CHAPTER - 1

INTRODUCTION AND BASIC CONCEPTS

1.1 FUZZY SETS, SOFT FUZZY SETS AND SOFT FUZZY SOFT SETS

Most of our traditional tools modeling, reasoning and computing are crisp, deterministic and precise in character. By crisp it means dichotomies, that is, yes-or-no type rather than more-or-less type. Certainty eventually indicates that one can assume the structures and parameters of the model to be definitely known and that there are no doubts about their values or their occurrences.

For factual model or modeling languages two major complications arise:

1. Real situations are very often not crisp and deterministic and they cannot be described precisely.

2. The complete description of a real system often would require by far more detailed data than a human being could ever recognize, process
and understand simultaneously.

In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition - an element either belongs or does not belong to the set.

To solve complicated problems in economics, engineering and environment, one cannot successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, interval mathematics and the theory of fuzzy sets which one can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties.

Theory of probabilities can deal only with stochastically stable phenomena. Without going into mathematical details, one can say that for a stochastically stable phenomenon there should exist a limit of the sample mean in a long series of trials. To test the existence of the limit, one must perform a large number of trials. One can do it in engineering, but one cannot do it in many economic, environmental or social problems.

Let us consider the characteristic features of real-world systems again: Real situations are very often uncertain or vague in a number of ways. Due to lack of information the future state of the system might not be known completely. This type of uncertainty (stochastic character) has long been handled appropriately by probability theory and statistics. This Kol-
mogoroff type probability is essentially frequent and is based on set-theoretic considerations. Koopman’s probability refers to the truth of statement and therefore it is based on logic. On both types of probabilistic approaches it is assumed, however, that the events (elements of sets) or the statements respectively, are well defined. One shall call this type of uncertainty or vagueness stochastic uncertainty by contrast to the vagueness concerning the description of the semantic meaning of the events, phenomena or statement themselves, which one shall call fuzziness.

Interval mathematics has arisen as a method of taking into account the errors of calculations by constructing an interval estimate for the exact solution of a problem. This is useful in many cases, but the methods of interval mathematics are not sufficiently adaptable for problems with different uncertainties. They cannot appropriately describe a smooth changing of information, unreliable, not adequate, and defective information, partially contradicting aims, and so on.

In 1923, the philosopher Russell, B. [66] referred to the first point when he wrote: All traditional logic habitually assumes that precise symbols are being employed. It is therefore not applicable to this terrestrial life but only to an imagined celestial existence.

L.Zadeh[78] referred to the second point when he wrote: "As the complexity of a system increases, our ability to make precise and yet signif-
ificant statement about its behaviour diminishes until a threshold is reached beyond which precision and significance (or relevance) become almost mutually exclusive characteristics”.

The most appropriate theory, for dealing with uncertainties is the theory of fuzzy sets developed by L.Zadeh[78]. Fuzziness can be found in many areas of daily life, such as in engineering (D.Blockley[7]), in medicine (M.A.Vila and M.Delgado [72]), in metereology (H.Cao and G.Chen[11]), in manufacturing (E.H.Mamdani[50]).

L.Zadeh[78] writes: “The notion of a fuzzy set provides a convenient point of departure for the construction of a conceptual framework which parallels in many respects the framework used in the case of ordinary sets but is more general than the latter and, potentially, may prove to have a much wider scope of applicability, particularly, in the fields of pattern classification and information processing. Essentially, such a framework provides a natural way of dealing with problems in which the source of imprecision in the absence of sharply defined criteria of class membership rather than the presence of random variables”.

Fuzzy set theory provides a strict mathematical framework in which vague conceptual phenomena can be precisely and rigorously studied. Fuzziness has so far not been defined uniquely, semantically and probably never will. It will mean different things, depending on the application
area and the way it is measured. In the meantime, numerous authors have contributed to this theory. Fuzzy sets are having useful and interesting applications in various fields including Probability theory, Information theory [68], Control [69] and Optimization techniques [6].

As an extension of classical set theory, fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval [0,1]. A fuzzy set \( F \) is described by its membership function \( \lambda_F \). At the present time, the theory of fuzzy sets is progressing rapidly. But there exists a difficulty: how to set the membership function in each particular case. One should not impose only one way to set the membership function. The nature of the membership function is extremely individual. Everyone may understand the notation \( \lambda_F (x) = 0.7 \) in his own manner. The reason for these difficulties is, possibly, the inadequacy of the parametrization tool of the theory.

