CHAPTER - 3

SOFT FUZZY SOFT TOPOLOGICAL FOLDINGS OF HYPERMANIFOLDS

The notion of differential manifolds was introduced and studied by Serge Lang [67]. A.K.Katsaras and D.B.Liu [34] introduced the concept of fuzzy topological vector spaces. M.Ferraro and David H.Foster [23] developed the concepts of fuzzy atlases and fuzzy diffeomorphisms. In this chapter, the concept of soft fuzzy soft manifolds of higher dimension is introduced. Some interesting properties of soft fuzzy soft retractions and soft fuzzy soft topological foldings of the soft fuzzy soft hypermanifolds are established.

3.1 SOFT FUZZY SOFT DIFFERENTIABLE MANIFOLDS

In this section, the notions of soft fuzzy soft topological vector spaces, soft fuzzy soft diffeomorphism, soft fuzzy soft atlases and soft fuzzy soft hypermanifolds are introduced. The relation between the soft fuzzy soft retractions and soft fuzzy soft topological foldings is obtained. In this connection, several properties are discussed.
**Definition 3.1.1.** Let $V$ be a vector space over the field $K$ of real numbers. Let $\{(\mu_p)_j\}_{j=1}^{n}$ be a family of soft fuzzy soft sets of a vector space $V$. The sum $\mu_p = \sum_{j=1}^{n} (\mu_p)_j$ of the family $\{(\mu_p)_j\}_{j=1}^{n}$ is the soft fuzzy soft set of $V$ such that $\mu_p(x) = H_j x_j = x_j H_1 \leq j \leq n \{(\mu_p)_j(x_j)\}$ for each $x \in V$.

Let $\alpha$ be any scalar in $K$ and $\mu_p$ be a soft fuzzy soft set of $V$. Then the **scalar product** $\alpha \mu_p$ is the soft fuzzy soft set of $V$ if $\alpha = 0$, $\alpha \mu_p(x) = \mu_p\left(\frac{x}{\alpha}\right)$ for all $x \in V$. For if $\alpha = 0$, 

$$\alpha \mu_p(x) = \begin{cases} \bigvee_{y \in V} \mu_p(y), & \text{for } x = 0 \\ (0, \varphi), & \text{otherwise} \end{cases}$$

**Definition 3.1.2.** Let $\tau_{\lambda P}$ be a soft fuzzy soft topology and $P$ be the set of all parameters. A **soft fuzzy soft topological vector space** is a vector space $V$ over the field $K$ of real numbers and $V$ equipped with $\tau_{\lambda P}$ and $K$ equipped with usual topology $\xi$ in an Euclidean space, such that the two functions

1. $(v_1, v_2) \rightarrow v_1 + v_2$ of $(V, \tau_{\lambda P}) \times (V, \tau_{\lambda P})$ into $(V, \tau_{\lambda P})$ and
2. $(k, v) \rightarrow kv$ of $(K, \omega(\xi)_Q) \times (V, \tau_{\lambda P})$ into $(V, \tau_{\lambda P})$

are the soft fuzzy soft continuous functions.

**Definition 3.1.3.** Let $P, Q$ and $R$ be the set of all parameters over the vector spaces $V, W$ and $I$ respectively. Let $(V, \tau_{\lambda P})$ and $(W, \sigma_{\mu_Q})$ be any two soft fuzzy soft topological vector spaces. Let $g(t)$ be any function of a real variable $t$ such that $\lim_{t \to 0} \frac{g(t)}{t} = 0$. The function $f : (V, \tau_{\lambda P}) \rightarrow (W, \sigma_{\mu_Q})$ is said to be
a soft fuzzy soft tangent to 0 if given a soft fuzzy soft open set $\lambda_Q$ in $(W, \sigma_{\mu_Q})$, there exists a soft fuzzy soft open set $\mu_P$ in $(V, \tau_{\lambda_P})$ such that $f(t, \mu_P) \prec g(t) \cdot \lambda_Q$, for some function $g(t)$.

