Chapter 4

Orthodox $\Gamma$-semirings

Orthodox semigroups were first studied by Yamada\cite{70} in 1967 and Hall\cite{27} in 1969. In this chapter, we introduce the notion of Orthodox $\Gamma$-semirings and extend important results of Yamada\cite{70,71}, Hall\cite{27} and Meakin\cite{44} to orthodox $\Gamma$-semirings. We also discuss homomorphisms on orthodox $\Gamma$-semirings.
4.1 Basic Definitions

In order to make this chapter reasonably self contained, we present here some basic definitions which we require for the development of this chapter.

**Definition 4.1.1.** Let $S$ and $\Gamma$ be two additive commutative semigroups. Then $S$ is called $\Gamma$-semiring if there exists a mapping $S \times \Gamma \times S \rightarrow S$ (image to be denoted by $aab$ for $a, b \in S, \alpha \in \Gamma$) satisfying the following conditions.

(i) $aa(b + c) = aab + aac$

(ii) $(a + b)\alpha c = aac + bac$

(iii) $a(\alpha + \beta)b = aab + a\beta b$

(iv) $aa(b\beta c) = (a\alpha b)\beta c$, for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

**Definition 4.1.2.** A non empty subset $A$ of a $\Gamma$-semiring $S$ is called a sub $\Gamma$-semiring of $S$ if $A$ is a sub semigroup of $S$ and $A\Gamma A \subseteq A$.

**Definition 4.1.3.** An element $e$ in a $\Gamma$-semiring $S$ is said to be an idempotent in $S$ if there exists an $\alpha \in \Gamma$ such that $e = eae$. In this case, we say that $e$ is an $\alpha$-idempotent. If every element of $S$ is an idempotent, then $S$ is called an idempotent $\Gamma$-semiring.

**Definition 4.1.4.** For an element $a$ in a $\Gamma$-semiring $S$, if there exists an element $b \in S$ and $\alpha, \beta \in \Gamma$ such that $a = aab\beta a$ and $b = b\beta aab$, then $b$ is said to be an $(\alpha, \beta)$ inverse of $a$. In this case, we write $b \in V^\beta_\alpha(a)$. we also denote it by $a_{\alpha,\beta}^{-1}$ i.e., $a_{\alpha,\beta}^{-1} \in V^\beta_\alpha(a)$.

**Definition 4.1.5.** An element $s$ in a $\Gamma$-semiring $S$ is said to be regular if $s \in s\Gamma S \Gamma s$, where $s\Gamma S \Gamma s = \{sax\beta s; x \in S; \alpha, \beta \in \Gamma\}$. A $\Gamma$-semiring $S$ is said to be regular if every element of $S$ is regular.
Definition 4.1.6. A non empty subset $T$ of a $\Gamma$-semiring $S$ is said to be a $\Gamma$-band in $S$ if $T$ satisfies the following conditions: (i) every element of $T$ is idempotent (ii) if $e$ is an $\alpha$-idempotent and $f$ is a $\beta$-idempotent, then $e\alpha f$ is a $\beta$-idempotent and $e\beta f$ is an $\alpha$-idempotent. A $\Gamma$-band $T$ of a $\Gamma$-semiring $S$ is said to be a rectangular $\Gamma$-band if for all $a, b \in S$ there exists $\alpha \in \Gamma$ such that $a\alpha a = e$ and $a\alpha b\alpha a = a$.

Definition 4.1.7. Let $S$ be a $\Gamma$-semiring and $S'$ be a $\Gamma'$-semiring. A pair of mappings $f_1 : S \to S'$ and $f_2 : \Gamma \to \Gamma'$ is said to be a homomorphism from $(S, \Gamma)$ into $(S', \Gamma')$ if (i) $f_1(a + b) = f_1(a) + f_1(b)$ (ii) $f_1(a\alpha b) = f_1(a)f_2(\alpha)f_1(b)$ for all $a, b \in S$ and $\alpha \in \Gamma$.