To overcome these difficulties, one must use an adequate parametrization tool which is soft set theory. The soft set is a parameterized family of subsets of the Universal set. For example the fact that the soft set describes the attractiveness of the houses with parameters expensive; beautiful; wooden; cheap; in the green surroundings; modern; in good repair; in bad repair which Mr.Y is going to buy. Soft set theory is the generalization of the fuzzy set theory since Zadeh’s [78] fuzzy set is considered as a special
case of the soft set. The soft set theory gives an opportunity to construct a new mathematical tool which keeps all good sides of choice function and eliminates its drawbacks.

The soft fuzzy set is represented as \((\mu, M)\) where \(\mu\) is a fuzzy set and \(M\) is the crisp subset of Universal set \(X\). For \(x \in X\), it follows that there are two possible states corresponding to \(x \in M\) and \(x \notin M\), associated with degree of membership \(\mu(x)\). In case \(x \in M\), \(\mu(x)\) has a realized or hard value, otherwise it will be a soft or unrealized. For this reason, one shall refer to the pairs \((\mu, M)\) a soft fuzzy set [70].

Soft fuzzy soft sets as a generalization of fuzzy sets can be useful in situations such as optimization problems, medical diagnosis, sales analysis, product marketing, financial services, etc.

### 1.2 REVIEW OF LITERATURE

The concept of a fuzzy set provides a natural framework for generalizing many of the concepts of a general topology. The theory of fuzzy topological spaces was introduced and developed by C.L.Chang [14]. Since then various notions in classical topology have been extended to fuzzy topological spaces by fuzzy topologists like K.K.Azad [2], L.Zadeh [78], T.Kubiak [37, 38, 39], T.H.Yalvac [76], J.G.Brown [9], J.A.Goguen [27, 28], B.W.Hutton and I.L.Reilly [32], Lowen et al. [43, 44, 45, 46], M.K.Chakrabarty and
T.M.G.Ahsanullah [13], S.E.Rodabaugh [62, 63, 64, 65], R.H.Warren [73], L.M.Friedler [25] and T.E.Gantner et al. [26].

The Russian researcher D.Molodtsov[54] introduced the concept of a soft set. P.K.Maji [48] and other scientists proposed the concept of fuzzy soft sets and gave its applications [49] in economics, engineering, sociology, decision making problem, medical diagnosing, etc. The concepts of soft fuzzy set over a poset I was introduced by I.U.Tiryaki [70] in his Ph.D Thesis.

Algebraic topology tries to connect topological spaces with algebraical objects in such a way that topological problems can be translated into algebraical problems which can possibly be easier to solve. Homotopy theory and fundamental group were introduced and developed by W.S.Massey [52]. G.Culvacioglu and M.Citil[18] introduced the concepts of fuzzy homotopy theory. E. Guner [29] discussed some properties on the fuzzy contractibility and fuzzy retraction.

The basic motivation for the concept of a manifold is to be able to talk about differentiability of functions defined on objects, which undergo some perturbation. The retraction of a manifold was defined and discussed as in [52, 55]. Most folding problems have close connections to important industrial applications, applications in robotics and hydraulic tube bending, shell structures in civil engineering and aerospace design, namely buckling instability. The folding of a manifold was introduced by S.A.Robertson [61].
Various folding problems arising in the physics of membrane and polymers are reviewed by P.Di.Francesco [24]. El-Ghoul et al. [20, 21] introduced and developed the concepts of foldings on fuzzy manifolds.

J.W.Alexander [1] was the first to show that knot theory is extremely important in the study of 3-dimensional topology. Knot theory moved from the realm of topology to mathematical physics. Later this was further underlined by Murasugi.K [56] that the knot theory is closely related to the solvable models of statistical mechanics. As knot theory grows and develops, its boundaries continue to shift.

Extending the concepts of manifolds to higher dimension through the topological vector space and differentiation, hypermanifolds (or) differentiable manifolds are studied. The notion of the differential manifold was introduced and studied by Serge Lang [67]. Fuzzy topological vector space and fuzzy differentiation were introduced and developed by Katsaras et al.[34]. M.Ferraro and D.H.Foster [23] discussed many properties of fuzzy differential manifolds.

Mathematical notions such as convergence, continuity, and separation axioms are usually associated with topological spaces. G.Choquet[15] approached these concepts using convergence instead of neighbourhood.