**Definition 3.1.4.** Let $(V, \tau_{\lambda_P})$ and $(W, \sigma_{\mu_Q})$ be any two soft fuzzy soft topological vector spaces and $v_{\lambda_P}$ be any soft fuzzy soft point of $V$. Let $f : (V, \tau_{\lambda_P}) \to (W, \sigma_{\mu_Q})$ be a soft fuzzy soft continuous function. Then $f$ is said to be a soft fuzzy soft differentiable function at a point $v_{\lambda_P}$, if there exists a linear soft fuzzy soft continuous function $h$ of $V$ into $W$ such that $f(v + w) = f(v) + h(w) + g(w)$, for $w \in V$ where $g$ is the soft fuzzy soft tangent to 0. Then the function $h$ is called the soft fuzzy soft derivative of $f$ at $v_{\lambda_P}$ denoted by $f'(v_{\lambda_P})$. Moreover $f$ is a soft fuzzy soft differentiable on $V$ if it is soft fuzzy soft differentiable at every point of $V$.

**Definition 3.1.5.** Let $(V, \tau_{\lambda_P})$ and $(W, \sigma_{\mu_Q})$ be any two soft fuzzy soft topological vector spaces. The bijective function $f : (V, \tau_{\lambda_P}) \to (W, \sigma_{\mu_Q})$ is said to be a soft fuzzy soft diffeomorphism, if $f$ and $f^{-1}$ are the soft fuzzy soft differentiable functions and $f'$ and $(f^{-1})'$ are the soft fuzzy soft continuous functions.

**Definition 3.1.6.** Let $J$ be an indexed set. Let $(X, \tau_{\lambda_P})$ be a soft fuzzy soft topological space. A soft fuzzy soft atlas $A$ on $X$ is a collection of pairs $((\lambda_P)_{j}, \psi_{j})$ for $j \in J$ which satisfies the following conditions:

1. Each $(\lambda_P)_j$ is a soft fuzzy soft open set in $(X, \tau_{\lambda_P})$ and $H_{j \in J} \{ (\lambda_P)_j(x) \} =$
\((\lambda_P(x), P)\) for all \(x \in X\).

(2) Each \(\psi_j\) is a bijection defined on the support of \((\lambda_P)_j\), \(\{x \in X : (\lambda_P)_j(x) + (0, \varphi)\}\) which maps \((\lambda_P)_j\) onto a soft fuzzy soft open set \(\psi_j((\lambda_P)_j)\) in some soft fuzzy soft topological vector space \(V_j\) and for each \(l \in J\), \(\psi_j((\lambda_P)_j \downarrow (\lambda_P)_l)\) is a soft fuzzy soft open set in \(V_j\).

(3) The function \(\psi_l \circ \psi_j^{-1}\) which maps \(\psi_j((\lambda_P)_j \downarrow (\lambda_P)_l)\) onto \(\psi_l((\lambda_P)_j \downarrow (\lambda_P)_l)\) is a soft fuzzy soft diffeomorphism for each pair of indices \(j, l\).

Each pair \(((\lambda_P)_j, \varphi_j)\) is called a soft fuzzy soft chart of the soft fuzzy soft atlas.

**Definition 3.1.7.** Let \((X, \tau_{\lambda_P})\) be a soft fuzzy soft topological space and \((V, \sigma_{\lambda_P})\) be a soft fuzzy soft topological vector space. Let \(\lambda_P\) be a soft fuzzy soft open set in \((X, \tau_{\lambda_P})\) and \(\psi\) be a soft fuzzy soft continuous bijective function defined on the support of \(\lambda_P\) which maps \(\lambda_P\) onto a soft fuzzy soft open set in some \((V, \sigma_{\lambda_P})\). Then \((\lambda_P, \psi)\) is said to be compatible with the soft fuzzy soft atlas \(\{((\lambda_P)_j, \psi_j) \mid j \in J\}\), if each function \(\psi_j \circ \psi_j^{-1}\) of \(\psi((\lambda_P) \downarrow (\lambda_P)_j)\) onto \(\psi_j((\lambda_P) \downarrow (\lambda_P)_j)\) is a soft fuzzy soft diffeomorphism.