4.2 Orthodox $\Gamma$-Semirings

Definition 4.2.1. A $\Gamma$-semiring $S$ is called orthodox if it is regular and if its idempotents form a sub $\Gamma$-semiring. In otherwords, a regular $\Gamma$-semiring $S$ is called an orthodox $\Gamma$-semiring if $e$ is an $\alpha$-idempotent and $f$ is a $\beta$-idempotent of $S$, then $e\alpha f, f\alpha e$ are $\beta$-idempotents($e\beta f, f\beta e$ are $\alpha$-idempotents).

Example 4.2.1. Let $Q^*$ be the set of all non-zero rational numbers and let $\Gamma$ be the set of all positive integers. Let $a, b \in Q^*$ and $\alpha \in \Gamma$. Let us define the mapping $S \times \Gamma \times S \to S$ by $a\alpha b \mapsto |a|ab$. For this operation, $Q^*$ is a $\Gamma$-semiring. Let $\frac{p}{q} \in Q^*$. Now, $\left|\frac{p}{q}\right| \frac{1}{\frac{1}{p}} \frac{1}{\frac{1}{q}} = \frac{p}{q}$. Hence $Q^*$ is a regular $\Gamma$-semiring. Here $\frac{1}{q}(q \in \Gamma)$ is a $q$-idempotent. These are the only idempotents of $Q^*$. Now, $\left|\frac{1}{q}\right| \frac{1}{\frac{1}{q}}$ is a $p$-idempotent. Hence $Q^*$ is an orthodox $\Gamma$-semiring.

Lemma 4.2.2. Let $S$ be a regular $\Gamma$-semiring with set $E$ of idempotents and let $\alpha, \beta \in \Gamma$. Suppose $e\alpha e = e, f\beta f = f \in E$. Then the set $S_{\alpha}^\beta(e, f)$ defined by $S_{\alpha}^\beta(e, f) = \{g \in V_{\beta}^\alpha(e\alpha f) \cap E; g\alpha e = f\beta g = g\}$ is not empty.
Proof. Let \( x \in V_\beta^\alpha(e\alpha f) \) and let \( g = f\beta x\alpha e \). Then

\[
(e\alpha f)\beta g(e\alpha f) = e\alpha(f\beta f)\beta x\alpha(eae)\alpha f,
\]

\[
= e\alpha f\beta x\alpha e\alpha f,
\]

\[
= e\alpha f.
\]

Now,

\[
g\alpha(e\alpha f)\beta g = f\beta x\alpha(eae)\alpha(f\beta f)\beta xae,
\]

\[
= f\beta(xae\alpha f\beta x)ae,
\]

\[
= f\beta xae,
\]

\[
= g.
\]

Hence \( g \in V_\beta^\alpha(e\alpha f) \).

Moreover,

\[
gag = f\beta(xae\alpha f\beta x)ae,
\]

\[
= f\beta xae,
\]

\[
= g.
\]

Consequently \( g \in E \).

Finally it is clear that

\[
gae = f\beta x\alpha(eae),
\]

\[
= f\beta xae,
\]

\[
= g.
\]
and

\[ f^\beta g = (f^\beta f)^\beta x^\alpha e \]
\[ = f^\beta x^\alpha e \]
\[ = g \]

Hence \( g \in S^\beta_{\alpha}(e,f) \).

Remark 4.2.1. The set \( S^\beta_{\alpha}(e,f) \) is called the \((\alpha,\beta)\) sandwich set of \( e \) and \( f \). It has an obvious alternative characterization \( S^\beta_{\alpha}(e,f) = \{ g\alpha g = g \in S; \ g\alpha e = g = f^\beta g, \ e\alpha g\alpha f = e\alpha f \} \)

Lemma 4.2.3. Let \( S \) be a regular \( \Gamma \)-semiring with set \( E \) of idempotents and let \( e, f \in E \). Then \( S^\beta_{\alpha}(e,f) \) is a sub \( \Gamma \)-semiring of \( S \) and it is a rectangular \( \Gamma \)-band.

Proof. Let \( g, h \in S^\beta_{\alpha}(e,f) \).