The notion of convergence is one of the basis notions in analysis. Convergence can be described in any topological space, by means of
nets or filters. The concept of filter was introduced by J.Dugundji [19]. M.A. de.Prada and M.Saralegui [58] proposed the notions of fuzzy filters. The concept of regularity in terms of convergent filters was introduced by C.H.Cook and H.R.Fisher[16]. Convergence space on the basis of filters was introduced and developed by D.C.Kent [35]. Lowen et al [42, 43, 46] introduced the concept of fuzzy convergence space. G.Richardson [59] studied several properties of the fuzzy convergence spaces. Later, G.Jager [33] defined fuzzy convergence space which is the generalization of Lowen’s [41] and Richardson's [59] fuzzy convergence spaces. The regularity in a fuzzy topological space was studied by B.W.Hutton and I.L.Reilly [32]. Minkler et al.[53] discussed the diagonal condition for the fuzzy convergence space and established the characterization of fuzzy regular convergence spaces.

The concept of $C$-set was introduced and developed by E.Hatir, T.Noiri and S.Yuksel [30]. Several properties of fuzzy product topological spaces were discussed by K.K.Azad[2]. The concept of fuzzy soft set was introduced and developed by P.K.Maji, R.Biswas and A.R.Roy[48]. The concept of soft fuzzy subset was introduced by I.U.Tiryaki [70]. The concept of fuzzy C-set was introduced by M.K.Uma, E.Roja and G.Balasubramanian [71]. The concepts of soft fuzzy $C$-open set was introduced by T.Yogalakshmi, E.Roja and M.K.Uma [77]. The notion of strong generalized topological space was introduced and discussed by A.Csaszar[17]. G.Palanichetty[57] introduced
and studied the concept of the fuzzy generalized topological spaces.

1.3 OUTLINE OF THE THESIS

This section presents a chapter wise summary of results obtained on a soft fuzzy soft set, soft fuzzy soft homotopy, soft fuzzy soft contractible spaces, soft fuzzy soft equivalent spaces, soft fuzzy soft path homotopy, soft fuzzy soft retraction, soft fuzzy soft deformation retraction, soft fuzzy soft topological foldings, soft fuzzy soft manifolds, soft fuzzy soft knot, soft fuzzy soft atlases, soft fuzzy soft hypermanifolds, soft fuzzy \( C \)-open sets, soft fuzzy B-sets, soft fuzzy \( C \)-continuous functions, soft fuzzy \( C \)-irresolute functions, soft fuzzy completely continuous functions, soft fuzzy strongly \( C \)-continuous functions, soft fuzzy \( C \)-connected spaces, soft fuzzy \( C \)-compact spaces, soft fuzzy \( C \)-filters, soft fuzzy \( C \)-convergence spaces, soft fuzzy \( C \)-regular convergence spaces, soft fuzzy product \( C \)-convergence spaces, soft fuzzy \( T_i \) \((i = 0, 1, 2, 2^1, 3)\) \( C \)-convergence spaces, soft fuzzy \( C \)-precompact sets, soft fuzzy locally \( C \)-precompact convergence spaces, soft fuzzy locally \( C \)-precompactification, soft fuzzy \( C \)-Tychonoff theorem, soft fuzzy product \( C \)-spaces, weakly induced soft fuzzy product \( C \)-spaces and soft fuzzy strong generalized product topological spaces. Several propositions and important results are obtained.

In Chapter 2, the concept of soft fuzzy soft sets is introduced. Soft
fuzzy soft topological space is studied as in [13]. Based on this concept, soft fuzzy soft homotopy, soft fuzzy soft fundamental groups, soft fuzzy soft topological foldings and soft fuzzy soft knots are introduced. In this connection, several properties are developed.

Soft fuzzy soft topological vector spaces, soft fuzzy soft differentiations, soft fuzzy soft diffeomorphisms, soft fuzzy soft hypermanifolds are introduced and some interesting properties on soft fuzzy soft retractions are discussed in Chapter 3. Further, the relation between soft fuzzy soft retractions and soft fuzzy soft topological foldings are established.

The concept of $C$-set was introduced by E.Hatir, T.Noiri and S.Yuksel [30]. I.U.Triyaki [70] introduced the concept of a soft fuzzy set. Motivated by these concepts, soft fuzzy $C$-open sets, soft fuzzy $C$-kernel, soft fuzzy $C$-neighbourhoods, soft fuzzy $C^*$-sets, soft fuzzy $C$-continuous functions, soft fuzzy $C$-connected spaces, soft fuzzy $C$-compact spaces and soft fuzzy $C$-normal spaces are introduced in Chapter 4. In this connection, several properties and interrelations are established. Counter examples are provided wherever necessary.