**Definition 3.1.8.** Two soft fuzzy soft atlases are compatible if each soft fuzzy soft chart of a soft fuzzy soft atlas is compatible with each soft fuzzy soft chart of the other soft fuzzy soft atlas. The relation of compatibility between the soft fuzzy soft atlases is an equivalence relation. The set of all equivalence
classes of the soft fuzzy soft atlases on a non-empty set $X$ is said to be a **soft fuzzy soft differentiable manifold** on $X$ which is also called as a **soft fuzzy soft hypermanifold**.

**Definition 3.1.9.** Let $(B, r_{A_p})$ and $(F, r_{A_p})$ be any two soft fuzzy soft topological spaces and the projection $P_2 : B \times F \to F$ be a soft fuzzy soft continuous surjective function. Then for any soft fuzzy soft point $x_{\delta_p} \in F$, $P_2^{-1}(x_{\delta_p})$ is soft fuzzy soft homeomorphic to $F$ and is called the **soft fuzzy soft fiber** over $x_{\delta_p}$.

**Notation 3.1.1.** $M_i$ and $\overline{M_i}$ denote the soft fuzzy soft lower hypermanifold and the soft fuzzy soft upper hypermanifold respectively.

**Proposition 3.1.1.** Let $M_i$ be a soft fuzzy soft hypermanifold in $\mathbb{R}^{n+1}$ and $M = \bigcup_i M_i$. Let $(M, r_{A_p})$ be the family of all soft fuzzy soft hypermanifold in $\mathbb{R}^{n+1}$. Then there are two types of soft fuzzy soft topological foldings on soft fuzzy soft hypermanifolds.

1. For all, $[A] \in M$, $\mu_p([A]) = (\lambda([A]), P)$. In this case, the family of all soft fuzzy soft hypermanifolds will be parallel spaces.

2. If $[A_i] = [A_j]$ with $\mu_p([A_i]) = \mu_p([A_j])$, $\forall [A_i], [A_j] \in M$, then the family of soft fuzzy soft hypermanifolds will be soft fuzzy soft diffeomorphic to the family of soft fuzzy soft hypermanifolds with common point.

**Proof.** (1) Let $M = \bigcup_i M_i$ be the soft fuzzy soft hypermanifold in $\mathbb{R}^{n+1}$. For
all \([A_i], [A_j] \in M\) with \(\mu_P([A_i]) = \mu_P([A_j]) = (\lambda([A]), P)\). Then the soft fuzzy soft hypermanifolds are parallel. Also for any point \([A_i] \in M_i\) with \(\mu_P([A_i]) = \mu_P([A]) + (\lambda([A]), P)\). Suppose if \([A_i], [A_j] \in M_j\) with \(\mu_P([A_i]) = \mu_P([A_j])\), then \(M_j = \overline{M}_j\) or \(M_j = \underline{M}_j\). Therefore for any point \([A] \in M = \cup_i M_i\), there exists a soft fuzzy soft fiber \(l_{[A]}\) over \([A]\) such that \(\forall [B] \in l_{[A]}, \mu_P([B]) + (\lambda([B]), P)\) and the value of \(\mu_P([B])\) decreases gradually to \((0, \varphi)\).