Then

\[ g\alpha h\alpha g = g\alpha(e\alpha h\alpha f)\beta g \]
\[ = (g\alpha e)\alpha(f^\beta g) \]
\[ = g\alpha g \]
\[ = g \quad (i) \]

It follows that \((g\alpha h)\alpha(g\alpha h) = g\alpha h\) and so \( g\alpha h \) is \( \alpha \)-idempotent.

Moreover,

\[ (g\alpha h)\alpha e = g\alpha(h\alpha e) \]
\[ = g\alpha h \]
\[ f^\beta(g\alpha h) = (f^\beta g)\alpha h \]
\[ = g\alpha h \]

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and

\[ e\alpha(gah)\alpha f = (e\alpha g\alpha f)\beta h\alpha f \]
\[ = e\alpha(f\beta h)\alpha f \]
\[ = e\alpha h\alpha f \]
\[ = e\alpha f \]

Hence \( gah \in S_{\alpha}^{\beta}(e, f) \). From (i), we deduce that \( S_{\alpha}^{\beta}(e, f) \) is a rectangular \( \Gamma \)-band.

\[ \square \]

Lemma 4.2.4. Let \( S \) be a regular \( \Gamma \)-semiring. Let \( a, b \in S \) and \( \alpha, \beta, \gamma, \delta \in \Gamma \). Suppose \( a' \in V_{\alpha}^{\beta}(a), b' \in V_{\gamma}^{\delta}(b) \). Then for each \( g \in S_{\alpha}^{\delta}(a'\beta a, b\gamma b') \), \( b'\delta g\alpha a' \in V_{\gamma}^{\beta}(a\alpha b) \).

Proof.

\[ (a\alpha b)\gamma(b'\delta g\alpha a')\beta(a\alpha b) = a\alpha(b\gamma b'\delta g)\alpha a'\beta a\alpha b \]
\[ = a\alpha(g\alpha a'\beta a)ab \]
\[ = a\alpha g\alpha ab \]
\[ = a\alpha(a'\beta a\alpha g\alpha b\gamma b')\delta b \]
\[ = (a\alpha a'\beta a)b\gamma b'\delta b \]
\[ = a\alpha a'\beta a \]
\[ = a\alpha a' \]

Moreover,

\[ (b'\delta g\alpha a')\beta(a\alpha b)\gamma(b'\delta g\alpha a') = b'\delta(g\alpha a'\beta a)b\gamma b'\delta g)\alpha a' \]
\[ = b'\delta(g\alpha g)\alpha a' \]
\[ = b'\delta g\alpha a' \]

Hence \( b'\delta g\alpha a' \in V_{\gamma}^{\beta}(a\alpha b) \). \[ \square \]
Theorem 4.2.5. Let $S$ be a regular $\Gamma$-semiring with set $E$ of idempotents. Then the following statements are equivalent:

(i) $S$ is orthodox

(ii) If $e = e\alpha e$ and $f = f\beta f$ are any two idempotents of $S$, then $f\beta e \in S^\beta_{\alpha}(e, f)$ where $\alpha, \beta \in \Gamma$.

(iii) For all $a, b \in S$, there exist $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $V^\beta_\gamma(b)\Gamma V^\beta_\alpha(a) \subseteq V^\beta_\gamma(aab)$.

(iv) For every $e \in E$, there exist $\alpha, \beta \in \Gamma$ such that $V^\beta_\alpha(e) \subseteq E$.

Proof.

(i) $\Rightarrow$ (ii): Suppose that $S$ is orthodox. Let $e = e\alpha e, f = f\beta f \in E$ and $g = f\beta e$.

Then

\[
\begin{align*}
  g\alpha e &= f\beta(e\alpha e) \\
  &= f\beta e \\
  &= g \\
  f\beta g &= (f\beta f)\beta e \\
  &= f\beta e \\
  &= g \\
  \text{and } e\alpha g\alpha f &= e\alpha f\beta e\alpha f \\
  &= e\alpha f
\end{align*}
\]

By remark 4.2.1, $g = f\beta e \in S^\beta_{\alpha}(e, f)$.

(ii) $\Rightarrow$ (iii): Let $a, b \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. Suppose $a' \in V^\beta_\alpha(a)$ and $b' \in V^\beta_\gamma(b)$.