Fuzzy convergence space is defined as the generalization of fuzzy topological spaces by Lowen et al.[41]. In Chapter 5, a new space called soft fuzzy $C$-convergence spaces in terms of soft fuzzy $C$-filters is introduced. Properties on soft fuzzy $C$-filters, soft fuzzy prime $C$-filters and soft fuzzy $C$-
regular convergence spaces are studied as in [42, 59]. A new characterization on a soft fuzzy $C$-regular convergence spaces is established. The concepts of soft fuzzy product $C$-convergence spaces and soft fuzzy $T_\alpha(i = 0, 1, 2, 2_\alpha, 3)$ $C$-convergence spaces are introduced and developed. Besides giving inter - relations among the spaces introduced, several counter examples are also given.

In Chapter 6, the concepts of the soft fuzzy $C$-convergence spaces, soft fuzzy locally $C$-precompact convergence spaces and soft fuzzy locally weakly $C$-precompact convergence spaces are introduced. In this connection, soft fuzzy locally $C$-precompactification is established. $C$-Tychonoff theorem for the soft fuzzy $C$-convergence spaces has been discussed as in [12] besides proving several other propositions.

In Chapter 7, the concepts of soft fuzzy product $C$-space and soft fuzzy strong generalized product topological spaces are introduced. Some interesting properties of the associated functions are studied and compactification of soft fuzzy product $C$-spaces are established.

1.4 BASIC CONCEPTS

In this section, some basic concepts of soft sets, soft fuzzy sets have been recalled. Also related results, propositions and important theorems are collected from various research articles.
Throughout the thesis, $X$ is a non-empty set, $I = [0, 1]$ is the unit interval and $J$ is the indexed set.

**Definition 1.4.1.** [78] A **fuzzy set** $\lambda$ is a function from a non-empty set $X$ to a unit interval $I = [0, 1]$. The family of all fuzzy sets, denoted by $I^X$.

**Definition 1.4.2.** [14] A **fuzzy topology** is a family $T$ of fuzzy sets in $X$ which satisfies the following conditions:

1. $\emptyset, X \in T$.
2. If $A, B \in T$ then $A \cap B \in T$.
3. If $A_i \in T$ for each $i \in I$, then $\cup_i A_i \in T$.

Then the pair $(X, T)$ is called a **fuzzy topological space**.

**Definition 1.4.3.** [54] Let $X$ be a non-empty set, $E$ be the set of all parameters for $X$ and $A \subseteq E$. A pair $(F, A)$ is called a **soft set** over $X$ if $F$ is a mapping defined by $F : A \rightarrow 2^X$, where $2^X$ is the power set of $X$.

**Definition 1.4.4.** [13] Let $\lambda$ be a fuzzy set of $X$. A collection $\tau$ of fuzzy sets of $\lambda$ satisfying

1. $0, \lambda \in \tau$.
2. $\mu_i \in \tau \ \forall i \in J \Rightarrow \vee \{\mu_i : i \in J\} \in \tau$.
3. $\mu, \lambda \in \tau \Rightarrow \mu \wedge \lambda \in \tau$. 
is called a **fuzzy topology** on $\lambda$. Then the pair $(\lambda, r)$ is called a fuzzy topological space.

**Definition 1.4.5.** [55] Let $X$ be a topological space with topology $T$. If $Y$ is a subset of $X$, then the collection

$$T_Y = \{Y \cap U : U \in T\}$$

is a topology on $Y$, called the **subspace topology**. Then the ordered pair $(Y, T_Y)$ is called a subspace of $X$.

**Definition 1.4.6.** [55] Let $A$ be the subset of $X$ and $\chi_A : X \to \{0, 1\}$ be a function. Then the **characteristic function** $\chi_A$ of $A$ is defined as

$$\chi_A(x) =
\begin{cases} 
1, & \text{if } x \in A, \\
0, & \text{otherwise}.
\end{cases}$$

**Definition 1.4.7.** [52] Let $f, g : X \to Y$ be any two continuous functions. We say that $f$ and $g$ are **homotopic**, and denote that by $f \succeq g$, if there exists a continuous function $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, for each $x \in X$. The function $F$ is called a **homotopy** between $f$ and $g$.

**Definition 1.4.8.** [52] Given two paths $f, g : I \to X$ beginning at $x_0$ and ending at $x_1$, it is said that $f$ and $g$ are **path homotopic** (notation $f \succeq g$) if there is a continuous function $F : I \times I \to X$ such that

1. $F(t, 0) = f(t)$ and $F(t, 1) = g(t)$ for all $t \in I$;
(2) \( F(0, s) = x_0 \) and \( F(1, s) = x_1 \) for all \( s \in I \).

**Definition 1.4.9.** [55] **Euclidean space** \( \mathbb{R} \) is the set of all real numbers together with the topology determined by the Euclidean metric, \( d(x, y) = |x - y| \) for all \( x, y \in \mathbb{R} \).

