CHAPTER-3
GALERKIN BASED WAVELET METHODS FOR SOLUTIONS OF DIFFERENTIAL EQUATIONS

Summary: Solutions of numerical differential equations based on orthogonal functions is a quite classical (old) method. Also wavelets being orthogonal functions have been applied to such problems. In recent years there has been increasing attempt to find solutions of differential equations using wavelet techniques. In this chapter, we elaborate various wavelet-Galerkin methods such as Amaratunga et al. method and fictitious boundary approach for ODEs and PDEs.

Obtaining solutions of ordinary differential equations through fictitious boundary or other approach of wavelet-Galerkin require use of connection coefficients or FFT which involves computational complexities and time consumption. In this chapter, finite difference based wavelet-Galerkin method has been developed for ordinary differential equations which is rather simple and avoids above complexities.

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1. Introduction

Since the contribution of orthogonal bases of compactly supported wavelet by Daubechies (1988) and multiresolution analysis based fast wavelet transform algorithm by Beylkin (1991), wavelet based approximation of ordinary and partial differential equations gained momentum in attractive way. Wavelets have the capability of representing the solutions at different levels of resolutions, which make them particularly useful for developing hierarchical solutions to engineering problems. Among the approximations, wavelet-Galerkin technique is the most frequently used scheme these days. Daubechies wavelets as bases in a Galerkin method to solve differential equations require a computational domain of simple shape. This has become possible due to the remarkable work by Amartunga et al. (1993, 1994 & 1996) [4, 3 & 2], Latto et al. (1992)[69], Xu et al. (1994) [122] and Williams et al. (1993 & 1994) [119 & 120]. Yet there is difficulty in dealing with boundary conditions. So far problems with periodic boundary conditions or periodic distribution have been dealt successfully.

Advantage of wavelet-Galerkin method over finite difference or element method has lead to tremendous applications in science and engineering. To a certain extent, the wavelet technique is a strong competitor to the finite element method. Although the wavelet method provided an efficient alternative technique for solving pdes numerically, it is not as easy to implement as the traditional finite-difference method. The reason is that the use of the wavelet-Galerkin method to solve pdes lead to the problem of computing integrals whose integrands involve products of compactly supported wavelets and their derivatives. These integrals are evaluated using what is known as the Connection Coefficient method. Notice that with increasing resolution (for Daub6) accuracy deteriorates, since condition number increases. But for higher order wavelets, condition number is consistently lower. Moreover, the condition number also depends on the order of derivatives, $\Omega^{0,d}$ increases with increase in derivatives $d$.

Wavelet-Galerkin methods such as Amaratunga et al. method, fictitious boundary approach, capacitance matrix method, split step methods, finite difference wavelet-Galerkin method and wavelet Taylor Galerkin approach for ODEs and PDEs are well known due to their own advantages (see, Barker et al. [8], Behiry et al. [9], Beylkin [14], Burgos et al. [17],
2. Condition Number of a Matrix [46, pp. 454-456]

We know that a linear system $AX = Y$ has a unique solution $X$ for every $Y$ if a square matrix $A$ is invertible. It is often observed that for two close values of $Y$ and $Y'$, $X$ and $X'$ are far apart. Such a linear system is called badly conditioned. Thus data $Y$ is expected to be fairly accurate. Condition number of $A$ is given by

$$C_\#(A) = \|A\| \|A^{-1}\|.$$  

If $A$ is not invertible, set $C_\#(A) = +\infty$. It is easy to prove that for $c \neq 0$,

$$C_\#(cA) = C_\#(A).$$

Also for any matrix $A$, $C_\#(A) \geq 1$.

Thus $C_\#(A)$, is the measure of stability of the linear system under perturbation of the data $Y$. Small condition number near 1 is desirable. In case it is high, replace the system by equivalent system $BAX = BY$, $B$ is a preconditioning matrix such that

$$C_\#(BA) < C_\#(A).$$

To facilitate easy calculation, $A$ is considered to be sparse, i.e. $A$ should have high proportion of entries 0. The best one is when $A$ is in diagonal form.

