Chapter 3

Inverse $\Gamma$-semirings

In [67, 68], Vagner initiated the study of inverse semigroups and in [52, 53] and [54], it was independently studied by Preston. In this chapter, we introduce a class of $\Gamma$-semirings known as inverse $\Gamma$-semiring and discuss some results of inverse $\Gamma$-semirings. We also discuss homomorphisms on inverse $\Gamma$-semirings.
3.1 Basic Definitions and Results

**Definition 3.1.1.** A regular $\Gamma$-semiring $S$ is called an inverse $\Gamma$-semiring if $|V_\alpha^\beta(a)| = 1$, for all $a \in S$ and for all $\alpha, \beta \in \Gamma$, whenever $V_\alpha^\beta(a) \neq \phi$. That is every element $a$ of $S$ has a unique $(\alpha, \beta)$ inverse whenever $(\alpha, \beta)$ inverse of $a$ exists.

**Theorem 3.1.1.** Let $S$ be a $\Gamma$-semiring. $S$ is an inverse $\Gamma$-semiring if and only if

(i) $S$ is regular and

(ii) if $e$ and $f$ are any two $\alpha$-idempotents of $S$, then $e\alpha f = f\alpha e$, where $\alpha \in \Gamma$.

**Proof.** Suppose $S$ is an inverse $\Gamma$-semiring. Then by definition 3.1.1, $S$ is regular. Let $e$ and $f$ be two $\alpha$-idempotents of $S$. Let $a \in V_\beta^\alpha(e\alpha f)$. Then $(e\alpha f)\beta a\gamma(e\alpha f) = e\alpha f$ and $a\gamma(e\alpha f)\beta a = a$. Now,

$$(f\beta a\gamma e)\alpha(f\beta a\gamma e) = f\beta(a\gamma e\alpha f\beta a)\gamma e$$

$$= f\beta a\gamma e$$

Hence, $f\beta a\gamma e$ is an $\alpha$-idempotent. Further,

$$(e\alpha f)\alpha(f\beta a\gamma e)\alpha(e\alpha f) = e\alpha f\beta a\gamma e\alpha f$$

$$= e\alpha f$$

and

$$(f\beta a\gamma e)\alpha(e\alpha f)\alpha(f\beta a\gamma e) = f\beta(a\gamma e\alpha f\beta a)\gamma e$$

$$= f\beta a\gamma e$$

Hence, $e\alpha f \in V_\alpha^\alpha(f\beta a\gamma e)$. But $f\beta a\gamma e$ being an $\alpha$-idempotent belongs to $V_\alpha^\alpha(f\beta a\gamma e)$. Hence $e\alpha f = f\beta a\gamma e$, since $S$ is an inverse $\Gamma$-semiring. Hence $e\alpha f$ is an $\alpha$-idempotent. Similarly we can show that $f\alpha e$ is an $\alpha$-idempotent of $S$. Now, $(e\alpha f)\alpha(f\alpha e)\alpha(e\alpha f) = (e\alpha f)\alpha(e\alpha f) = e\alpha f$ and $(f\alpha e)\alpha(e\alpha f)\alpha(f\alpha e) = (f\alpha e)\alpha(f\alpha e) = f\alpha e$. So, $f\alpha e \in V_\alpha^\alpha(e\alpha f)$. But $e\alpha f \in V_\alpha^\alpha(e\alpha f)$. Hence $e\alpha f = f\alpha e$.}

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Conversely let conditions (i) and (ii) hold. We shall prove that \( S \) is an inverse \( \Gamma \)-semiring. Let \( a \in S \) and \( b, c \in V_{\alpha}^{\beta}(a) \). Then \( a\alpha b\beta a = a \) and \( b\beta a\alpha b = b \). Also \( a\alpha c\beta a = a \) and \( c\beta a\alpha c = c \). Now, each of \( a\alpha b \) and \( a\alpha c \) is \( \beta \)-idempotent of \( S \) and each of \( b\beta a \) and \( c\beta a \) is \( \alpha \)-idempotent of \( S \). Further, \( a\alpha b\alpha c\alpha c = a\alpha c\beta a\alpha b \). Therefore, \( a\alpha c = a\alpha b \). Similarly \( b\beta a\alpha c\alpha b = c\beta a\alpha b \). Hence \( b\beta a = c\beta a \). Then \( b = b\beta a\alpha b = b\beta a\alpha c = c\beta a\alpha c = c \). Thus \( S \) is an inverse \( \Gamma \)-semiring.

\[ \square \]

**Theorem 3.1.2.** Let \( S \) be an inverse \( \Gamma \)-semiring. If \( e \) is an \( \alpha \)-idempotent and \( f \) is a \( \beta \)-idempotent of \( S \) such that \( e\alpha f = e\beta f \), \( f\beta e = f\alpha e \), then \( e\alpha f = f\alpha e \).