Then by lemma 4.2.4, $b'\delta g\alpha a' \in V^\beta_\gamma(aab)$ for all $g$ in $S^\delta_{\alpha}(a'\beta a, b\gamma b')$. From (ii), it follows that $(b\gamma b')\delta(a'\beta a) \in S^\delta_{\alpha}(a'\beta a, b\gamma b')$. By lemma 4.2.4, $b'\delta(b\gamma b'\delta a'\beta a)a'a' \in V^\beta_\gamma(aab)$. Hence $b'\delta a' \in V^\beta_\gamma(aab)$. 

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(iii) ⇒ (iv): Let \( e \in E \) and \( \alpha, \beta \in \Gamma \). Suppose \( x \in V^\beta_\alpha(e) \). Then \( x\beta eax = x \) and \( eax\beta e = e \). Since \( x\beta e \) is \( \alpha \)-idempotent and \( eax \) is \( \beta \)-idempotent, \( x\beta e \in V^\alpha_\alpha(x\beta e) \) and \( eax \in V^\beta_\beta(eax) \). By (iii), \((eax)\beta(x\beta e) \in V^\alpha_\beta(x\beta eax)\) which implies \((eax)\beta(x\beta e) \in V^\alpha_\beta(x)\). Now,

\[
x = x\beta(eax\beta x\beta e)\alpha x
= (x\beta eax)\beta(x\beta eax)
= x\beta x
\]

Hence \( x \) is idempotent.

(iv) ⇒ (i): Let \( eae = e, f\beta f = f \in E \) and \( \alpha, \beta \in \Gamma \). By lemma 4.2.2, there exists an idempotent \( g \) in \( V^\beta_\alpha(f\beta e) \). Since \( (f\beta e) \) being an \( (\beta, \alpha) \) inverse of the idempotent \( g \) is itself an idempotent, \( S \) is orthodox. \( \square \)

**Theorem 4.2.6.** Let \( S \) be an orthodox \( \Gamma \)-semiring with set \( E \) of idempotents. Let \( \alpha, \beta, \gamma \in \Gamma \). For all \( a \) in \( S \), \( e \) in \( E \) and \( a' \in V^\gamma_\alpha(a) \), the element \( aae\beta a' \) is \( \gamma \)-idempotent and the element \( a'\beta eaa \) is \( \alpha \)-idempotent.

**Proof.**

\[
(aae\beta a')\gamma(aae\beta a') = a\alpha(e\beta a'\gamma aae\beta a'\gamma a)\alpha a'
= aae\beta(a'\gamma aaea')
= aae\beta a'
\]

and \((a'\beta eaa)\alpha(a'\beta eaa) = a'\gamma(aa\beta eaa\alpha a'\beta_e)a\alpha
= (a'\gamma aae)\beta eae
= a'\beta eaa
\]

as required. \( \square \)
Theorem 4.2.7. Every inverse $\Gamma$-semiring $S$ is an orthodox $\Gamma$-semiring.

Proof. Let $e$ be an $\alpha$-idempotent and $f$ be a $\beta$-idempotent of $S$. We first show that $eaf$ is a $\beta$-idempotent. Now $eaf \in S$. There exist $\gamma, \delta \in \Gamma$ and $x \in S$ such that $x \in V_\delta^\gamma(eaf)$. Then $(eaf)\delta x\gamma(eaf) = eaf$ and $x\gamma(eaf)\delta x = x$. Let $g = f\delta x\gamma eaf$ and $h = f\delta x\gamma e$. It can be seen easily that $g\beta g = g$.