**Lemma 1.** Suppose that $A$ is an $n$ by $n$ normal invertible matrix. Let
\[
|\lambda|_{\text{max}} = \max \{ |\lambda| : \lambda \text{ is an eigenvalue of } A \}
\]

and

\[
|\lambda|_{\text{min}} = \min \{ |\lambda| : \lambda \text{ is an eigenvalue of } A \}
\]

Then

\[
C_\#(A) = \frac{|\lambda|_{\text{max}}}{|\lambda|_{\text{min}}}.
\]

**Lemma 2.** Suppose that \( A \) is an \( n \) by \( n \) invertible matrix, \( x, y, \delta x, \delta y \in C^n \). \( x \neq 0 \), \( Ax = y \) and \( A\delta x = \delta y \). Then

\[
\frac{\|\delta x\|}{\|y\|} \leq C_\#(A) \frac{\|\delta x\|}{\|y\|}.
\]

Moreover, there exist non zero \( x, y, \delta x, \delta y \in C^n \) such that \( Ax = y \), \( A\delta x = \delta y \) and equality is attained in relation (1). Hence \( C_\#(A) \) cannot be replaced in relation (1) by any smaller number.

### 3. 1D Wavelet-Galerkin Technique

**Galerkin Method.** It was Russian engineer V.I. Galerkin who proposed a projection method based on weak form. In it a set of test functions are selected such that residual of differential equation becomes orthogonal to test functions [12].

Consider one-dimensional differential equation [46, pp. 451-479]

\[
Lu(x) = f(x), \quad 0 \leq x \leq 1
\]

with Dirichlet boundary conditions

\[
u(0) = a, u(1) = b.
\]
$f$ is real valued and continuous function of $x$ on $[0,1]$. $L$ is a uniformly elliptic differential operator.

Suppose that $\{v_j\}$ is a complete orthonormal system for $L^2([0,1])$ and that every $v_j$ is $C^2$ on $[0,1]$ such that

$$v_j(0) = a, \; v_j(1) = b.$$  

Select a finite set $\Lambda$ of indices $j$ and consider the subspace  

$$S = \text{span} \{v_j : j \in \Lambda\}.$$  

Let the approximate solution $u_s$ of the given equation be  

$$u_s = \sum_{k \in \Lambda} x_k v_k \in S, \text{ for each scalar } x_k. \quad (3)$$  

We would like to determine $x_k$ in a way that $u_s$ behaves as if is a true solution on $S$, i.e.

$$\langle Lu_s, v_j \rangle = \langle f, v_j \rangle \; \forall \; j \in \Lambda \quad \text{ (4)}$$

such that the boundary conditions $u_s(0) = a, u_s(1) = b$ are satisfied. Substituting $u_s$ in (4),

$$\sum_{k \in \Lambda} \langle Lv_k, v_j \rangle x_k = \langle f, v_j \rangle \; \forall \; j \in \Lambda \quad \text{ (5)}$$

Let $X$ and $Y$ denote the vectors $(x_k)_{k \in \Lambda}$ and $(y_k)_{k \in \Lambda} = \langle f, v_k \rangle$, and $A$ the matrix

$$A = \begin{bmatrix} a_{j,k} \end{bmatrix}_{j,k \in \Lambda}, \text{ where } a_{j,k} = \langle Lv_k, v_j \rangle.$$  

(5) reduces to the system of linear equations

$$\sum_{k \in \Lambda} a_{j,k} x_k = y_j \text{ or equivalently } AX = Y. \quad \text{ (6)}$$

Thus in Galerkin method, for each subset $\Lambda$, we find an approximation $u_s$ in $S$ to $u$ by solving (6) for $X$ and then substituting its components in (3). It is expected that as we increase $\Lambda$ in some systematic way, $u_s$ converges to $u$, the actual solution.
Let \( \psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k) \) be a wavelet basis for \( L^2([0,1]) \) with boundary conditions

\[
\psi_{j,k}(0) = \psi_{j,k}(1) = 0.
\]

For each \((j,k) \in \Lambda\), \( \psi_{j,k} \) is \( C^2 \).