**Proof.** Let \( a \in V_{\gamma}^{\delta}(e\alpha f) \). Then \( (e\alpha f)\gamma\delta (e\alpha f) = e\alpha f \) and \( a\delta (e\alpha f)\gamma a = a \). Now, \( f\gamma a\delta e \) is \( \alpha \)-idempotent. Also, \( e\alpha f \in V_{\alpha}^{\beta}(f\gamma a\delta e) \). Again,

\[
(f\gamma a\delta e)\alpha (f\gamma a\delta e)\beta (f\gamma a\delta e) = f\gamma a\delta (e\alpha f)\gamma a\delta (e\beta f)\gamma a\delta e
\]

\[= f\gamma a\delta (e\alpha f)\gamma a\delta (e\alpha f)\gamma a\delta e\]

\[= f\gamma a\delta e\alpha f\gamma a\delta e\alpha f\gamma a\delta e\]

\[= f\gamma a\delta e\]

\[
(f\gamma a\delta e)\beta (f\gamma a\delta e)\alpha (f\gamma a\delta e) = f\gamma a\delta (e\beta f)\gamma a\delta (e\alpha f)\gamma a\delta e
\]

\[= f\gamma a\delta (e\alpha f)\gamma a\delta (e\alpha f)\gamma a\delta e\]

\[= (f\gamma a\delta e)\alpha (f\gamma a\delta e)\alpha (f\gamma a\delta e)\]

\[= f\gamma a\delta e\]

Hence \( f\gamma a\delta e \in V_{\alpha}^{\beta}(f\gamma a\delta e) \). Since \( S \) is an inverse \( \Gamma \)-semiring, \( e\alpha f = f\gamma a\delta e \). Therefore, \( e\alpha f \) is an \( \alpha \)-idempotent of \( S \). Similarly we can show that \( f\alpha e \) is an \( \alpha \)-idempotent of \( S \).
Further,
\[(e\alpha f)\beta(f\alpha e)\alpha(e\alpha f) = (e\alpha f)\alpha(e\alpha f)\]
\[= e\alpha f\]
and \[(f\alpha e)\alpha(e\alpha f)\beta(f\alpha e) = (f\alpha e)\alpha(f\alpha e)\]
\[= f\alpha e\]
Hence, \(f\alpha e \in V_{\alpha}^\beta(e\alpha f)\). But \(e\alpha f \in V_{\beta}^\alpha(e\alpha f)\). Hence \(e\alpha f = f\alpha e\).

\[\square\]

**Theorem 3.1.3.** Let \(S\) be an inverse \(\Gamma\)-semiring. If \(e\) is an \(\alpha\)-idempotent and \(f\) is a \(\beta\)-idempotent of \(S\), then \(e\alpha f, f\alpha e\) are \(\beta\)-idempotents and \(e\beta f, f\beta e\) are \(\alpha\)-idempotents of \(S\).

**Proof.** Let \(e\) be an \(\alpha\)-idempotent and \(f\) be a \(\beta\)-idempotent of \(S\). We first show that \(e\alpha f\) is a \(\beta\)-idempotent. Now \(e\alpha f \in S\). There exist \(\gamma, \delta \in \Gamma\) and \(x \in S\) such that \(x \in V_{\delta}^\gamma(e\alpha f)\). Then \((e\alpha f)\delta x\gamma(e\alpha f) = e\alpha f\) and \(x\gamma(e\alpha f)\delta x = x\). Let \(g = f\delta x\gamma e\alpha f\) and \(h = f\delta x\gamma e\). It can be seen easily that \(g\beta g = g\).