2. Let \(M_i\) be a soft fuzzy soft hypermanifold such that \(\mu_P([A_i]) = \mu_P([A])\), \(\forall [A_i], [A_j] \in M_i\). Therefore for any point \([B] \in M_i\) in which \(\mu_P = l_{[A]}(\mu_P)\). Thus for any point \([A] \in M\), there is an induced soft fuzzy soft fiber \(l_{[A]}\) with \(\mu_P([A]) + \mu_P([A]) + \mu_P([B])\), for all \([A_i], [A_j] \in l_{[A]}\). This implies that there is no common point except \([B]\). Also for any horizontal soft fuzzy soft fiber, the soft fuzzy soft fiber at \([B]\) has a maximum value of \(\mu_P\), that is, \(\lambda_P\).

Generalizing the above discussion, the family of soft fuzzy soft hypermanifold has a common point \([B]\). □

**Proposition 3.1.2.** Let \(M_i\) be a soft fuzzy soft hypermanifold in \(\mathbb{R}^{n+1}\) and \(M = \cup_i M_i\). If \(\bar{r}\) is a soft fuzzy soft retraction of \(M\) onto \(A \subseteq M\), then there are induced soft fuzzy soft retractions, \(\bar{r}_i\) such that \(\bar{r}_i : (M_i, \tau_{\lambda_P}) \rightarrow (A_i, \tau_{\lambda_P})\) and these soft fuzzy soft retractions are the soft fuzzy soft topological foldings if \(\dim A_i = \dim M\) and there is a common point.

**Proof.** Let \(\psi : \cup_i M_i \rightarrow \cup_i \overline{M}_i\) such that \(\psi(M_i) = \overline{M}_i\) and \(\mu_P(M_i) = \mu_P(\overline{M}_i)\).

Then \(\psi(\cup_i M_i) = \cup_i \overline{M}_i\). In the first case, if there is no common point, then
by the Proposition 3.1.1., the family of soft fuzzy soft hypermanifold $\cup_i M_i$ will be a parallel space and for any soft fuzzy soft retraction, $\tilde{r}_i : M_i \to A_i$, there are induced soft fuzzy soft retractions $\tilde{r}_i : (M_i, \tau_{A_i}) \to (A_i, \tau_{A_i})$ and $\dim A_i = \dim M_i$. Generalizing this discussion, there are induced soft fuzzy soft retractions $\tilde{r}_i : (M_i, \tau_{A_i}) \to (A_i, \tau_{A_i})$ and $\dim A_i = \dim \cup_i M_i$. Therefore soft fuzzy soft retractions $\tilde{r}_i$ are the soft fuzzy soft topological foldings.

But in the second case, if $[B]$ is a common point of each $M_i$ and $\tilde{r}_i : M \to [B]$, then there are induced soft fuzzy soft retractions $\tilde{r}_i : M_i \to [B]$ and $\dim [B] = \dim M_i$. Moreover the soft fuzzy soft retraction $\tilde{r}_i : M_i \to [B]$ is the minimum soft fuzzy soft retraction and it is not a type of soft fuzzy soft topological folding, since $\dim [B] = \dim \cup_i M_i$.

\[ \square \]

**Note 3.1.1.** $(b) = (b_1, b_2, \ldots, b_n)$ denotes the n-tuple point in $R^n$.

**Proposition 3.1.3.** Let $(M_i, \tau_{A_i})$ be a soft fuzzy soft hypermanifold in $R^n$, soft fuzzy soft diffeomorphic to a soft fuzzy soft hypersphere $S^n_i$. Let

$$\mu_P = \begin{cases} (r, P)^{-} \quad \text{if } 0 < r \leq \gamma_P \\ (1, P)^{-} \quad \text{if } \gamma_P < r < \infty \end{cases}$$

where $\gamma_P = H_{(\alpha_i) \in S^n_i} \lambda_{P}(\alpha_i)$ and $r$ be the radius of the soft fuzzy soft hypersphere. Let $\mu_P = (0, \varphi)^{-}$, if $r = 0, \infty$. Then $\cup_i S^n_i$ is a system of nested $n$-soft fuzzy soft hyperspheres with common center $(b)$. For any soft fuzzy soft retraction of $(D^n - (b) \mathcal{J})$ onto $S^{n-1}$, there are induced soft fuzzy soft retractions of $(D^n_i - (b) \mathcal{J}^i)$ onto $S_i^{n-1}$ and any soft fuzzy soft topological foldings $\psi : S_i^n \to \overline{S}_i^n$ must be defined such that $\psi(S^n) = S^n$. 