The scale of \( \psi \) approximates \( 2^{-j} \) and is centralized near point \( 2^{-j} k \) and equates to zero outside the interval centred at \( 2^{-j} k \) of length proportional to \( 2^{-j} \).

In Wavelet-Galerkin method (3) and (4) may thus be replaced by

\[
u_s = \sum_{j,k \in \Lambda} x_{j,k} \psi_{j,k}
\]

and

\[
\sum_{j,k \in \Lambda} \langle L \psi_{j,k}, \psi_{l,m} \rangle x_{j,k} = \langle f, \psi_{l,m} \rangle \quad \forall (l,m) \in \Lambda.
\]

So that \( AX = Y \). \hspace{1cm} (7)

Here \( A = \left[ a_{l,m;j,k} \right]_{(l,m),(j,k) \in \Lambda} \), \( X = (x_{j,k})_{(j,k) \in \Lambda} \), \( Y = (y_{l,m})_{(l,m) \in \Lambda} \).

In it \( a_{l,m;j,k} = \langle L \psi_{j,k}, \psi_{l,m} \rangle \), \( y_{l,m} = \langle f, \psi_{l,m} \rangle \).

The pairs \((l,m)\) and \((j,k)\) represent respectively row and column of \( A \).

We would like \( A \) to be sparse and have a low condition number. In fact \( A \) itself does not have a low condition number but we can replace the system \( AX = Y \) by an equivalent system \( MZ = V \), with matrix \( M \) has the desired properties. Firstly, let the diagonal matrix

\( D = \left[ d_{l,m;j,k} \right]_{(l,m),(j,k) \in \Lambda} \) by

\[
d_{l,m;j,k} = \begin{cases} 
2^j & \text{if } (l,m) = (j,k) \\
0 & \text{if } (l,m) \neq (j,k)
\end{cases}
\]

Define

\( M = \left[ m_{l,m;j,k} \right]_{(l,m),(j,k) \in \Lambda} \).
by

\[ M = D^{-1}AD^{-1}. \]  

(9)

This gives

\[ m_{l,m,j,k} = 2^{-j-i} a_{l,m,j,k} = 2^{-j-i} \{ L\psi_{j,k} \psi_{l,m} \}. \]  

(10)

The system \( AX = Y \) is equivalent to

\[ D^{-1}AD^{-1}DX = D^{-1}Y. \]

Setting \( Z = DX \) and \( V = D^{-1}Y \),

\[ MZ = V. \]  

(11)

Hence the norm equivalence has the consequence that the system (11) is well conditioned. The process of changing an ill conditioned system into a well conditioned system is a variation on the preconditioning process.

The matrix \( M \) in the preconditioned system (11) has condition number bounded independently of \( \Lambda \). So as we increase \( \Lambda \) to approximate solution with more accuracy, the condition number maintains its boundedness, which is much better than the finite difference method in which case condition number grows as \( N^2 \). Thus the data errors, may be due to rounding off, has no effect in wavelet-Galerkin solution over \([0, 1]\) as we approach for better and better accuracy.

4. Wavelet Methods for ODEs

4.1 Amaratunga et al. Based Wavelet-Galerkin Method ([3], see also [18] & [45])

Consider the equation

\[ \frac{\partial^2 u}{\partial x^2} + au = f, \]  

(12)
where \( u, f \) are periodic in \( x \) of period \( d \in \mathbb{Z} \).

The wavelet–Galerkin solution of periodic problem is slightly more complicated than the finite difference approach as the former involves solving a set of simultaneous equations in wavelet space and not in physical space. The solution in wavelet space is then transformed back to physical space by FFT.

Let the wavelet expansion \( u(x) \) at scale \( j \) be

\[
 u(x) = \sum_k c_k 2^{j/2} \varphi(2^j x - k), \quad k \in \mathbb{Z}
\]

(13)

where \( c_k \)s are periodic wavelet coefficients of \( u \), periodic in \( x \).