Also,
\[(f\delta x\gamma e\alpha f)\beta(f\delta x\gamma e)\alpha(f\delta x\gamma e\alpha f) = f\delta(x\gamma e\alpha f\delta x)\gamma(e\alpha f\delta x\gamma e\alpha f)\]
\[= f\delta x\gamma e\alpha f\]
\[= g\]
This shows that \(g\beta h \alpha g = g\). Similarly \(h \alpha g \beta h = h\). Hence, \(g \in V_{\alpha}^\beta(h)\). Also, \(e\alpha f \in V_{\beta}^\alpha(h)\). Since \(S\) is an inverse \(\Gamma\)-semiring, \(g = e\alpha f\). Hence \(e\alpha f\) is a \(\beta\)-idempotent. Proceeding as above, we can show that \(f\alpha e\) is a \(\beta\)-idempotent and both \(e\beta f\) and \(f\beta e\) are \(\alpha\)-idempotents.
Theorem 3.1.4. Let $S$ be an inverse $\Gamma$-semiring. Let $a \in S$. If $a' \in V_\gamma^\delta(a)$, $\gamma, \delta \in \Gamma$, then for any $\alpha$-idempotent $e$ of $S$,

(i) $a\gamma e a'$, $a\alpha e a'$ are $\delta$-idempotents

(ii) $a'\delta e a$, $a'\alpha e a$ are $\gamma$-idempotents.

Proof. Let $\gamma, \delta \in \Gamma$ such that $V_\gamma^\delta(a) \neq \phi$. Let $a' \in V_\gamma^\delta(a)$. Then $a'\delta a$ is a $\gamma$-idempotent. From theorem 3.1.3, it follows that $e\alpha a'\delta a$ is a $\gamma$-idempotent, $e\gamma a'\delta a$ is an $\alpha$-idempotent and $a\gamma a'$ is a $\delta$-idempotent. Hence $e\alpha a'\delta a$ is a $\gamma$-idempotent and $e\delta a\gamma a'$ is an $\alpha$-idempotent. Now,

$$(a\gamma e a')\delta(a\gamma e a') = a\gamma (e\alpha a'\delta a)\gamma (e\gamma a'\delta a)\gamma a' \text{ since } a' \in V_\gamma^\delta(a)$$

$$= a\gamma e a'\delta a a' \text{ since } e\alpha a'\delta a \text{ is a } \gamma \text{-idempotent}$$

$$= a\gamma e a'$$

$$(a\alpha e a')\delta(a\alpha e a') = a\alpha (e\gamma a'\delta a)\alpha (e\gamma a'\delta a)\gamma a' \text{ since } a' \in V_\gamma^\delta(a)$$

$$= a\alpha e a'\delta a a' \text{ since } e\gamma a'\delta a \text{ is an } \alpha \text{-idempotent}$$

$$= a\alpha e a'$$

Similarly we can prove (ii).

Theorem 3.1.5. Let $S$ be an inverse $\Gamma$-semiring. Let $a, b \in S$ and $a' \in V_{\alpha_1}^{\alpha_2}(a)$, $b' \in V_{\beta_1}^{\beta_2}(b)$ where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Gamma$. Then $b'\beta_2 a' \in V_{\beta_1}^{\alpha_2}(a\alpha_1 b)$ and $b'\alpha_1 a' \in V_{\beta_1}^{\alpha_2}(a\beta_1 b)$.

Proof. Since $a' \in V_{\alpha_1}^{\alpha_2}(a)$ and $b' \in V_{\beta_1}^{\beta_2}(b)$, $a\alpha_1 a'\alpha_2 a = a$, $a'\alpha_2 a a_1 a' = a'$ and $b\beta_1 b'\beta_2 b = b$, $b'\beta_2 b\beta_1 b' = b'$. Now $a'\alpha_2 a$ is an $\alpha_1$-idempotent and $b\beta_1 b'$ is a $\beta_2$-idempotent. From theorem 3.1.3, it follows that $a'\alpha_2 a a_1 b\beta_1 b'$ is a $\beta_2$-idempotent,
$b\beta_1 b' \beta_2 a' \alpha_2 a$ is an $\alpha_1$-idempotent, $a' \alpha_2 a \beta_2 b \beta_1 b'$ is an $\alpha_1$-idempotent and $b\beta_1 b' \alpha_1 a' \alpha_2 a$ is a $\beta_2$-idempotent.