**Definition 1.4.10.** [75] Let \( (X, T) \) be a topological spaces. The collection \( \tilde{T} = \{ D : D = X_{D_0} \text{ is a fuzzy set of } X \text{ and } D_0 \in T \} \) is a fuzzy topology on \( X \), called the **fuzzy topology on \( X \) introduced by \( T \)**. \( (X, \tilde{T}) \) is called the fuzzy topological space introduced by \( (X, T) \).

**Definition 1.4.11.** [75] Let \( (X, \tau) \) and \( (Y, \sigma) \) be the fuzzy topological spaces and \( (I, \varepsilon_I) \) be the fuzzy topological space introduced by the Euclidean space \((I, \varepsilon_I)\). Let \( f, g : (X, \tau) \rightarrow (Y, \sigma) \) be two fuzzy continuous mappings. If there exists a fuzzy continuous mapping \( F : (X, \tau) \times (I, \varepsilon_I) \rightarrow (Y, \sigma) \) such that \( F(x_{\lambda}, 0) = f(x_{\lambda}) \) and \( F(x_{\lambda}, 1) = g(x_{\lambda}) \) for every fuzzy point \( x_{\lambda} \) in \( (X, \tau) \), then we say that \( f \) is **fuzzy homotopic** to \( g \). The mapping \( F \) is called a **fuzzy homotopy** between \( f \) and \( g \) and write \( f \sim g \).

**Definition 1.4.12.** [29] A fuzzy set \( (A, \mu) \) of a fuzzy manifold \( (M, \mu) \) is called a **fuzzy retraction** if there exist a continuous map \( \tilde{r} : (M, \mu) \rightarrow (A, \mu) \) such that \( \tilde{r}(a, \mu(a)) = (a, \mu(a)) \), for all \( a \in A \), \( \mu \in [0, 1] \).

**Definition 1.4.13.** [55] A **manifold** is a Hausdorff space \( X \) with a countable basis such that each point \( x \) of \( X \) has a neighbourhood that is homeomorphic with an open subset of \( \mathbb{R}^n \).
Definition 1.4.14. [21] Let \((X, T)\) be a topological space. A map \(f : (X, T) \to (X, T)\) is said to be folding of a topological space into itself if \(f(X) \subset X\) and either for all \(G \in T\), \(f(G) = U \subset G\), \(U \in T\) or for all \(G \in T\), \(f(G) = G\).

Definition 1.4.15. [34] A fuzzy topological vector space is a vector space \(E\) over the field \(K\) of real or complex numbers, \(E\) equipped with a fuzzy topology \(T\) and \(K\) equipped with the usual topology \(K\) such that the two mappings

\[
\begin{align*}
(1) \quad (x, y) &\mapsto x + y \text{ of } (E, T) \times (E, T) \text{ into } (E, T), \\
(2) \quad (\alpha, x) &\mapsto \alpha \cdot x \text{ of } (K, K) \times (E, T) \text{ into } (E, T).
\end{align*}
\]

are fuzzy continuous.

Definition 1.4.16. [23] Let \(E\) and \(F\) be any two fuzzy topological vector spaces and let \(\varphi\) be a function from \(E\) into \(F\). The function \(\varphi\) is said to be tangent to 0 if given a neighbourhood \(W\) of \(0_\delta\), \(0 < \delta \leq 1\), in \(F\) there exists a neighbourhood \(V\) of \(0_\lambda\), for every \(\lambda, 0 < \lambda < \delta\), in \(E\) such that \(\varphi[V] \subset o(t)W\), for some function \(o(t)\), where \(o(t)\) is the function of a real variable \(t\) such that \(\lim_{t \to 0} \frac{o(t)}{t} = 0\).

Definition 1.4.17. [23] Let \(E\) and \(F\) be any two fuzzy topological vector spaces, each endowed with a \(T_1\) fuzzy topology. Let \(f : E \to F\) be a fuzzy continuous function. Then \(f\) is said to be fuzzy differentiable at a point \(x \in E\) if there exists a linear fuzzy continuous function \(u\) of \(E\) into \(F\) such
that \( f(x + y) = f(x) + u(y) + \varphi(y), \ y \in E \), where \( \varphi \) is tangent to 0. Then the mapping \( u \) is called the **fuzzy derivative of** \( f \) at \( x \). The fuzzy derivative of \( f \) at \( x \) is denoted by \( f'(x) \).

**Definition 1.4.18.** [60] A **knot** \( K \) is a simple closed curve in \( \mathbb{R}^3 \) that can be broken into a finite number of straight line segments \( e_1, e_2, \ldots, e_n \) such that the intersection of any segment \( e_k \) with the other segments is exactly one end point of \( e_k \) intersecting an endpoint of \( e_{k-1} \) and the other endpoint of \( e_k \) interesting an end point of \( e_{k+1} \).