Put \( y = 2^j x \) so that

\[
 U(y) = u(x) = \sum_k C_k \varphi(y - k), \quad C_k = 2^{j/2} c_k
\]

If \( d \) is the period of \( u \), then \( U(y) \) and so also \( C_k \) is periodic in \( y \) with period \( 2^j d \). Let us discretize \( U(y) \) at all dyadic points \( x = 2^{-j} y, y \in \mathbb{Z} \)

\[
 U_i = \sum_k C_k \varphi_{i-k} = \sum_k C_{i-k} \varphi_k, \quad i = 0, 1, 2, ..., n-1
\]

The matrix representation is \( U = k_\varphi * C \), where \( k_\varphi \) is the convolution kernel, i.e., the first column of the scaling function matrix.

Similarly the wavelet expansion for \( f(x) \)

\[
 f(x) = \sum_k d_k 2^{j/2} \varphi(2^j x - k), \quad k \in \mathbb{Z}
\]

(14)

\[
 F(y) = f(x) = \sum_k D_k \varphi(y - k), \quad D_k = 2^{j/2} d_k
\]

And the matrix representation is

\[
 F = k_\varphi * D
\]
Substitute the expansions of $u(x)$ and $f(x)$ in (12) and then take inner product on both sides with $\varphi(y - j)$, $j \in \mathbb{Z}$.

Use $\Omega_{j-k} = \int \varphi^{(\alpha)}(y-k)\varphi(y-j)dy$ and $\delta_{jk} = \int \varphi(y-k)\varphi(y-j)dx$, we obtain $k_{\Omega}C = g$. Now take FTs

$$\hat{U} = \hat{k}_{\varphi} \hat{C}$$

$$\hat{F} = \hat{k}_{\varphi} \hat{D}$$

$$\hat{k}_{\Omega} \hat{C} = \hat{g}$$

Subsequently, $\hat{U} = \hat{F} / \hat{k}_{\Omega}$. Inverse FT will give $U$.

Wherein . and denote component by component multiplication and division respectively.

### 4.2 Fictitious Boundary Approach [34]

Consider the equation

$$\frac{\partial^2 u}{\partial x^2} + au = 0, \quad x \in [0,1]$$

(15)

with Dirichlet’s boundaries $u(0), u(1)$.

$u$ in (15) is periodic in $x$ of period $d \in \mathbb{Z}$ (k varies from $-N+1$ to $2^j$).

The boundaries of the support of (15) are $\frac{-N+1}{2^j}$ and $\frac{N-1+2^j}{2^j}$. Subsequently, the original boundaries 0 & 1 now changes to the Fictitious Boundaries, i.e. boundary on both sides of 0 and 1 are extended by an amount $\frac{N-1}{2^j}$.

$$\varphi(0) = u\left(\frac{-N+1}{2^j}\right)$$

$$\varphi(N-1) = u\left(\frac{N-1+2^j}{2^j}\right)$$
without affecting the solution within $[0,1]$, the affected solution is within $[-N+1/2^j,0]$ and $[1, N+1/2^j]$. The eq.(15) now reduces to

$$2^{2j} \sum_{k=-N+1}^{2j} C_k \phi''(X-k) + \alpha \sum_{k=-N+1}^{2j} C_k \phi(X-k) = 0 \quad 2^j x = X.$$  

(16)

Inner product is taken on both sides of (16) with $\phi(X-n)$ taking the integration limits to $[-N+1/2^j, N+1/2^j]$. We obtain

$$2^{2j} \sum_{k=-N+1}^{2j} C_k \Omega_{n-k} + \alpha \sum_{k=-N+1}^{2j} C_k \delta_{n,k} = 0.$$  

(17)

Wherein,

$$\Omega_{n-k} = \int \phi_{nk}(X-k)\phi(X-n) dX$$  and  $$\delta_{n,k} = \int \phi(X-n)\phi(X-k) dX.$$

$$C_k = 2^{j/2} c_k.$$  

The Dirichlet boundary conditions give equations

$$\sum_{k=-N+1}^{2j} C_k \phi(-k) = u(0)$$  

(18a)

$$\sum_{k=-N+1}^{2j} C_k \phi(1-k) = u(1)$$  

(18b)

First and last equations are replaced by the boundary conditions in (17). The place of corresponding connection coefficients in first and last rows are determined by inner product of equations (18a) & (18b). Appropriate connection coefficients are used to solve the ill-conditioned system for $C_k$. 