$$(a\alpha_1 b) \beta_1 (b' \beta_2 a') \alpha_2 (a\alpha_1 b) = a\alpha_1 a' \alpha_2 a \alpha_1 b \beta_1 b' \beta_2 a' \alpha_2 \alpha_1 b \beta_2 b$$

$$= a\alpha_1 (a' \alpha_2 a \alpha_1 b \beta_1 b') \beta_2 (a' \alpha_2 a \alpha_1 b \beta_1 b') \beta_2 b$$

$$= a\alpha_1 a' \alpha_2 a \alpha_1 b \beta_1 b' \beta_2 b$$ since $a' \alpha_2 a \alpha_1 b \beta_1 b'$ is a $\beta_2$-idempotent

$$= a\alpha_1 b$$

$$(b' \beta_2 a') \alpha_2 (a\alpha_1 b) \beta_1 (b' \beta_2 a') = b' \beta_2 b \beta_1 \beta_2 a' \alpha_2 a \alpha_1 b \beta_2 a' \alpha_2 a \alpha_1 a'$$

$$= b' \beta_2 (b \beta_1 \beta_2 a' \alpha_2 a) a_1 (b \beta_1 \beta_2 a' \alpha_2 a) \alpha_1 a'$$

$$= b' \beta_2 b \beta_1 \beta_2 a' \alpha_2 a \alpha_1 a'$$ since $b \beta_1 \beta_2 a' \alpha_2 a$ is an $\alpha_1$-idempotent

$$= b' \beta_2 a'$$

Hence $b' \beta_2 a' \in V_{\beta_1}^{a_2} (a\alpha_1 b)$. Similarly we can prove that $b' \alpha_1 a' \in V_{\beta_1}^{a_2} (a\beta_2 b)$.

3.2 Natural Partial Order on an Inverse $\Gamma$-semiring

In this section we want to study partial order relation ‘$\leq$’ as defined in the definition 2.4.1 when $S$ is an inverse $\Gamma$-semiring.

**Lemma 3.2.1.** Let $S$ be an inverse $\Gamma$-semiring. If $e$ is an $\alpha$-idempotent in $S$ and $a$ is an element of $S$ such that $V_{\gamma}^{\delta} (a) \neq \phi$ for some $\gamma, \delta \in \Gamma$, then

(i) $e a \alpha = a \gamma e_1$ for some $\gamma$-idempotent $e_1$.

(ii) $a a e = e_2 \delta a$ for some $\delta$-idempotent $e_2$.

**Proof.** Let $a' \in V_{\gamma}^{\delta} (a)$. Then $a' \gamma a' \delta a = a$ and $a' \delta a \gamma a' = a'$.
(i)
\[ e \alpha a = (e\alpha a' \gamma) \delta (a \gamma a') \delta a \]
\[ = (a \gamma a') \delta (e\alpha a' \gamma) \delta a \] since by theorem 3.1.1, \( e\alpha a' \gamma a' \) and \( a \gamma a' \) are both \( \delta \)-idempotents
\[ = a \gamma a' \delta e \alpha a \]
\[ = a \gamma e_1 \] since by theorem 3.1.4, \( e_1 = a' \delta e \alpha a \) is a \( \gamma \)-idempotent

(ii)
\[ a \alpha e = a \gamma (a' \delta a) \gamma (a' \delta a \alpha e) \]
\[ = a \gamma (a' \delta a \alpha e) \gamma (a' \delta a) \] since by theorem 3.1.1, \( a' \delta a \) and \( a' \delta a \alpha e \) are both \( \gamma \)-idempotents
\[ = (a \alpha e \gamma a') \delta a \]
\[ = e_2 \delta a \] since by theorem 3.1.4, \( e_2 = a \alpha e \gamma a' \) is a \( \delta \)-idempotent

\[ \square \]

**Lemma 3.2.2.** Let \( S \) be an inverse \( \Gamma \)-semiring. Suppose that \( \alpha, \beta \in \Gamma \) and \( a' \in V_\alpha^\beta(a) \) for \( a \in S \). Then \( a \alpha a' = b \alpha a' \) if and only if \( a' \beta a = a' \beta b \), where \( b \in S \).

**Proof.** Let \( a \alpha a' = b \alpha a' \). Since \( a' \in V_\alpha^\beta(a) \), it follows that \( a \alpha a' \) is a \( \beta \)-idempotent. Hence \( b \alpha a' \) is a \( \beta \)-idempotent. Now,
\[ (a' \beta b) \alpha (a' \beta b) = a' \beta (b \alpha a') \beta b \]
\[ = a' \beta (a \alpha a') \beta b \] since \( a \alpha a' = b \alpha a' \)
\[ = a' \beta b \]

which shows that \( a' \beta b \) is an \( \alpha \)-idempotent.
Then

\[ a' \beta a = a' \beta a a' \beta a = a' \beta (a a a') \beta a \]
\[ = a' \beta (b a a') \beta a \text{ since } a a a' = b a a' \]
\[ = (a' \beta b) \alpha (a' \beta a) \]
\[ = (a' \beta a) \alpha (a' \beta b) \text{ since theorem 3.1.1 both } a' \beta a \text{ and } a' \beta b \text{ are } \alpha \text{-idempotents} \]
\[ = a' \beta b \]

Similarly we can prove the converse part.

\[ \square \]

**Theorem 3.2.3.** Let \( S \) be an inverse \( \Gamma \)-semiring and let \( a, b \in S \). Then the following conditions are equivalent:

(i) \( a \leq b \) in \( S \).