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Proof. Let \((M, \tau_{\mathcal{P}})\) be a soft fuzzy soft hypermanifold in \(\mathbb{R}^n\) which is soft fuzzy soft diffeomorphic to a soft fuzzy soft hypersphere, \(S^n\) with radius \(r\). Let \(\gamma_P = H_{\mathcal{P}(\mathcal{P})} \lambda_P (\mathcal{P})\) and \(D^n\) be any soft fuzzy soft closed ball. Let

\[
\mu_P = \begin{cases} (r, P) & \text{if } (0 < r \leq \gamma_P) \\ (1, r, P) & \text{if } (\gamma_P < r < \infty) \end{cases}
\]

and \(\mu_P = (0, r, \varphi)^\sim\), if \(r = 0, \infty\). Then \(S^n = \cup_i S^n_i\) is a system of nested \(n\)-soft fuzzy soft hyperspheres with common point \((\mathcal{b})\). If the system consists of soft fuzzy soft hyperspheres inside \(S^n\) with common point \((\mathcal{b}) \in S^n\), there is a soft fuzzy soft fiber \(I_{\mathcal{b}}\) with \(\mu_P(\mathcal{b}) + \mu_P(\mathcal{b}) + (\lambda_P(\mathcal{b}), P)\) for all \((\mathcal{b}) \in S^n_i\) and \((\mathcal{b}) \in S^n_i\). If the soft fuzzy soft topological foldings, \(\psi : S^n_i \rightarrow S^n_i\) such that \(\psi(S^n_i) = S^n_i\) and \(\mu_P(S^n_i) = \mu_P(S^n_i)\), then there is an induced soft fuzzy soft topological folding \(\psi : D^n_i \rightarrow D^n_i\) and soft fuzzy soft retraction \(\bar{r} : D^n_i - \{\mathcal{b}\} \rightarrow S^n_i\) for which there are induced soft fuzzy soft retractions \(\bar{r}_i : D^n_i - \{\mathcal{b}\} \rightarrow S^n_i\) and \(\bar{r}_i : D^n_i - \{\mathcal{b}\} \rightarrow S^n_i\). \qed

Proposition 3.1.4. Let \(M\) be a soft fuzzy soft hypermanifold soft fuzzy soft diffeomorphic to a soft fuzzy soft hypersphere \(S^n\). Let \((\mathcal{b}) \in S^n\) be a common point such that \(\mu_P(\mathcal{b}) = H_{\mathcal{P}(\mathcal{P})} \mu_P(\mathcal{b})\). For any soft fuzzy soft topological folding \(\psi : S^n_i \rightarrow S^n_i\), then there are induced soft fuzzy soft topological foldings \(\psi : S^n_i \rightarrow S^n_i\) and also for any soft fuzzy soft retraction \(\bar{r} : S^n - \{\mathcal{b}\} \rightarrow A\), then there are induced soft fuzzy soft retractions \(\bar{r}_i : S^n_i - \{\mathcal{b}\} \rightarrow A_i\) such that \(\psi(S^n) = S^n\). That is, there is a system of soft fuzzy soft hyperspheres having common point \((\mathcal{b})\).
**Proof.** Consider the soft fuzzy soft hypersphere \( S^n \subseteq \mathbb{R}^{n+1} \). Take a point \((b) \in S^n\) such that \( \mu_P((b)) = H_{(b) \in S^n} \mu_P((b)) = (0, \varphi) \). Then by the Proposition 3.1.3., there is a system of soft fuzzy soft hyperspheres \( S_i^n \) and \( S'_i^n \) with a common point \((b) \in S^n\). Therefore the soft fuzzy soft topological folding \( \psi: S_i^n \rightarrow S'_i^n \) with \( \psi(S^n) = S^n \) such that \( \psi \) folds at the points which equal \( \mu_P \).