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4.3 Wavelet-Galerkin Finite Difference Method

Here we develop a wavelet-Galerkin finite difference method [WGFDM], based on finite difference approach, to find wavelet solution of certain ODEs.

**Lemma.** For large $j \in \mathbb{Z}_+$, $f^{(n)}(x) = 2^n \sum_{i=0}^{n} (-1)^{n+i} \binom{n}{i} f \left( x + \frac{i}{2^j} \right)$. 

**Proof.** For $\frac{1}{2^j}$ small, 

$$f'(x) = \frac{f\left(x + \frac{1}{2^j}\right) - f(x)}{\frac{1}{2^j}} = 2^j \left[ f \left( x + \frac{1}{2^j} \right) - f(x) \right]$$

using forward difference Taylor’s expansion.

$$f''(x) = 2^j \left[ f' \left( x + \frac{1}{2^j} \right) - f'(x) \right]$$

$$= 2^{2j} \left[ \left[ f \left( x + \frac{2}{2^j} \right) - 2f \left( x + \frac{1}{2^j} \right) + f(x) \right] \right]$$

$$f''(x) = 2^{2j} \left[ f' \left( x + \frac{2}{2^j} \right) - 2f' \left( x + \frac{1}{2^j} \right) + f'(x) \right]$$

$$= 2^{3j} \left[ f \left( x + \frac{3}{2^j} \right) - 3f \left( x + \frac{2}{2^j} \right) + 3f \left( x + \frac{1}{2^j} \right) - f(x) \right]$$

Proceeding in this way,

$$f^{(n)}(x) =$$

$$2^{nj} \left[ f \left( x + \frac{n}{2^j} \right) - \binom{n}{1} f \left( x + \frac{n-1}{2^j} \right) + \binom{n}{2} f \left( x + \frac{n-2}{2^j} \right) + \right.$$  

$$\left. \ldots + \binom{n}{n-1} f \left( x + \frac{1}{2^j} \right) - f(x) \right]$$

$$= 2^n \sum_{i=0}^{n} (-1)^{n+i} \binom{n}{i} f \left( x + \frac{i}{2^j} \right), \ j \in \mathbb{Z}_+$$

(19)
Remark. For large \( j \in \mathbb{Z}_+ \), \( f^{(n)}(x) = 2^n \sum_{i=0}^{n} (-1)^i \binom{n}{n-i} f\left(x - \frac{i}{2^j}\right) \).

This can easily be proved by letting \( f'(x) = 2^j \left[f(x) - f\left(x - \frac{1}{2^j}\right)\right] \).

Applications in ODEs

To solve nth order linear ODE

\[
\sum_{l=0}^{n} A_l f^{(l)}(x) = B(x).
\] (20)

Let

\[
f(x) = \sum \alpha_{j,k} \varphi_{jk}(x) = \sum \alpha_{j,k} 2^{j/2} \varphi(2^j x - k)
\] (21)

Using above Lemma,

\[
f^{(l)}(x) = 2^l \sum_{i=0}^{l} (-1)^{\frac{l+i}{2}} \binom{l}{i} \sum \alpha_{j,k} 2^{j/2} \varphi\left(2^j \left(x + \frac{i}{2^j}\right) - k\right)
\]

\[
= 2^l \sum_{i=0}^{l} (-1)^{\frac{l+i}{2}} \binom{l}{i} \sum \alpha_{j,k} 2^{j/2} \varphi\left(2^j x + i - k\right)
\]

\[
= 2^l \sum_{i=0}^{l} (-1)^{\frac{l+i}{2}} \binom{l}{i} \sum \alpha_{j,k+i} \varphi_{jk}
\]

Substituting in (20),

\[
\sum_{l=0}^{n} A_l 2^l \sum_{i=0}^{l} (-1)^{\frac{l+i}{2}} \binom{l}{i} \sum \alpha_{j,k+i} \varphi_{jk} = B(x).
\]