(ii) \( a = e \delta b \) for some \( \delta \)-idempotent \( e \in S \).

(iii) \( a = b \gamma_1 f \) for some \( \gamma_1 \)-idempotent \( f \in S \).

*Proof.*

(i) \( \Rightarrow \) (ii): Let \( a \leq b \). Then by definition 2.4.1 there exist \( \gamma, \delta \in \Gamma \) and \( a' \in V_{\gamma}^{\delta}(a) \) such that \( a' \delta a = a' \delta b \), \( a \gamma a' = b \gamma a' \). Hence \( a = a \gamma (a' \delta a) = a \gamma (a' \delta b) = (a \gamma a') \delta b = e \delta b \), where \( e = a \gamma a' \) is a \( \delta \)-idempotent.

(ii) \( \Rightarrow \) (iii): Let \( a = e \delta b \) for some \( \delta \)-idempotent \( e \in S \). Suppose that there exist \( \gamma_1, \delta_1 \in \Gamma \) such that \( V_{\gamma_1}^{\delta_1}(b) \neq \phi \). Let \( b' \in V_{\gamma_1}^{\delta_1}(b) \). From lemma 3.2.1 it follows that \( e \delta b = b \gamma_1 f \) for some \( \gamma_1 \)-idempotent \( f \in S \). Hence \( a = e \delta b = b \gamma_1 f \) for some \( \gamma_1 \)-idempotent \( f \in S \).
(iii) \(\Rightarrow\) (i): Let \(a = b\gamma_1 f\) for some \(\gamma_1\)-idempotent \(f \in S\). Suppose that there exist \(\delta_1, \delta_2 \in \Gamma\) such that \(V_{\delta_1}^\beta(b) \neq \phi\). Let \(b' \in V_{\delta_1}^\beta(b)\). Also, \(f \in V_{\gamma_1}^\beta(f)\). Then from theorem 3.1.5 it follows that \(f\delta_1 b' \in V_{\gamma_1}^\beta(b\gamma_1 f)\). That is \(f\delta_1 b' \in V_{\gamma_1}^\beta(a)\).

Now

\[
a\gamma_1(f\delta_1 b') = (a\gamma_1 f)\delta_1 b' = (b\gamma_1 f\gamma_1 f)\delta_1 b' = (b\gamma_1 f)\delta_1 b' = b\gamma_1(f\delta_1 b')
\]

That is \(a\gamma_1 a' = b\gamma_1 a'\), where \(a' = f\delta_1 b'\). From lemma 3.2.2, it then follows that \(a'\delta_2 b = a'\delta_2 b\). Hence \(a \leq b\).

**Theorem 3.2.4.** Let \(S\) be an inverse \(\Gamma\)-semiring with \(E\) as the set of idempotents. Then the restriction of the partial order relation \(\leq\) to \(E\) is identical with the natural partial order relation \(\equiv\) (as defined in the definition 2.5.1).

**Proof.** Let \(e\) be an \(\alpha\)-idempotent and \(f\) be a \(\beta\)-idempotent in \(E\) such that \(e \leq f\). Then from theorem 3.2.3 there exist \(\delta\)-idempotent \(g\) and \(\gamma\)-idempotent \(h\) such that \(e = g\delta f\) and \(e = f\gamma h\). Then \(e\beta f = (g\delta f)\beta f = g\delta(f\beta f) = g\delta f = e\). Similarly \(f\beta e = e\). Now, \(g\) is a \(\delta\)-idempotent and \(f\) is a \(\beta\)-idempotent. From theorem 3.1.3, it follows that \(e = g\delta f\) is a \(\beta\)-idempotent. Since both \(e\alpha f\) and \(e\) are \(\beta\)-idempotents, it follows from theorem 3.1.1 that \(e\beta(e\alpha f) = (e\alpha f)\beta e = e\).
Then

\[ e\alpha f = (e\beta e)\alpha f \]
\[ = e\beta(e\alpha f) \]
\[ = (e\alpha f)\beta e \text{ by (i)} \]
\[ = e\alpha(f\beta e) \]
\[ = e\alpha e \]
\[ = e \]

Thus \( e = e\alpha f = f\beta e \). Hence \( epf \).