**Definition 1.4.19.** [60] A **simple closed curve** is a curve that does not intersect itself.

**Definition 1.4.20.** [23] Let \( X \) be a non-empty set. A **fuzzy atlas** \( A \) of class \( C^1 \) on \( X \) is a collection of pairs \( (A_j, \varphi_j) \) (\( j \) ranging here and subsequently in some index set) which satisfies the following conditions:

1. Each \( A_j \) is a fuzzy set in \( X \) and \( \sup_j \{ \mu_{A_j} \} = 1 \), for all \( x \in X \).

2. Each \( \varphi_j \) is a bijection, defined on the support of \( A_j \), \( \{ x \in X : \mu_{A_j}(x) > 0 \} \), which maps \( A_j \) onto an open fuzzy set \( \varphi_j(A_j) \) in some fuzzy topological vector space \( E_j \) and for each \( l \) in the index set \( \varphi_j(A_j \cap A_l) \) is an open fuzzy set in \( E_j \).

3. The mapping \( \varphi_l \circ \varphi_j^{-1} \) which maps \( \varphi_j(A_j \cap A_l) \) onto \( \varphi_l(A_j \cap A_l) \) is a \( C^1 \) fuzzy diffeomorphism for each pair of indices \( j, l \).
Each pair \((A_j, \varphi_j)\) is called a **fuzzy chart** of the fuzzy atlas. If a point \(x \in X\) lies in the support of \(A_j\), then \((A_j, \varphi_j)\) is said to be a fuzzy chart at \(x\). An equivalence class of \(C^1\) fuzzy atlases on \(X\) is said to define a \(C^1\) fuzzy manifold on \(X\). In the sequel we refer simply to **fuzzy manifold**.

**Definition 1.4.21.** [52] Let \((B, T_1)\) and \((F, T_2)\) be the topological spaces and the projection \(P_2 : B \times F \to F\) be a continuous surjective function. Then for any point \(x \in F\), \(P_2^{-1}(x)\) is homeomorphic to \(F\) and is called the **fiber over** \(x\).

**Definition 1.4.22.** [70] Let \(X\) be a nonempty set and \(I = [0, 1]\) be the unit interval. Let \(\mu\) be a fuzzy subset of \(X\) such that \(\mu : X \to [0, 1]\) and \(M\) be any subset of \(X\). Then, the pair \((\mu, M)\) is called a **soft fuzzy set** of \(X\). The family of all soft fuzzy sets of \(X\), denoted by \(SF(X)\).

**Definition 1.4.23.** [70] Let \(x \in X\) and \(\lambda : X \to [0, 1]\). Define,

\[
x_{\lambda}(y) = \begin{cases} 
\lambda (0 < \lambda \leq 1), & \text{if } x = y \\
0, & \text{otherwise}
\end{cases}
\]

Then the soft fuzzy set \((x_{\lambda}, \{x\})\) is called a **soft fuzzy point** in \(SF(X)\), with support \(x\) and base value \(\lambda\).

**Definition 1.4.24.** [70] The **relation** \(\pm\) on \(SF(X)\) is given by \((\mu, M) \pm (\lambda, N) \Leftrightarrow \mu(x) \leq \lambda(x), \forall x \in X\) and \(M \subseteq N\).

**Proposition 1.4.1.** [70] If \(\{\mu_j, M_j\}_{j \in J} \in SF(X),\) then the family \(\{\mu_j, M_j\} : j \in J\) has a meet, that is, g.l.b in \((SF(X), \pm)\) denoted by \(H_{j \in J}(\mu_j, M_j)\) such
that \( H_{j \in J}(\mu_j, M_j) = (\mu, M) \) where \( \mu(x) = \land \mu_j(x) \), for \( j \in J \), \( \forall x \in X \) and \( M = \cap M_j \), for \( j \in J \)

**Proposition 1.4.2.** [70] If \( \{(\mu_j, M_j)_{j \in J} \in SF(X) \) , then the family \( \{(\mu_j, M_j) : j \in J \} \) has a join, that is, \( \text{l.u.b.} \) in \((SF(X), \pm)\) denoted by \( H_{j \in J}(\mu_j, M_j) \) such that \( H_{j \in J}(\mu_j, M_j) = (\mu, M) \) where \( \mu(x) = \lor \mu_j(x) \), for \( j \in J \), \( \forall x \in X \) and \( M = \cup M_j \), for \( j \in J \)

**Definition 1.4.25.** [70] Let \( X \) be a non-empty set. Then the complement of a soft fuzzy set \((\mu, M)\) is defined as \((\mu, M)' = (1 - \mu, X \setminus M)\), where \( X \setminus M \) is the complement of \( M \), \( M \subset X \).