If \( \tilde{r}: S^n - \{(b)\} \rightarrow A \) is a soft fuzzy soft retraction, then there are induced soft fuzzy soft retractions \( \bar{r}_i: S_i^n - \{(b_i)\} \rightarrow A_i \) and \( \tilde{r}_i: S'_i^n - \{(b_i)\} \rightarrow \bar{A}_i \). This soft fuzzy soft retraction is a type of soft fuzzy soft topological foldings and \( \psi \circ \tilde{r} = \bar{r} \circ \psi, \lim_{i \to \infty} \bar{r}_i((S^n - \{(b)\}) \cup (S'_i^n - \{(b_i)\}) \cup (S_i^n - \{(b_i)\}) = \{(b)\} \). \( \square \)

**Proposition 3.1.5.** The soft fuzzy soft topological folding of soft fuzzy soft hypersphere \( S^n = \cup S_i^n \subseteq \mathbb{R}^{n+1} \) into itself induces two chains of soft fuzzy soft topological foldings \( \cup \bar{S}_i^n \) and \( \cup S_i^n \) which are soft fuzzy soft retractions.

**Proof.** Let \( \psi_1: S^n \rightarrow S^n \) be a soft fuzzy soft topological folding from \( S^n \) into \( S^n \) such that \( \psi_1(S^n) = S^n \) and \( \bar{r}_1 \) be a soft fuzzy soft retraction with a common point. This type of soft fuzzy soft topological folding induces two chains of soft fuzzy soft topological foldings \( \underline{\psi}_1: \bar{S}^n \rightarrow S^n \) such that \( \underline{\psi}_1(\bar{S}^n) = \bar{S}^n \) and \( \bar{\psi}_1: \bar{S}^n \rightarrow \bar{S}^n \) such that \( \bar{\psi}_1(\bar{S}^n) = \bar{S}^n \). Then by Proposition 3.1.2., there induced two chains of soft fuzzy soft retractions \( \bar{r}_1 \) and \( \bar{r}_2 \) which are the soft fuzzy soft topological foldings. Hence the soft fuzzy soft topological foldings coincide with the soft fuzzy soft retractions. Also, let \( \psi_2: \psi_1(S^n) \rightarrow \psi_1(S^n) \) be another soft fuzzy soft topological foldings from \( \psi_1(S^n) \) to \( \psi_1(S^n) \). This soft
fuzzy soft topological folding induces

$$\psi_2 : \psi_1(S^n) \to \psi_1(S^n),$$

$$\overline{\psi}_2 : \overline{\psi}_1(S^n) \to \overline{\psi}_1(S^n).$$

Similarly, $$\psi_3 : \psi_2(\psi_1(S^n)) \to \psi_2(\psi_1(S^n))$$ induces

$$\psi_3 : \psi_2(\psi_1(S^n)) \to \psi_2(\psi_1(S^n)),$$

$$\overline{\psi}_3 : \overline{\psi}_2(\overline{\psi}_1(S^n)) \to \overline{\psi}_2(\overline{\psi}_1(S^n));$$

continuing this process, the soft fuzzy soft topological foldings

$$\psi_n : \psi_{n-1}(\psi_{n-2}(\cdots \psi_2(\psi_1(S^n)) \cdots )) \to \psi_{n-1}(\psi_{n-2}(\cdots \psi_2(\psi_1(S^n)) \cdots ))$$

induce two chains of soft fuzzy soft topological foldings,

$$\psi_n : \psi_{n-1}(\psi_{n-2}(\cdots \psi_2(\psi_1(S^n)) \cdots )) \to \psi_{n-1}(\psi_{n-2}(\cdots \psi_2(\psi_1(S^n)) \cdots )),$$

$$\overline{\psi}_n : \overline{\psi}_{n-1}(\overline{\psi}_{n-2}(\cdots \overline{\psi}_2(\overline{\psi}_1(S^n)) \cdots )) \to \overline{\psi}_{n-1}(\overline{\psi}_{n-2}(\cdots \overline{\psi}_2(\overline{\psi}_1(S^n)) \cdots ))$$

