Chapter 4

Stability of Generalized Quadratic Functional Equation in Non-Archimedean $\ell$-Fuzzy Normed Spaces

4.1 Introduction

One of the problems in $\ell$-fuzzy topology is to obtain an appropriate concept of $\ell$-fuzzy metric spaces and $\ell$-fuzzy normed spaces. R. Saadati and J. Park [158], respectively, introduced and studied a notion of intuitionistic fuzzy metric (normed) spaces and then
Deschrijver et al.\cite{39} and Saadati generalized the concept of intuitionistic fuzzy metric (normed) spaces and introduced and studied a notion of $\ell$-fuzzy metric spaces and $\ell$-fuzzy normed spaces\cite{39,159}.

**Definition 4.1.1** Let $K$ be a field. A non-Archimedean absolute value on $K$ is a function $\|\cdot\| : K \to [0, +\infty)$ such that, for any $a, b \in K$,

- $(i)$ $|a| \geq 0$ and the equality holds if and only if $a = 0$,
- $(ii)$ $|ab| = |a||b|$,
- $(iii)$ $|a + b| \leq \max \{|a|, |b|\}$ (the strict triangle inequality).

Note that $|n| \leq 1$ for each integer $n$. We always assume, in addition, that $\|\cdot\|$ is non-trivial, i.e., there exists an $a_0 \in K$ such that $|a_0| \neq 0, 1$.

**Definition 4.1.2** A non-Archimedean $\ell$-fuzzy normed space is a triple $(\nu, p, \tau)$, where $\nu$ is a vector space, $\tau$ is a continuous $t$-norm on $\ell$ and $p$ is an $\ell$-fuzzy set on $\nu \times (0, +\infty)$ satisfying the following conditions: for all $x, y \in \nu$ and $t, s \in (0, +\infty),$

(a) $0_\ell \leq p(x, t);$

(b) $p(x, t) = 1_\ell$ if and only if $x = 0;$

(c) $p(\alpha x, t) = p\left(x, \frac{t}{|\alpha|}\right)$ for all $\alpha \neq 0;$
(d) \( \tau(p(x, t), p(y, s)) \leq \ell p(x + y, \max(t, s)) \)

(e) \( p(x, \cdot) : ]0, \infty[ \to \ell \) is continuous;

(f) \( \lim_{t \to 0} p(x, t) = 0 \) and \( \lim_{t \to 0} p(x, t) = 1 \).

**Example 4.1.3** Let \((X, \|\cdot\|)\) be a non-Archimedean normed linear space. Then the triple \((X, p, min)\), where

\[
p(x, t) = \begin{cases} 
0 & \text{if } t \leq \|x\|; \\
1 & \text{if } t > \|x\|, 
\end{cases}
\]

is a non-Archimedean \(\ell\)-fuzzy normed space in which \(\ell = [0, 1]\).

**Example 4.1.4** Let \((X, \|\cdot\|)\) be a non-Archimedean normed linear space. Denote \(\tau_m(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})\) for all \(a = (a_1, a_2), b = (b_1, b_2) \in \ell^*\) and let \(p_{\mu, \nu}\) be the intuitionistic fuzzy set on \(X \times ]0, +\infty[\) defined as follows:

\[
p_{\mu, \nu}(x, t) = \left( \frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|} \right)
\]

for all \(t \in R^+\). Then \((X, p_{\mu, \nu}, \tau_M)\) is a non-Archimedean intuitionistic fuzzy normed space.

In this chapter, author investigates the general solution of following new generalized quadratic functional equations

\[
f(ax - y) \pm a f(ax \pm y) = (a \pm 1)[af(x) \pm f(y)]
\]

for any fixed integer \(a\) with \(a \neq -1, 0, 1\). We also study the Hyers-Ulam-Rassias stability of the functional equation (4.1) in non-Archimedean \(\ell\)-fuzzy normed spaces.
4.2 The General Solution of the Functional Equation (4.1)

Let $X$ and $Y$ be a linear spaces. In this section author will find out the general solution of (4.1).

**Theorem 4.2.1** If an function $f : X \to Y$ satisfies

\[ f(ax - y) \pm af(ax \pm y) = (a \pm 1)[af(x) \pm f(y)] \quad (4.2) \]

then $f$ is quadratic.