Taking inner product with \( \varphi_{j,m} \),

\[
\sum_{l=0}^{n} A_l 2^l \sum_{i=0}^{l} (-1)^{\frac{l+i}{2}} \binom{l}{i} \alpha_{j,m+i} = C(x),
\] (22)

where \( C(x) = \int_{R} B(x) \varphi_{j,m} \, dx \).
5. 2D Multiresolution Analysis

2D Multiresolution Analysis MRA ([26], see also [54]) can be constructed by taking tensor product \([F \otimes G = \{f(x)g(y) : f \in F, g \in G\}\]) of 1D ones. Let \(L^2(R^2)\) be the space of two-dimensional square integrable function. Given an MRA \(\{V_j\}_{j \in \mathbb{Z}}\) with scaling function \(\phi_{n,k}\) and corresponding wavelet space \(\{W_j\}_{j \in \mathbb{Z}}\) with wavelet \(\psi_{n,k}\), define

\[
\phi_{n,(k_1,k_2)}(x,y) = \phi_{n,k_1}(x) \otimes \phi_{n,k_2}(y)
\]

\[
\psi_{n,(k_1,k_2)}^1(x,y) = \phi_{n,k_1}(x) \otimes \psi_{n,k_2}(y)
\]

\[
\psi_{n,(k_1,k_2)}^2(x,y) = \psi_{n,k_1}(x) \otimes \phi_{n,k_2}(y)
\]

\[
\psi_{n,(k_1,k_2)}^3(x,y) = \psi_{n,k_1}(x) \otimes \psi_{n,k_2}(y)
\]

\(V_j \subset L^2(R^2)\) will be spanned by \(\phi_{n,(k_1,k_2)}(x,y)\) and each one of new wavelets will span its corresponding wavelet space, \(V_j^1 = V_j \otimes W_j, W_j^2 = W_j \otimes V_j\) and \(W_j^3 = W_j \otimes W_j\).

Furthermore \(\otimes_{j \in \mathbb{Z}, i \in \{1,2,3\}} W_j^i = L^2(R^2)\).

For any function \(f(x,y) \in L^2(R^2)\), projection of \(f\) onto scaling space \(V_m\) at resolution \(m\) may be defined by

\[
P_m f(x,y) = \sum_k \sum_l c_{k,l} \phi_{m,k}(x) \phi_{m,l}(y).
\]

6. 2D Wavelet-Galerkin Technique [26]

Consider the following problem:

\[
L[u(x,y)] = 0 \text{ on the region } S(x,y)
\]  \(23\)

with boundary conditions \(D(u) = 0\) on the boundary \(\tau\) of \(S\).

Assume that \(u(x,y)\) can be represented accurately by a set of analytic function \(\{g_l(x,y)\}_{l=1}^N\) such that
\[ u(x, y) \cong u_0(x, y) + \sum_{i=1}^{N} a_i g_i(x, y) = u_a(x, y). \]

\( u_0 \) is so chosen as to satisfy the initial conditions. For approximation \( u_a \) to be good, the residual \( R \) of

\[ L[u_0(x, y)] + L\left[ \sum_{i=1}^{N} a_i g_i(x, y) \right] = R(a_1, \ldots, a_N, x, y). \]

must reduce to minimum, for that we use the simplest Galerkin Method, namely Ritz-Raleigh. This method minimizes residual to the effect that

\[ \langle R(a_1, \ldots, a_N, x, y), g_i(x, y) \rangle_{L^2} = 0, \quad i = 1, 2, \ldots, N \]

which in turn gives

\[ \sum_{i=1}^{N} a_i \left\langle L[g_i(x, y)], g_j(x, y) \right\rangle + \left\langle L[u_a(x, y)], g_j(x, y) \right\rangle = 0. \]  \hspace{1cm} (24)

Next step is to find \( \{a_i\}_{i=1}^{N} \) form matrix system of (23)

\[ GA = U, \quad G_{i,j} = L[g_i, g_j], \quad a = [a_i], \quad U_j = L[u_a, g_j]. \]