Also,

$$(f\delta x\gamma eaf)\beta(f\delta x\gamma e)(f\delta x\gamma eaf) = f\delta(x\gamma eaf\delta x)\gamma(eaf\delta x\gamma eaf)$$

$$= f\delta x\gamma eaf$$

$$= g$$

This shows that $g\beta hg = g$. Similarly $h\alpha g\beta h = h$. Hence, $g \in V_\alpha^\beta(h)$. Also, $eaf \in V_\alpha^\beta(h)$. Since $S$ is an inverse $\Gamma$-semiring, $g = eaf$. Hence $eaf$ is a $\beta$-idempotent. Proceeding as above, we can show that $fae$ is a $\beta$-idempotent and both $e\beta f$ and $f\beta e$ are $\alpha$-idempotents. Hence $S$ is an orthodox $\Gamma$-semiring. \hfill \Box

Theorem 4.2.8. A regular $\Gamma$-semiring $S$ is an orthodox $\Gamma$-semiring if and only if for any $\alpha$-idempotent $e \in S$, where $\alpha \in \Gamma$, if $V_\alpha^\beta(e) \neq \phi$ and $V_\beta^\alpha(e) \neq \phi$, then each member of $V_\alpha^\beta(e)$ and $V_\beta^\alpha(e)$ is a $\beta$-idempotent.

Proof. Suppose $S$ is an orthodox $\Gamma$-semiring. Let $e$ be an $\alpha$-idempotent of $S$ and let $x \in V_\alpha^\beta(e)$. Then $eax\beta e = e$ and $x\beta eax = x$. Now, $eax$ is a $\beta$-idempotent and $x\beta e$ is an $\alpha$-idempotent. Then $x = (x\beta e)\alpha(eax)$ is a $\beta$-idempotent. Next let $y \in V_\beta^\alpha(e)$. Then $e\beta yae = e$ and $yae\beta ey = y$. Now $yae$ is a $\beta$-idempotent and $e\beta y$ is an $\alpha$-idempotent. Then $y = (yae)\alpha(e\beta y)$ is a $\beta$-idempotent. Conversely suppose that $S$ satisfies the given conditions. Let $e$ be an $\alpha$-idempotent and $f$ a $\beta$-idempotent of $S$.\hfill \Box
Let us now consider $e\alpha f$. Now $e\alpha f \in S$ and since $S$ is regular, there exists $x \in S$ and $\gamma, \delta \in \Gamma$ such that $(e\alpha f)\gamma x\delta(e\alpha f) = e\alpha f$, $x\delta(e\alpha f)\gamma x = x$. Let $g = f\gamma x\delta e$. Then

$$g\alpha g = f\gamma x\delta e \alpha f\gamma x\delta e$$

$$= f\gamma x\delta e \alpha f\gamma x\delta e$$

$$= f\gamma x\delta e$$

$$= g$$

Further

$$(e\alpha f)\beta (f\gamma x\delta e)\alpha (e\alpha f) = e\alpha f\gamma x\delta e \alpha f$$

$$= e\alpha f$$

and $$(f\gamma x\delta e)\alpha (e\alpha f)\beta (f\gamma x\delta e) = (f\gamma x\delta e)\alpha (f\gamma x\delta e)$$

$$= f\gamma x\delta e$$

Hence $e\alpha f \in V^\beta(f\gamma x\delta e)$. Then by the given condition, $e\alpha f$ is a $\beta$-idempotent. Dually, we can prove that $f\alpha e$ is a $\beta$-idempotent.

\[\square\]

**Lemma 4.2.9.** Let $S$ be an inverse $\Gamma$-semiring. Let $a, b \in S$ and $a' \in V^\alpha_{\alpha_1}(a)$, $b' \in V^\beta_{\beta_1}(b)$ where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Gamma$. Then $b'\beta_2 a' \in V^\alpha_{\alpha_1}(a) b$ and $b'\alpha_1 a' \in V^\beta_{\beta_1}(a) b'$.

**Proof.** Since $a' \in V^\alpha_{\alpha_1}(a)$ and $b' \in V^\beta_{\beta_1}(b)$, $a\alpha_1 a'\alpha_2 a = a$, $a'\alpha_2 a\alpha_1 a' = a'$ and $b\beta_1 b'\beta_2 b = b$, $b'\beta_2 b\beta_1 b' = b'$. Now $a'\alpha_2 a$ is an $\alpha_1$-idempotent and $b\beta_1 b'$ is a $\beta_2$-idempotent. From theorem 4.2.7 it follows that $a'\alpha_2 a\alpha_1 b\beta_1 b'$ is a $\beta_2$-idempotent, $b\beta_1 b'\beta_2 a'\alpha_2 a$ is an $\alpha_1$-idempotent, $a'\alpha_2 a\beta_2 b\beta_1 b'$ is an $\alpha_1$-idempotent and $b\beta_1 b'\alpha_1 a'\alpha_2 a$ is a $\beta_2$-idempotent.