**Proof.** Setting $x = y = 0$ in (4.2), we obtain $f(0) = 0$ and setting $(x, y) = (x, 0)$ and $(x, y) = (0, x)$ in (4.1), we obtain

\[ f(ax) = a^2 f(x) \quad \text{and} \quad f(-x) = f(x) \quad (4.3) \]

respectively, for all $x \in X$. Therefore, $f$ is even. Replacing $y$ by $-y$ in (4.2), we obtain

\[ f(ax + y) \pm a \ f(ax \mp y) = (a \pm 1)[af(x) \pm f(y)] \quad (4.4) \]

for all $x, y \in X$. Adding (4.2) and (4.4) and in resultant again using (4.2), we arrive

\[ f(ax + y) + f(ax - y) \pm a \ f(x + y) \pm a \ f(x - y) = 2f(ax - y) \pm 2af(x \pm y) \quad (4.5) \]

for all $x, y \in X$. If we choose either first order sign or the second order sign in the equation (4.5), in both cases, we obtain

\[ f(ax + y) + af(x - y) = f(ax - y) + a \ f(x + y) \quad (4.6) \]
for all $x, y \in X$. Replacing $y$ by $ax + y$ in (4.6), we obtain

$$f (2ax + y) + af (ax - (x - y)) = f (y) + af (ax + (x + y)) \quad (4.7)$$

for all $x, y \in X$. Replacing $y$ by $-y$ in (4.7), we obtain

$$f (2ax - y) + af (ax - (x + y)) = f (y) + af (ax + (x - y)) \quad (4.8)$$

for all $x, y \in X$. Adding equations (4.7) and (4.8), and using (4.6) in the resultant, we arrive

$$f (2ax + y) + f (2ax - y) + 2a^2 f (y) = a^2 f (2x + y) + a^2 f (2x - y) + 2f (y) \quad (4.9)$$

for all $x, y \in X$. Replacing $y$ by $x + ay$ in (4.6), we obtain

$$f (a(x + y) + x) + af (ay) = f (a(x - y) - x) + af (2x + ay) \quad (4.10)$$

for all $x, y \in X$. Replacing $y$ by $-y$ in (4.10), we obtain

$$f (a(x - y) + x) + af (ay) = f (a(x + y) - x) + af (2x - ay) \quad (4.11)$$

for all $x, y \in X$. Adding equations (4.10) and (4.11), and using (4.6) in the resultant, we arrive

$$f (2x + y) + f (2x - y) + 2f (y) [a^2 - 1] = f (2x + ay) + f (2x - ay) \quad (4.12)$$

for all $x, y \in X$. Setting $(x, y) = (\frac{y}{2}, 2x)$ in (4.12), we get

$$f (2x + y) + f (2x - y) + 2(a^2 - 1)f (2x) = f (2ax + y) + f (2ax - y) \quad (4.13)$$
for all $x, y \in X$. Using (4.13) in (4.9), we arrive

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

for all $x, y \in X$. This shows that $f$ is quadratic, which completes the proof of Theorem.

### 4.3 $\ell$–Fuzzy Hyers-Ulam-Rassias Stability of the Functional Equation (4.1)

Let $K$ be a non-Archimedean field, $X$ a vector space over $K$ and $(Y, p, \tau)$ a non-Archimedean $\ell$–fuzzy Banach space over $K$.

In this section, author proves the Hyers-Ulam-Rassias stability of the quadratic functional equation (4.1). We define an $\ell$–fuzzy approximately quadratic mapping. Let $\Psi$ be an $\ell$–fuzzy set on $X \times X \rightarrow [0, \infty)$ such that $\Psi(x, y, \cdot)$ is nondecreasing,

$$\psi(cx, cx, t) \geq \ell \psi \left( x, x, \frac{t}{|c|} \right), \quad \forall \ x \in X, \ c \neq 0$$

and

$$\lim_{t \to \infty} (x, y, t) = 1_\ell, \quad \forall \ x, y \in X, \ t > 0.$$ 

**Definition 4.3.1** A mapping $f : X \rightarrow Y$ is said to be $\psi$–approximately quadratic if

$$\rho(f(ax - y) \pm a \ f(ax \pm y) - (a \pm 1)[af(x) \pm f(y)], t) \geq \ell \psi(x, y, t)$$

(4.14)
for all \( x, y \in X \) and \( t > 0 \).

Author now investigates the generalized Hyers-Ulam stability problem for functional equation (4.1). The following is one of our main results in this section.