**Proposition 1.4.3.** [70] Let \( f : X \rightarrow Y \) be a function. If \((\lambda, N)\) is a soft fuzzy set of \( Y \), then its pre-image under \( f \), denoted \( f^-(\lambda, N) \) is defined as, \( f^-(\lambda, N) = (\lambda \circ f, f^{-1}(N)) \) where \( f^{-1}(N) = \{x \in X : f(x) = y, \forall y \in N \} \).

**Note:** \( f^-(\lambda, N) = f^{-1}(\lambda, N) \)

**Proposition 1.4.4.** [70] Let \( f : X \rightarrow Y \) be a function. If \((\mu, M)\) is a soft fuzzy set of \( X \), then its image under \( f \), denoted \( f^-(\mu, M) \) is defined as, \( f^-(\mu, M) = (y, L) \) where, \( y(y) = f(\mu)(y) = \sup \{\mu(x) : x \in f^{-1}(y)\} \) and \( L = \{f(x) : x \in M \text{ and } y(f(x)) = \mu(x)\} \).

**Note:** \( f^-(\lambda, N) = f(\lambda, N) \)

**Definition 1.4.26.** [70] A soft fuzzy topology on a non-empty set \( X \) is a family \( t \) of soft fuzzy sets in \( X \) satisfying the following axioms:
(1) \((0, \varnothing), (1, X) \in \tau.\)

(2) For any family of soft fuzzy sets \((\lambda_j, N_j) \in \tau, j \in J, \Rightarrow H_{j \in J}(\lambda_j, N_j) \in \tau.\)

(3) For any finite number of soft fuzzy sets \((\lambda_j, N_j) \in \eta, j = 1, 2, 3, \ldots, n, \Rightarrow\)
\[H_{j=1}^n(\lambda_j, N_j) \in \tau.\]

Then the pair \((X, \tau)\) is called a **soft fuzzy topological space**. (in short, SFTS). The members of \(\tau\) is said to be a **soft fuzzy open set** (in short, SFOS) of \(X\). The complement of a soft fuzzy open set is a **soft fuzzy closed set** (in short, SFCS) in \(X\).

**Definition 1.4.27.** [70] Let \((X, \tau)\) be a soft fuzzy topological space and \((\lambda, N)\) be a soft fuzzy set. Then the **soft fuzzy interior** and **soft fuzzy closure** of \((\lambda, N)\) are defined by

\[
\overline{\overline{\lambda, N}} = H\{(\mu, M) : (\mu, M) \text{ is a soft fuzzy closed set and } (\lambda, N) = (\mu, M)\}
\]

\[
(\lambda, N)^o = H\{(\gamma, L) : (\gamma, L) \text{ is a soft fuzzy open set and } (\lambda, N) = (\gamma, L)\}
\]

Clearly, \(\overline{\overline{\lambda, N}} = (\lambda, N)\) if and only if \((\lambda, N)\) is a soft fuzzy closed set. and \((\lambda, N)^o = (\lambda, N)\) if and only if \((\lambda, N)\) is a soft fuzzy open set.

**Note 1.4.1.** \(\overline{\overline{\lambda, N}} = cl(\lambda, N)\) and \((\lambda, N)^o = int(\lambda, N)\).

**Definition 1.4.28.** [70] Let \(T\) be a soft fuzzy topology on \(X\). \(B \subseteq T\) is called a **base** for \(T\) if and only if given \((\mu, M) \in T\) and \((x_r, \{x_j\} \in (\mu, M), \text{ there exists } (v, N) \in B\) with \((x_r, \{x_j\}) \in (v, N) \pm (\mu, M)\).
Definition 1.4.29. [5] Let \((X, r)\) be a fuzzy topological space. A fuzzy set \(\lambda : X \rightarrow [0, 1]\) is said to be **fuzzy semi-open**, if \(\lambda \leq cl(int(\lambda))\). The complement of fuzzy semi-open is called a fuzzy semi-closed set.

Definition 1.4.30. [4] Let \((X, r)\) be a fuzzy topological space. A fuzzy set \(\lambda : X \rightarrow [0, 1]\) is said to be **fuzzy \(\beta\)-open**, if \(\lambda \leq cl(int(cl(\lambda)))\). The complement of fuzzy \(\beta\)-open is called a fuzzy \(\beta\)-closed set.

Remark 1.4.1. [5] Every fuzzy semi-open set is fuzzy \(\beta\)-open.