Conversely assume that \( e \) is an \( \alpha \)-idempotent and \( f \) is a \( \beta \)-idempotent of \( S \) such that \( epf \). Then \( e = e\alpha f = f\beta e \). Then it follows from theorem 3.2.3 that \( e \leq f \). This partial order \( \leq \) on an inverse \( \Gamma \)-semiring \( S \) is called the natural partial order on \( S \). \( \square \)

**Definition 3.2.1.** A \( \Gamma \)-semiring \( S \) together with a partial order relation \( \leq \) on the set \( S \) is said to be a partially ordered \( \Gamma \)-semiring if for \( a, b \in S \), \( a \leq b \) implies that \( c\alpha a \leq c\alpha b \) and \( a\alpha c \leq b\alpha c \) for all \( c \in S \), \( \alpha \in \Gamma \).

**Example 3.2.5.** Let \((A, \leq_1)\) and \((B, \leq_2)\) be two partially ordered sets. A mapping \( f : A \rightarrow B \) is said to be an isotone mapping if \( x \leq_1 y \) implies \( f(x) \leq_2 f(y) \), \( x, y \in A \).

Let \( S \) be the set of all isotone mappings \( f : A \rightarrow B \) and \( \Gamma \) be the set of all isotone mappings \( \alpha : B \rightarrow A \). Let \( f, g \in S \) and \( \alpha \in \Gamma \). Define \( f\alpha g : A \rightarrow B \) by \((f\alpha g)(x) = f(\alpha(g(x)))\) for all \( x \in A \). Then we can show easily that \( S \) is a \( \Gamma \)-semiring. Define a relation \( \leq \) on \( S \) by \( f \leq g \) if and only if \( f(x) \leq g(x) \) for all \( x \in A \). This is a partial order relation on \( S \). Also it can be shown that for this relation \( \leq \), \( S \) is a partially ordered \( \Gamma \)-semiring.
Theorem 3.2.6. An inverse $\Gamma$-semiring $S$ together with the natural partial order relation $\leq$ is a partially ordered $\Gamma$-semiring.

Proof. Let $a, b \in S$ and let $a \leq b$. Then there exists an $\alpha$-idempotent $e$ such that $a = e\alpha b$. Then $a\gamma c = e\alpha b\gamma c$. Hence $a\gamma c \leq b\gamma c$. Further, $c\gamma a = c\gamma e\alpha b$. Since $S$ is an inverse $\Gamma$-semiring, there exist $\beta, \delta \in \Gamma$ and $d \in S$ such that $b = b\beta d\delta b$ and $d = d\delta b\beta d$. From lemma 3.2.1, it follows that $e\alpha b = b\beta e_1$ for some $\beta$-idempotent $e_1 \in S$. Hence $c\gamma a = c\gamma e\alpha b = c\gamma b\beta e_1$. Hence $c\gamma a \leq c\gamma b$. 

3.3 Homomorphisms on Inverse $\Gamma$-semirings

Definition 3.3.1. Let $S$ be a $\Gamma$-semiring and $S'$ be a $\Gamma'$-semiring. A pair of mappings $f_1 : S \to S'$ and $f_2 : \Gamma \to \Gamma'$ is said to be a homomorphism from $(S, \Gamma)$ into $(S', \Gamma')$ if

(i) $f_1(a + b) = f_1(a) + f_1(b)$ 
(ii) $f_1(aab) = f_1(a)f_2(\alpha)f_1(b)$ for all $a, b \in S$ and $\alpha \in \Gamma$.

Lemma 3.3.1. Let $S$ be a regular $\Gamma$-semiring and let $S'$ be a $\Gamma'$-semiring. Let $(f,g)$ be a homomorphism from $(S, \Gamma)$ onto $(S', \Gamma')$. Then $S'$ is a regular $\Gamma'$-semiring.

Proof. Let $a' \in S'$. There exists $a \in S$ such that $f(a) = a'$. Since $S$ is a regular $\Gamma$-semiring, there exists $b \in S$ and $\alpha, \beta \in \Gamma$ such that $a = aab\beta a$.