**Theorem 4.3.2** Let \( K \) be a non-Archimedean field, \( X \) a vector space over \( K \) and \((Y,p,\tau)\) a non-Archimedean \( \ell \)-fuzzy Banach space over \( K \). Let \( f : X \to Y \) be a \( \psi \)-approximately quadratic mapping and \( f(0) = 0 \). If there exist an \( \alpha \in R(\alpha > 0) \) and an integer \( k, k \geq 2 \) with \( |a^k| < \alpha, |a| \neq 1 \) and \( a \neq 0 \) such that

\[
\psi(a^{-k}x, a^{-k}y, t) \geq \ell \psi(x, y, \alpha t) \quad \forall \ x, y \in X, \ t > 0 \quad (4.15)
\]

and

\[
\lim_{n \to \infty} \tau_{j=n}^\infty M \left( x, \frac{\alpha^j t}{|a|^{kj}} \right) = 1 \quad \forall \ x \in X, \ t > 0
\]

then there exists a unique quadratic mapping \( Q : X \to Y \) such that

\[
p(f(x) - Q(x), t) \geq \ell \tau_{j=1}^\infty M \left( x, \frac{\alpha^{j+1} t}{|a|^{kj}} \right) \quad \forall \ x \in X, \ t > 0 \quad (4.16)
\]

where

\[
M(x, t) = \tau(\psi(x, 0, t), \psi(ax, 0, t) \cdots \psi(a^{k-1}x, 0, t)) \quad \forall \ x \in X, \ t > 0.
\]

**Proof.** First, we show, by induction on \( j \), that, for all \( x \in X, \ t > 0 \) and \( j \geq 1 \).

\[
p(f(a^j x) - a^{2j} f(x), t) \geq \ell M_j(x, t) = T(\psi(x, 0, t), \ldots, \psi(a^{j+1} x, 0, t)). \quad (4.17)
\]


Putting \( y = 0 \) in (4.14), we obtain

\[ p(f(ax) - a^2 f(x), t) \geq \psi(x, 0, t) \]

for all \( x \in X, t > 0 \). This proves (4.17) for \( j = 1 \). Let (4.17) hold for some \( j > 1 \).

Replacing \( y \) by 0 and \( x \) by \( ax \) in (4.14), we get

\[ p(f(a^{j+1}x) - a^2 f(a^j x), t) \geq \psi(a^j x, 0, t) \]

for all \( x \in X, t > 0 \). Since \( \ell \leq 1 \), it follows that

\[ p \left( f(a^{j+1}x) - a^{2(j+1)} f(x), t \right) \]

\[ \geq T \left( p(f(a^{j+1}x) - a^2 f(a^j x), t), p(a^2 f(a^j x) - a^{2(j+1)} f(x), t) \right) \]

\[ = T \left( p(f(a^{j+1}x) - a^2 f(a^j x), t), p(f(a^j x) - a^{2j} f(x), \frac{t}{|I|^2}) \right) \]

\[ \geq T \left( p(f(a^{j+1}x) - a^2 f(a^j x), t), p(f(a^j x) - a^{2j} f(x), t) \right) \]

\[ \geq T \left( \psi(a^j x, 0, t), M_j(x, t) \right) \]

\[ = M_{j+1}(x, t) \]

for all \( x \in X, t > 0 \). Thus (4.17) holds for all \( j \leq 1 \). In particular, we have

\[ p(f(a^k x) - a^{2k} f(x), t) \geq M(x, t) \tag{4.18} \]

for all \( x \in X, t > 0 \). Replacing \( x \) by \( a^{-(kn+k)} x \) in (4.18) and using inequality (4.15), we
obtain

\[
p \left( f \left( \frac{x}{a^{kn}} \right) - a^{2k} f \left( \frac{x}{a^{kn+k}} \right), t \right) \\
\geq l \ M \left( \frac{x}{a^{kn+k}}, t \right) \\
\geq l \ M \left( x, \alpha^{n+1} t \right)
\]

forall \( x \in X, \ t > 0, \ n \geq 0, \) and so

\[
p \left( (a^{2k})^n f \left( \frac{x}{(a^k)^n} \right) - (a^{2k})^{n+1} f \left( \frac{x}{(a^k)^{n+1}}, t \right) \right) \\
\geq l \ M \left( x, \frac{\alpha^{n+1}}{|(a^{2k})^n|} t \right) \\
\geq l \ M \left( x, \frac{\alpha^{n+1}}{|(a^k)^n|} t \right)
\]