Let \( \{V_j\}_{j \in \mathbb{Z}} \) be an MRA with scaling function \( \varphi_k(x) = \sum a_k \varphi(2x - k). \) \( \{\varphi_{j,k} = 2^{j/2} \varphi(2^j x - k), j, k \in \mathbb{Z}\} \) acts as orthonormal basis for \( V_j. \) At each approximation level \( j, \)

orthogonal projection of \( u(x, y) \) onto \( V_j \) is taken in the manner (fit \( y \) fixed)

\[ u(x, y) \cong \Gamma_j u(x, y) = \sum_{k} a_{j,k} \varphi_{j,k}(x), = \langle u(x, y), \varphi_{j,k}(x) \rangle. \] \hspace{1cm} (25)

For some \( j, V_j \) will capture all details of the original function. Select \( k \in \{0, 1, \ldots, 2^n - 1\}. \)

Substituting (31) in (23) and forcing the condition (24), we find

\[ \langle L[g]u(x, y) \rangle + Lu_0(x, y), \varphi_{j,k} \rangle = 0. \] \hspace{1cm} (26)

Clearly this is 1D problem and cannot be applied to 2D. Let us assume \( a_{j,k} \) function of \( y \) and for each \( y_j, \) we can solve the system as for 1D problem. Ultimately we obtain
\[ BA = R, B_{l,m} = \langle L[\varphi_{j,l}(x)], \varphi_{j,m} \rangle, A_l = a_{j,l}(y), R_l = \langle Lu_0(x,y), \varphi_{j,l}(x) \rangle \]  

(27)

Solving (27), we find for each \( y \) the coefficients \( a_{j,k}(y) \) and thus the solution to (23).

For details refer to [26].

7. Wavelet Methods for PDEs

7.1 Wavelet-Galerkin Solution of the Periodic Problem [4]

Consider the two dimensional Poisson’s equation

\[ u_{xx} + u_{yy} = f, \]  

(28)

where \( u = u(x,y), f = f(x,y) \) are periodic in \( x, y \) of period \( d_x, d_y \in \mathbb{Z} \).

Let the approximate solution \( u(x,y) \) at scale \( m \) be

\[ u(x,y) = \sum_k \sum_l c_{k,l} 2^{m/2} \varphi(2^m x - k) 2^{m/2} \varphi(2^m y - l), \quad k, l \in \mathbb{Z} \]

where \( c_{k,l} \) are periodic wavelet coefficients of \( u \).

Put \( X = 2^m x, Y = 2^m y \) so that

\[ U(X,Y) = u(x,y) = \sum_k \sum_l C_{k,l} \varphi(X - k) \varphi(Y - l), C_{k,l} = c_{k,l} 2^{m/2}. \]  

(29)

\( U(X,Y) \) and also \( C_{k,l} \) are periodic in \( X, Y \) with periods \( n_x = 2^m d_x, n_y = 2^m d_y \). Let us discretize \( U(X,Y) \) at all dyadic points \( x, y = 2^{-m} X, 2^{-m} Y, X, Y \in \mathbb{Z} \)

\[ U_{i,j} = \sum_k \sum_l C_{k,l} \varphi_{i-k} \varphi_{j-l} = \sum_k \sum_l C_{i-k} \varphi_{j-l}, i = 0, 1, 2, \ldots, n_x - 1, j = 0, 1, 2, \ldots, n_y - 1. \]

(30)

The matrix representation is
\[ U = k_{\varphi_x} \ast k_{\varphi_y} \ast C, \]  

where \( k_{\varphi_x}, k_{\varphi_y} \) are the convolution kernels, i.e. the first column of the scaling function matrices and \( C \) is the wavelet coefficient matrix.

