. a semigroup \( S \) is called a \( \Gamma \)-semiring if \( \Gamma \) is a semigroup and there exists a mapping \( S \times \Gamma \times S \to S \) satisfying certain properties. It generalizes not only semiring but also \( \Gamma \)-ring. M.M.Rao studied \( \Gamma \)-semirings with left(right) unity, sub \( \Gamma \)-semirings, ideals, prime ideals, \( k \)-ideals, \( h \)-ideals in \( \Gamma \)-semirings, regular \( \Gamma \)-semirings, etc. Changing slightly the defining conditions of \( \Gamma \)-semirings of Rao and introducing the notion of operator semirings of a \( \Gamma \)-semiring as J.Luh did in the theory of \( \Gamma \)-rings, Dutta and Sardar enriched the theory of \( \Gamma \)-semirings in 2000. Moreover using the nice interplay between the operator semirings and \( \Gamma \)-semirings, they obtained many relationships. They have also studied for ideals, prime ideals, prime radical, Jacobson radical, Levitzki radical, semiprime ideals, irreducible ideals of \( \Gamma \)-semirings. The authors have studied orthodox \( \Gamma \)-semirings and discussed commuting regularity in \( \Gamma \)-semirings. The authors have also investigated pseudo symmetric and semi pseudo symmetric ideals in \( \Gamma \)-semirings and discussed many results in pseudo
symmetric and semipseudo symmetric Γ-semirings. The notion of a both sided Γ-semiring i.e. Nobusawa Γ-semiring was introduced by S.K.Sardar and B.C.Saha [59] as a continuation of their study of Γ-semirings [58]. It may be recalled here that though semiring is a generalization of ring, its ideal structures differ a lot, for example an ideal of a semiring needs to be the kernal of some semiring morphism. To amend this gap different types of ideals namely $k$-ideals and $h$-ideals were introduced by Lattore [39] and Iizuka [33]. Olson et al [48, 49] by using the notion of $k$-ideal and $k$-closure, introduced the notion of pre-prime and pre-semiprime ideals in semirings. S.K.Sardar and B.C.Saha [59] extended the same notion in the general setting of Γ-semirings and also introduced the notion of $h$-prime and $h$-semiprime ideals in semirings, Γ-semirings by using the notion of $h$-ideal and $h$-closure.