Then

$$a' = f(a)$$
$$= f(aab\beta a)$$
$$= f(a)g(\alpha)f(b)g(\beta)f(a)$$
$$= a'g(\alpha)f(b)g(\beta)a'$$

Thus $a'$ is a regular element of $S'$. Hence $S'$ is a regular $\Gamma'$-semiring. 

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Lemma 3.3.2. Let $S$ be a regular $\Gamma$-semiring and $S'$ be a $\Gamma'$-semiring. Let $(f,g)$ be a homomorphism from $(S,\Gamma)$ onto $(S',\Gamma')$. Let $e'$ be an $\alpha'$-idempotent of $S'$. Then $f^{-1}(e')$ contains an idempotent of $S$.

Proof. Let $a \in S$ be such that $f(a) = e' = e'\alpha'e'$ where $\alpha' \in \Gamma'$. Let $\alpha \in \Gamma$ be such that $g(\alpha) = \alpha'$. Now let us consider the element $a\alpha a$. As $S$ is a regular $\Gamma$-semiring, there exist $\beta, \gamma \in \Gamma$ and $c \in S$ such that $(a\alpha a)\beta c \gamma (a\alpha a) = a\alpha a$ and $c \gamma (a\alpha a) \beta c = c$. Now, $a\beta c \gamma a$ is an $\alpha$-idempotent in $S$ since $(a\beta c \gamma a)\alpha (a\beta c \gamma a) = a\beta (c \gamma a \alpha \beta c) \gamma a = a\beta c \gamma a$. Also,

$$f(a\beta c \gamma a) = f(a)g(\beta)f(c)g(\gamma)f(a)$$

$$= e'\alpha' e'g(\beta)f(c)g(\gamma)e'\alpha'e'$$

$$= f(a\alpha a)g(\beta)f(c)g(\gamma)f(a\alpha a)$$

$$= f(a\alpha a)\beta c \gamma a\alpha a$$

$$= f(a\alpha a)$$

$$= e'\alpha' e'$$

$$= e'$$

Hence $f^{-1}(e')$ contains an idempotent of $S$. \qed

Theorem 3.3.3. Let $S$ be an inverse $\Gamma$-semiring and $S'$ be a $\Gamma'$-semiring. Let $(f,g)$ be a homomorphism from $(S,\Gamma)$ onto $(S',\Gamma')$. Then $S'$ is an inverse $\Gamma'$-semiring. Moreover in any homomorphism the $(\alpha, \beta)$ inverse of an element of $S$ is mapped into the corresponding inverse of the image of that element.

Proof. By lemma 3.3.1, it follows that $S'$ is a regular $\Gamma'$-semiring. We shall show that any two $\alpha'$-idempotents of $S'$ are $\alpha'$-commutative where $\alpha' \in \Gamma'$. Let $e'_1, e'_2$ be two
α′-idempotents of $S'$. Since $(f, g)$ is onto homomorphism, there exist $e_1$ and $e_2$ of $S$ and $\alpha \in \Gamma$ such that $e_1$ and $e_2$ are $\alpha$-idempotents of $S$ and $f(e_i) = e_i', g(\alpha) = \alpha'$, $i = 1, 2$ by lemma 3.3.2. As $S$ is an inverse $\Gamma$-semiring, $e_1^\alpha e_2 = e_2^\alpha e_1$. Therefore $f(e_1^\alpha e_2) = f(e_2^\alpha e_1)$. So, $e_1' \alpha' e_2' = e_2' \alpha' e_1'$. Hence from theorem 3.1.1, it follows that $S'$ is an inverse $\Gamma'$-semiring. Moreover for $a \in S$, $f(a^{-1}_{a,\beta}) = f(a)_{g(\alpha), g(\beta)}^{-1}$. Indeed, $a a a^{-1}_{a,\beta} \beta a = a$ and $a^{-1}_{a,\beta} \beta a a a^{-1}_{a,\beta} = a^{-1}_{a,\beta}$. Therefore $f(a)g(\alpha)f(a^{-1}_{a,\beta})g(\beta)f(a) = f(a)$ and $f(a^{-1}_{a,\beta})g(\beta)f(a)g(\alpha)f(a^{-1}_{a,\beta}) = f(a^{-1}_{a,\beta})$. Thus $f(a^{-1}_{a,\beta}) = f(a)_{g(\alpha), g(\beta)}^{-1}$.