forall \( x \in X, \ t > 0, \ n \geq 0. \) Hence it follows that

\[
p \left( (a^{2k})^n f \left( \frac{x}{(a^k)^n} \right) - (a^{2k})^{n+p} f \left( \frac{x}{(a^k)^{n+p}}, t \right) \right) \\
\geq l \ T_{j=n}^{n+p} \left( p \left( (a^{2k})^j f \left( \frac{x}{(a^k)^j} \right) - (a^{2k})^{j+p} f \left( \frac{x}{(a^k)^{j+p}}, t \right) \right) \right) \\
\geq l \ T_{j=n}^{n+p} M \left( x, \frac{\alpha^{j+1}}{|(a^k)^j|} t \right)
\]

forall \( x \in X, \ t > 0, \ n \geq 0. \) Since \( \lim_{n \to \infty} T_{j=n}^\infty M(x, \frac{\alpha^{j+1}}{|(a^k)^j|} t) = 1_l \ \forall x \in X, \ t > 0, \)

\( \left\{ \left( a^{2k} \right)^n f \left( \frac{x}{(a^k)^n} \right) \right\}_{n \in \mathbb{N}} \) is a Cauchy sequence in the Non-Archimedean \( \ell \)-fuzzy Banach space \((Y, P, T)\). Hence we can define a mapping \( Q : X \to Y \) such that

\[
\lim_{n \to \infty} p((a^{2k})^n f \left( \frac{x}{(a^k)^n} \right) - Q(x), t) = 1_l, \ \forall x \in X, \ t > 0. \quad (4.19)
\]
Next, for all \( n \geq 1, x \in X \) and \( t > 0 \), we have

\[
p\left(f(x) - (a^{2k})^n f\left(\frac{x}{(a^k)^n}\right), t\right) = p\left(\sum_{i=0}^{n-1} (a^{2k})^i f\left(\frac{x}{(a^k)^i}\right) - (a^{2k})^{i+1} f\left(\frac{x}{(a^k)^{i+1}}, t\right)\right) 
\geq_t T_{i=0}^{n-1} \left(p\left(\left(\frac{x}{(a^k)^i}\right) - (a^{2k})^{i+1} f\left(\frac{x}{(a^k)^{i+1}}, t\right)\right)\right) 
\geq_t T_{i=0}^{n-1} M\left(x, \frac{\alpha^{i+1}t}{|a^k|^i}\right)
\]

and so

\[
p(f(x) - Q(x), t) 
\geq_t T\left(p\left(f(x) - (a^{2k})^n f\left(\frac{x}{(a^k)^n}\right), t\right), p\left((a^{2k})^n f\left(\frac{x}{(a^k)^n}\right) - Q(x), t\right)\right) 
= p\left(T_{i=0}^{n-1} M\left(x, \frac{\alpha^{i+1}t}{|a^k|^i}\right), p\left((a^{2k})^n f\left(\frac{x}{(a^k)^n}\right) - Q(x), t\right)\right) \quad (4.20)
\]

Taking the limit as \( n \to \infty \) in (4.20), we obtain

\[
P(f(x) - Q(x), t) \geq T_{i=1}^\infty M\left(x, \frac{\alpha^{i+1}t}{|a^k|^i}\right)
\]

which proves (4.16). As \( T \) is continuous, from a well-known result in \( \ell \)–fuzzy (probabilistic) normed space, it follows that

\[
\lim_{n \to \infty} p\left((a^{2k})^n f(a^{-kn}(ax - y)) \pm a(a^{2k})^n f(a^{-kn}(x \pm y)) \right) - (a \pm 1)[a(a^{2k})^n f(a^{-kn}x) \pm (a^{2k})^n f(a^{-kn}y)], t = p(Q(ax - y) \pm aQ(x \pm y) - (a \pm 1)(aQ(x) \pm Q(y)), t)
\]
for almost all $t > 0$. On the other hand, replacing $x, y$ by $a^{-kn}x, a^{-kn}y$ in equations (4.14) and (4.15), we get.

\[
p \left( (a^{2k})^n f(a^{-kn}(ax - y)) \pm a(a^{2k})^n f(a^{-kn}(x \pm y)) \right) \\
= - (a \pm 1) [a(a^{2k})^n f(a^{-kn}x) \pm (a^{2k})^n f(a^{-kn}y), t] \\
\geq \psi \left( a^{-kn}x, a^{-kn}x, \frac{t}{a^{2k}} \right), \forall x \in X, \ t > 0 \\
\geq \psi \left( x, y, \frac{\alpha^nt}{|a^k|^{n}} \right),
\]

since \( \lim_{n \to \infty} \psi(x, y, \frac{\alpha^nt}{|a^k|^{n}}) = 1 \), we infer that $Q$ is a quadratic mapping.