That is, the soft fuzzy soft topological foldings

$$\psi_n : \psi_{n-1}(\psi_{n-2}(\cdots \psi_2(\psi_1(\cup_i S^n_i)) \cdots )) \to \psi_{n-1}(\psi_{n-2}(\cdots \psi_2(\psi_1(\cup_i S^n_i)) \cdots ))$$

induce two chains of soft fuzzy soft topological retractions,

$$\psi_n : \psi_{n-1}(\psi_{n-2}(\cdots \psi_2(\psi_1(\cup_i S^n_i)) \cdots )) \to \psi_{n-1}(\psi_{n-2}(\cdots \psi_2(\psi_1(\cup_i S^n_i)) \cdots )),$$

$$\psi_n : \psi_{n-1}(\psi_{n-2}(\cdots \psi_2(\psi_1(\cup_i S^n_i)) \cdots )) \to \psi_{n-1}(\psi_{n-2}(\cdots \psi_2(\psi_1(\cup_i S^n_i)) \cdots ))$$

\[ \square \]

**Proposition 3.1.6.** If the soft fuzzy soft retraction of the soft fuzzy soft hypersphere $$S^n \subseteq \mathbb{R}^{n+1}$$ is $$\overline{r} : S^n - \{b\} \to S^{n-1}$$ and the soft fuzzy soft topological folding $$\psi : S^n - \{b\} \to S^n - \{b\}$$, then there are induced two chains of soft fuzzy soft retractions and soft fuzzy soft topological foldings such that

$$\overline{r}_2 \circ \overline{r}_1 = \overline{r}_2 \circ \overline{r}_1$$

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and \( \psi_2 \circ \bar{r}_1 = \bar{r}_2 \circ \psi_1 \).
Proof. Let the soft fuzzy soft retraction of $S^n$ is defined by $\tilde{r}_1 : S^n - \{(b)\} \to S^{n-1}$ and the soft fuzzy soft topological foldings be given by $\psi_1 : S^n - \{(b)\} \to S^n - \{(b)\}$, $\psi_2 : S^{n-1} \to S^{n-1}$. Also, $\tilde{r}_2 : \psi_1(S^n - \{(b)\}) \to S^{n-1}$. Then there are induced two chains of soft fuzzy soft retractions and soft fuzzy soft topological foldings are given by

$$
\tilde{r}_1 : S^n - \{(b)\} \to S^{n-1}, \quad \tilde{r}_1 : S^n - \{(b)\} \to S^{n-1};
$$

$$
\tilde{r}_2 : \psi_1(S^n - \{(b)\}) \to S^{n-1}, \quad \tilde{r}_2 : \psi_1(S^n - \{(b)\}) \to S^{n-1};
$$

$$
\psi_1 : S^n - \{(b)\} \to S^n - \{(b)\}, \quad \overline{\psi}_1 : S^n - \{(b)\} \to S^n - \{(b)\};
$$

$$
\psi_2 : S^{n-1} \to S^{n-1}, \quad \overline{\psi}_2 : S^{n-1} \to S^{n-1}.
$$

Hence the following diagrams commute.
Proposition 3.1.7. The generalization of Proposition 3.1.6. represented by the following chains $\bar{r}_{i+1} \circ \psi_i = \varphi_{i+1} \circ \bar{r}_i$, $\bar{r}_{i+1} \circ \psi_j = \psi_{j+1} \circ \bar{r}_j$ and $\bar{r}_{i+1} \circ \bar{\psi}_i = \bar{\psi}_{i+1} \circ \bar{r}_i$, for $i = 1, 2, 3, ..., n$.