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Now,
\[(a_1b)\beta_1(b_2a')\alpha_2(a_1b) = a_1a'\alpha_2a_1b\beta_1b_2a'\alpha_2a_1b\beta_1b_2b\]
\[= a_1(a_1'\alpha_2a_1b\beta_1b_2(a_2\alpha_1b\beta_1b'))\beta_2b\]
\[= a_1a'\alpha_2a_1(b\beta_1b_2b)\]
\[= a_1a\]

\[(b'\beta_2a')\alpha_2(a_1b)\beta_1(b_2a') = b'\beta_2b_1b_2a'\alpha_2a_1b_1b_2a'\alpha_2a_1a'
\[= b'\beta_2(b\beta_1b_2a')\alpha_2(b\beta_1b\beta_2a)a_1a'
\[= (b'\beta_2b_1b')\beta_2(a'\alpha_2a_1a')
\[= b'\beta_2a'
\]
Hence \(b'\beta_2a' \in V_{\beta_1}^{a_2}(a_1b)\). Similarly we can prove that \(b'\alpha_1a' \in V_{\beta_1}^{a_2}(a\beta_2b)\). \hfill \Box

**Theorem 4.2.10.** A regular \(\Gamma\)-semiring \(S\) is an orthodox \(\Gamma\)-semiring if and only if for \(a, b \in S\), \(\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Gamma\), \(a' \in V_{\alpha_1}^{a_2}(a)\), \(b' \in V_{\beta_1}^{\beta_2}(b)\) imply that \(b'\beta_2a' \in V_{\beta_1}^{a_2}(a_1b)\) and \(b'\alpha_1a' \in V_{\beta_1}^{a_2}(a\beta_2b)\).

**Proof.** Let us assume that \(S\) is an orthodox \(\Gamma\)-semiring. Let \(a' \in V_{\alpha_1}^{a_2}(a)\) and \(b' \in V_{\beta_1}^{\beta_2}(b)\). Then \(a_1a'\alpha_2a = a, a'\alpha_2a_1a' = a', b_1b_2b = b, b'\beta_2b_1b' = b'\). Then by lemma 4.2.9, \(b'\beta_2a' \in V_{\beta_1}^{a_2}(a_1b)\) and \(b'\alpha_1a' \in V_{\beta_1}^{a_2}(a\beta_2b)\).

Conversely assume that the given conditions hold in \(S\). Let \(e\) be an \(\alpha\)-idempotent and \(f\) be a \(\beta\)-idempotent of \(S\). Now, \(f \in V_{\beta}^{\beta}(f)\) and \(e \in V_{\alpha}^{\alpha}(e)\). Then by the given conditions \(e\alpha f \in V_{\alpha}^{\beta}(f\beta e)\) and \(e\beta f \in V_{\alpha}^{\beta}(f\alpha e)\). From (i), we get \((e\alpha f)\beta(f\beta e)\alpha(e\alpha f) = e\alpha f\). Hence \((e\alpha f)\beta(e\alpha f) = e\alpha f\). Thus \(e\alpha f\) is a \(\beta\)-idempotent. From (ii), we get \((f\alpha e)\alpha(e\beta f)\beta(f\alpha e) = f\alpha e\). Hence, \((f\alpha e)\beta(f\alpha e) = f\alpha e\). So, \(f\alpha e\) is a \(\beta\)-idempotent. Hence \(S\) is an orthodox \(\Gamma\)-semiring. \hfill \Box
4.3 Homomorphisms on Orthodox $\Gamma$-semirings

In this section we deal with homomorphisms on orthodox $\Gamma$-semirings.