**Definition 3.3.2.** Let $S$ be a $\Gamma$-semiring. For any ideal $I$ of $S$, we define $\rho_I = I_S \cup (I \times I)$ where $I_S$ is the identity relation on $S$. The relation $\rho_I$ is a congruence on $S$ whose equivalence classes consist of the ideal $I$ and singletons. We define it as Rees $\Gamma$-congruence on $S$ and the quotient $S/\rho_I$ as Rees $\Gamma$-quotient of $S$ by $I$.

**Lemma 3.3.4.** Let $S$ be a $\Gamma$-semiring and $I$ be an ideal of $S$. Then the Rees $\Gamma$-quotient $S/\rho_I$ is a $\Gamma$-semiring and $S/\rho_I$ is a homomorphic image of $S$.

**Proof.** Let $a \rho_I, b \rho_I \in S/\rho_I$ and $\alpha \in \Gamma$. Then by definition 3.3.2 $(a \rho_I)\alpha(b \rho_I) = (aab)\rho_I$. It can be easily seen that $S/\rho_I$ is a $\Gamma$-semiring. Let us now consider the pair of mappings $(f, g)$, where $f : S \to S/\rho_I$ and $g : \Gamma \to \Gamma$ are defined by $f(a) = a \rho_I$ and $g(\alpha) = \alpha$ for all $a \in S$ and for all $\alpha \in \Gamma$.

Then

$$f(aab) = (aab)\rho_I$$
$$= (a \rho_I)\alpha(b \rho_I)$$
$$= f(a)g(\alpha)f(b) \quad \text{for all } a, b \in S \text{ and for all } \alpha \in \Gamma.$$
Moreover,

\[ f(a + b) = (a + b)\rho_I \]
\[ = a\rho_I + b\rho_I \]
\[ = f(a) + f(b) \]

Thus \((f, g)\) is a homomorphism from \((S, \Gamma)\) into \((S/\rho_I, \Gamma)\). Further for any \(a\rho_I \in S/\rho_I\), there exists \(a \in S\) such that \(f(a) = a\rho_I\). Hence \(f\) is an onto mapping and \(g\) is obviously an onto mapping. Hence \(S/\rho_I\) is a homomorphic image of \(S\).

**Theorem 3.3.5.** Let \(S\) be a \(\Gamma\)-semiring. Let \(I\) be an ideal of \(S\). Then \(S\) is an inverse \(\Gamma\)-semiring if and only if \(I\) and the Rees \(\Gamma\)-quotient \(S/\rho_I\) are both inverse \(\Gamma\)-semirings.

**Proof.** Let \(S\) be an inverse \(\Gamma\)-semiring and \(I\) be an ideal of \(S\). By lemma 3.3.4, \(S/\rho_I\) is a homomorphic image of \(S\). Then by theorem 3.3.3, \(S/\rho_I\) is an inverse \(\Gamma\)-semiring. Further let \(a \in I\). Then there exist \(\alpha, \beta \in \Gamma\) and \(b \in S\) such that \(a = a\alpha b\beta a\) and \(b = b\beta a\alpha b\). Since \(I\) is an ideal of \(S\) and \(a \in I, b \in b\Gamma\Gamma b \subseteq S\Gamma\Gamma S \subseteq I\). Thus \(I\) is an inverse \(\Gamma\)-semiring.

Conversely, let \(I\) be an ideal of the \(\Gamma\)-semiring \(S\) and let \(I\) and \(S/\rho_I\) are both inverse \(\Gamma\)-semirings. Let \(a \in S\). If \(a \in I\), then for \(\alpha, \beta \in \Gamma\) such that \(V_\alpha^\beta(a) \neq \phi\), there exists unique \(b \in V_\alpha^\beta(a)\) in \(I\) such that \(a\alpha b\beta a = a\) and \(b\beta a\alpha b = b\). Further since \(I\) is an ideal of \(S\), \(b\beta a\alpha b \in I\) for every \(b \in S\). Hence there exists unique \(b \in V_\alpha^\beta(a)\) in \(S\). If \(a \in S \setminus I\), then since \(S/\rho_I\) is an inverse \(\Gamma\)-semiring, for \(\gamma, \delta \in \Gamma\) such that \(V_\gamma^\delta(a) \neq \phi\), there exists unique \(c \in V_\gamma^\delta(a)\) in \(S \setminus I\) such that \(a\gamma c\delta a = a\) and \(c\delta a\gamma c = c\). Further any such \(c\) in \(S\) must belong to \(S \setminus I\) as \(I\) is an ideal of \(S\). Hence \(S\) is an inverse \(\Gamma\)-semiring.