Proposition 3.1.8. The relation between the soft fuzzy soft retraction and the limit of the soft fuzzy soft topological foldings discussed from the following commutative diagram

\[
\begin{array}{ccc}
S^n - \{\emptyset_n\} & \xrightarrow{\bar{r}_1} & S^{n-1} - \{\emptyset_{n-1}\} \\
\downarrow{\bar{\psi}_1} & & \downarrow{\bar{\psi}_2} \\
S^n - \{\emptyset_n\} & \xrightarrow{\bar{r}_2} & S^{n-1} - \{\emptyset_{n-1}\}
\end{array}
\]

and the corresponding relation between the two chains of soft fuzzy soft retractions and limit of soft fuzzy soft topological foldings are described from the two induced diagrams

\[
\begin{array}{ccc}
S^n - \{\emptyset_n\} & \xrightarrow{\bar{r}_1} & S^{n-1} - \{\emptyset_{n-1}\} \\
\downarrow{\lim_{m \to \infty} \psi_m} & & \downarrow{\lim_{m \to \infty} \psi_{m+1}} \\
S^n - \{\emptyset_n\} & \xrightarrow{\bar{r}_2} & S^{n-1} - \{\emptyset_{n-1}\}
\end{array}
\]

\[
\begin{array}{ccc}
S^{n-1} - \{\emptyset_{n-1}\} & \xrightarrow{\bar{r}_j} & S^{n-2} - \{\emptyset_{n-2}\} \\
\downarrow{\lim_{m \to \infty} \psi_m} & & \downarrow{\lim_{m \to \infty} \psi_{m+1}} \\
S^{n-1} - \{\emptyset_{n-1}\} & \xrightarrow{\bar{r}_j} & S^{n-2} - \{\emptyset_{n-2}\}
\end{array}
\]

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\[
\mathbb{S}^n - \{\bar{\partial}_n\} \xrightarrow{\bar{r}_1} \mathbb{S}^{n-1} - \{\bar{\partial}_{n-1}\}
\]

\[
\lim_{m \to \infty} \bar{\psi}_m \xrightarrow{\ast} \lim_{m \to \infty} \bar{\psi}_{m+1}
\]

\[
\mathbb{S}^{n-1} - \{\bar{\partial}_{n-1}\} \xrightarrow{\bar{r}_j} \mathbb{S}^{n-2} - \{\bar{\partial}_{n-2}\}
\]

**Proposition 3.1.9.** The end of the limits of the soft fuzzy soft topological foldings and soft fuzzy soft retractions of the system will be

\[
\mathbb{S}^2 - \{\bar{\partial}_2\} \xrightarrow{\bar{r}_k} \mathbb{S}^1 - \{\bar{\partial}_1\}
\]

\[
\lim_{m \to \infty} \bar{\psi}_m \xrightarrow{\ast} \lim_{m \to \infty} \bar{\psi}_{m+1}
\]

\[
\mathbb{S}^1 - \{\bar{\partial}_1\} \xrightarrow{\bar{r}_j} 0
\]

Also the induced relation between the two chains of the end of the limits of the soft fuzzy soft topological foldings and the soft fuzzy soft retractions of the system are defined by

\[
\mathbb{S}^2 - \{\bar{\partial}_2\} \xrightarrow{\bar{r}_k} \mathbb{S}^1 - \{\bar{\partial}_1\}
\]

\[
\lim_{m \to \infty} \bar{\psi}_m \xrightarrow{\ast} \lim_{m \to \infty} \bar{\psi}_{m+1}
\]

\[
\mathbb{S}^1 - \{\bar{\partial}_1\} \xrightarrow{\bar{r}_j} 0
\]