Similarly, RHS of (28) can be expressed for

\[ f(x, y) = \sum_k \sum_{l} d_{k,l} 2^{m/2} \varphi(2^m x - k) \; 2^{m/2} \varphi(2^m y - l), k, l \in \mathbb{Z}. \]

as

\[ F(X, Y) = f(x, y) = \sum_k \sum_{l} D_{k,l} \varphi(X - k) \; \varphi(Y - l), D_{k,l} = d_{k,l} 2^{m/2}. \tag{32} \]

(32) takes the form as

\[ F = k_{\varphi_x} \ast k_{\varphi_y} \ast D. \tag{33} \]

Substitute the expansion of \( u(x, y) \) and \( f(x, y) \) into the given differential equation (28) and then take inner product on both sides with \( \varphi(X - p), \varphi(Y - q), (p, q \in \mathbb{Z}). \)

Use \( \Omega_{j-k} = \int \varphi^n(y - k)\varphi(y - j)dy \) and \( \delta_{jk} = \int \varphi(y - k)\varphi(y - j)dy. \)

We obtain, \( k_{\Omega}xC = \frac{1}{2^{2m}}g. \)

Taking FTs of (31), (33) and (34), we find

\[ \hat{U} = \frac{1}{2^{2m}} \hat{F}. \]

Inverse FT gives the solution \( U. \)

### 7.2 Wavelet-Galerkin Method for 2D [4]

Consider the equation (28).

Let approximate solution be

\[ u(x, y) = \sum_k \sum_{l} c_{k,l} 2^{m/2} \varphi(2^m x - k) \; 2^{m/2} \varphi(2^m y - l); \; k, l \in \mathbb{Z}. \]
\[ \sum_{k} \sum_{l} C_{k,l} \varphi(X - k) \varphi(Y - l), C_{k,l} = c_{k,l} 2^{m/2}, \quad 2^m x = X, \quad 2^m y = Y \]

Similarly \( f(x, y) = \sum_{k} \sum_{l} D_{k,l} \varphi(X - k) \varphi(Y - l), D_{k,l} = d_{k,l} 2^{m/2}. \)

Substituting in the given equation (28) and letting inner product with \( \varphi(Y - q) \) \( (p, q \in \mathbb{Z}) \) gives

\[
\sum_{k} \sum_{l} 2^{2m} C_{k,l} \int \varphi''(X - k) \varphi(X - p) \, dx \int \varphi(Y - l) \varphi(Y - q) \, dy \\
+ \sum_{k} \sum_{l} 2^{2m} C_{k,l} \int \varphi(X - k) \varphi(X - p) \, dx \int \varphi''(Y - l) \varphi(Y - q) \, dy \\
= \sum_{k} \sum_{l} D_{k,l} \int \varphi(X - k) \varphi(X - p) \, dx \int \varphi(Y - l) \varphi(Y - q) \, dy.
\]

Or

\[
\sum_{k} C_{k,q} \int \varphi''(X - k) \varphi(X - p) \, dx + \sum_{l} C_{p,l} \int \varphi''(Y - l) \varphi(Y - q) \, dy = \frac{1}{2^{2m}} D_{p,l}.
\]

That is,

\[
\sum_{k} C_{k,q} \Omega_{p-k} + \sum_{l} C_{p,l} \Omega_{q-l} = \frac{1}{2^{2m}} D_{p,l}, \quad \text{(34)}
\]

where \( \Omega_{p-k} = \int \varphi''(X - k) \varphi(X - p) \, dx \)

\( \Omega_{q-l} = \int \varphi''(Y - l) \varphi(Y - q) \, dy \)

are the connection coefficients.

Solving (34) will give the coefficients and hence the solution.
8. Conclusion

A. Use of Daubechies wavelet of orthonormal basis in ODEs has advantages that the matrix $A$ associated with it is sparse and a diagonal matrix $D$ can be constructed such that the condition number in $M$ is bounded by a constant independent of size of $A$.

The condition number in wavelet-Galerkin method remains bounded with increase of resolution and much better than the finite difference method. Rounding off has least effect in this approach and gives for better solution than finite difference.

Amaratunga et al. method has a few advantages:

- It is best suitable for use, less computational cost is involved and computation is faster due to FFT.
- The scale of computation (number of mesh or grid points) can be readily changed. At high frequency the method works well with greater accuracy

B. In order to find solution of ODEs through wavelet-Galerkin method (1) one needs to find connection coefficients for desired scale (2) use of scaling functions corresponding to that value of scale whereas in finite difference based wavelet-Galerkin method don’t require these.