For the uniqueness of $Q$, let $Q' : X \to Y$ be another quadratic mapping such that

\[
p(Q'(x) - f(x), t) \geq M(x, t),
\]

for all $x \in X, t > 0$. Then we have, for all $x, y \in X, t > 0$,

\[
p(Q(x) - Q'(x), t) \\
\geq \psi \left( x, y, \frac{\alpha^nt}{|a^k|^{n}} \right).
\]

Therefore, from (4.19), we conclude that $Q = Q'$. This completes the proof.

**Corollary 4.3.3** Let $K$ be a non-Archimedean field, $X$ a vector space over $K$ and $(Y, p, \tau)$ a non-Archimedean $\ell$–fuzzy Banach space over $K$ under a $t$–norm $\tau \in H$. Let $f : X \to Y$ be a $\psi$–approximately quadratic mapping and $f(0) = 0$. If there exist an $\alpha \in R(\alpha > 0)$ and an integer $k, k \geq 2$ with $|a^k| < \alpha, |a| \neq 1$ and $a \neq 0$ such that

\[
\psi(a^{-k}x, a^{-k}y, t) \geq \psi(x, y, \alpha t) \quad \forall \ x, y \in X, \ t > 0
\]
then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$p(f(x) - Q(x), t) \geq \tau \sum_{j=1}^{\infty} M \left( x, \frac{\alpha^j t}{|a|^{k_j}} \right) \quad \forall \ x \in X, \ t > 0$$

where

$$M(x, t) = \tau(\psi(x, 0, t), \psi(ax, 0, t) \cdots \psi(a^{k-1}x, 0, t)) \quad \forall \ x \in X, \ t > 0.$$ 

**Proof.** Since

$$\lim_{n \to \infty} M \left( x, \frac{\alpha^j t}{|a|^{k_j}} \right) = 1, \quad \forall x \in X, \ t > 0$$

it follows that

$$\lim_{n \to \infty} T_{j=n} M \left( x, \frac{\alpha^j t}{|a|^{k_j}} \right) = 1, \quad \forall x \in X, \ t > 0.$$ 

Now, if we apply Theorem 4.3.2, we get the conclusion.

Now, author gives an example to validate the main result as follows:

**Example 4.3.4** Let $(X, P, T)$ be a non-Archimedean Banach space, $(X, p_{\mu, \nu}, \tau_m)$ a non-Archimedean $\ell$–fuzzy normed space (intuitionistic fuzzy normed space) in which

$$p_{\mu, \nu}(x, t) = \left( \frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|} \right)$$

for all $x \in X, \ t > 0$ and let $(Y, p_{\mu, \nu}, \tau_m)$ be a complete non-Archimedean $\ell$–fuzzy normed space (intuitionistic fuzzy normed space) (see Example 4.1.4). Define

$$\psi(x, y, t) = \left( \frac{t}{1 + t}, \frac{1}{1 + t} \right)$$
It is easy to show that (4.15) holds for \( \alpha = 1 \) (note that \(|a| \neq 1, a \neq 0\). Also, since
\[
M(x, t) = \left( \frac{t}{1 + t}, \frac{1}{1 + t} \right)
\]
we have
\[
\lim_{n \to \infty} T_j^\infty M \left( x, \frac{\alpha^j t}{|a|^k} \right) = \lim_{n \to \infty} \left( \lim_{m \to \infty} T_{M^j = n}^m M \left( x, \frac{t}{|a|^k} \right) \right),
\]
\[
= \lim_{n \to \infty} \lim_{m \to \infty} \left( \frac{t}{t + |a|^k}, \frac{|a|^k}{t + |a|^k} \right)
\]
\[
= (1, 0) = 1_{t^*}, \forall x \in X, t > 0.
\]

Let \( f : X \to Y \) be a \( \psi \)-approximately quadratic mapping. Therefore, all the conditions of Theorem 4.3.2 hold and so there exists a unique quadratic mapping \( Q : X \to Y \) such that
\[
p_{\mu, \nu}(f(x) - Q(x), t) \geq t^* \left( \frac{t}{t + |a|^k}, \frac{|a|^k}{t + |a|^k} \right).
\]