**Lemma 4.3.1.** Let $S$ be a regular $\Gamma$-semiring and $S'$ be a $\Gamma'$-semiring. Let $(f,g)$ be a homomorphism from $(S,\Gamma)$ onto $(S',\Gamma')$. Let $e'$ be an $\alpha'$-idempotent of $S'$. Then $f^{-1}(e')$ contains an idempotent of $S$.

**Proof.** Let $a \in S$ be such that $f(a) = e' = e'\alpha' e'$ where $\alpha' \in \Gamma'$. Let $\alpha \in \Gamma$ be such that $g(\alpha) = \alpha'$. Now let us consider the element $aaa$. As $S$ is a regular $\Gamma$-semiring, there exist $\beta, \gamma \in \Gamma$ and $c \in S$ such that $(aaa)\beta c \gamma (aaa) = aaa$ and $c \gamma (aaa) \beta c = c$. Now, $a\beta c \gamma a$ is an $\alpha$-idempotent in $S$ since $(a\beta c \gamma a)\alpha (a\beta c \gamma a) = \alpha (c \gamma a a \beta c) \gamma a = a \beta c \gamma a$. Also,

$$f(a\beta c \gamma a) = f(a)g(\beta)f(c)g(\gamma)f(a)$$

$$= e'\alpha' e' g(\beta)f(c)g(\gamma)e'\alpha' e'$$

$$= f(a a a)g(\beta)f(c)g(\gamma)f(a a a)$$

$$= f(a a a \beta c \gamma a a a)$$

$$= f(a a a) = e'\alpha' e' = e'$$

Hence $f^{-1}(e')$ contains an idempotent of $S$. 

**Lemma 4.3.2.** Let $S$ be a regular $\Gamma$-semiring and let $S'$ be a $\Gamma'$-semiring. Let $(f,g)$ be a homomorphism from $(S,\Gamma)$ onto $(S',\Gamma')$. Then $S'$ is a regular $\Gamma'$-semiring.

**Proof.** Let $a' \in S'$. There exists $a \in S$ such that $f(a) = a'$. Since $S$ is a regular $\Gamma$-semiring, there exists $b \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a a b \beta a$. 

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Then

\[
a' = f(a) \\
= f(a\alpha b\beta a) \\
= f(a)g(\alpha)f(b)g(\beta)f(a) \\
= a'g(\alpha)f(b)g(\beta)a'
\]

Thus \(a'\) is a regular element of \(S'\). Hence \(S'\) is a regular \(\Gamma'\)-semiring.

\[\square\]

**Theorem 4.3.3.** Let \(S\) be an orthodox \(\Gamma\)-semiring and \(S'\) be a \(\Gamma'\)-semiring. Let \((\varphi, \psi)\) be a homomorphism from \((S, \Gamma)\) onto \((S', \Gamma')\). Then \(S'\) is an orthodox \(\Gamma'\)-semiring.

**Proof.** Since \(S\) is regular, by lemma 4.3.2 it follows that \(S'\) is also regular \(\Gamma'\)-semiring. Let \(e' = e'\alpha' e'\) and \(f' = f'\beta' f'\) where \(\alpha', \beta' \in \Gamma'\) be two arbitrary idempotents of \(S'\). By lemma 4.3.1, \(\varphi^{-1}(e')\) and \(\varphi^{-1}(f')\) both contain idempotents of \(S\). Let \(E_S\) denote the set of idempotents of \(S\). Let \(e \in \varphi^{-1}(e') \cap E_S\) and \(eae = e\), where \(\psi(\alpha) = \alpha'\) and \(f \in \varphi^{-1}(f') \cap E_S\) and \(f\beta f = f\) where \(\psi(\beta) = \beta'\). Now since \(S\) is an orthodox \(\Gamma\)-semiring, \(eaf, fae\) are \(\beta\)-idempotents and \(e\beta f, f\beta e\) are \(\alpha\)-idempotents. Then \(\varphi(eaf) = e'\alpha' f' \in E_{S'}\) and \(e'\alpha' f'\) is \(\beta'\)-idempotent. Similarly \(f'\alpha' e'\) is a \(\beta'\)-idempotent. Hence \(S'\) is an orthodox \(\Gamma'\)-semiring.

\[\square\]