Definition 1.4.31. [71] Let \((X, r)\) be a fuzzy topological space. Let \(\lambda\) be any fuzzy set of \(X\). Then, the fuzzy set \(\lambda : X \rightarrow [0, 1]\) is said to be

(1) **fuzzy semi-regular set** if it is both fuzzy semi-open and fuzzy semi-closed.

(2) **fuzzy AB-set** if \(\lambda = \mu \wedge \gamma\)

where \(\mu\) is a fuzzy open set, \(\gamma\) is a fuzzy semi-regular set.

(3) **fuzzy t-set**, if \(int(\lambda) = int(cl(\lambda))\).

(4) **fuzzy \(a'\)-set**, if \(int(\lambda) = int(cl(int(\lambda)))\).

(5) **fuzzy B-set** if \(\lambda = \mu \wedge \gamma\)

where \(\mu\) is a fuzzy open set, \(\gamma\) is a fuzzy t-set.

(6) **fuzzy \(\beta\)-regular set** if it is both fuzzy \(\beta\)-open and fuzzy \(\beta\)-closed.
(7) weak fuzzy AB-set if \( \lambda = \mu \land \gamma \)

where \( \mu \) is a fuzzy open set, \( \gamma \) is a fuzzy \( \beta \) regular set.

(8) fuzzy C-set if \( \lambda = \mu \land \gamma \)

where \( \mu \) is a fuzzy open set, \( \gamma \) is a fuzzy \( \alpha^{*} \) set.

**Definition 1.4.32.** [3] Let \((X, \tau)\) be a fuzzy topological space. A fuzzy set \( \lambda : X \rightarrow [0, 1] \) is said to be a **fuzzy \( G_{\delta} \)-set**, if \( \lambda = \wedge_{i=1}^{\infty} \lambda_{i} \), where each \( \lambda_{i} \in \tau \).

**Remark 1.4.2.** [3] Every fuzzy open set is a fuzzy \( G_{\delta} \)-set.

**Definition 1.4.33.** [10] Let \((X, T)\) be a topological space and \( A \) be a subset of \( X \). Then the **kernel** of \( A \) is defined by \( \text{Ker}(A) = \cap \{U \in T \mid A \subset U\} \).

**Definition 1.4.34.** [22] A fuzzy topological space \((X, \tau)\) is said to be a **fuzzy connected space** if and only if the only fuzzy sets which are both fuzzy open and fuzzy closed are \( 0_{X} \) and \( 1_{X} \).

**Definition 1.4.35.** [70] Let \((X, \tau)\) be a soft fuzzy topological space and \( J \) be an indexed set. \((X, \tau)\) is said to be a **soft fuzzy compact space** whenever \( H_{j \in J}(\lambda_{j}, N_{j}) = (1, X) \), \( (\lambda_{j}, N_{j}) \in \tau, j \in J \), there is a finite subset \( F \) of \( J \) with \( H_{j \in F}(\lambda_{j}, N_{j}) = (1, X) \).

**Definition 1.4.36.** [70] A soft fuzzy topological space \((X, \tau)\) is said to be a **soft fuzzy normal space** if given \((\lambda, N) \in \tau^{'}, (\mu, M) \in \tau \) with \((\lambda, N) \pm (\mu, M) \) there exists a fuzzy open set \((\delta, L)\) satisfying \((\lambda, N) \pm (\delta, L) \pm (\delta, L) \pm (\mu, M) \).
Definition 1.4.37. [19] Let $X$ be any non-empty set. A filter, $F$ is a non-empty collection of subsets of $X$ provided:

1. $\varnothing \notin F$.
2. $M, N \in F \Rightarrow M \cap N \in F$.
3. $N \in F$ and $N \subseteq M \Rightarrow M \in F$.

The family of all filters on $X$ is denoted by $F(X)$.

Definition 1.4.38. [19] An ultrafilter on a non-empty set $X$ is a filter $F$ such that there is no filter on $X$ which is strictly finer than $F$. The family of all ultrafilters on $X$ is denoted by $U(X)$.

Definition 1.4.39. [35] Let $F(X)$ be the family of filters on $X$. A convergence space is a pair $(X, q)$, where $X$ is a set and $q : F(X) \to 2^X$ satisfies the following conditions:

(CS1) $x \in q(\hat{x})$, for each $x \in X$, where $\hat{x}$ is the ultrafilter containing $\{x\}$.

(CS2) $q(F) \subseteq q(G)$, when $F \subseteq G$.

(CS3) $q(F) = \cap \{q(G) : G$ is an ultrafilter containing $F\}$

Quite often $x \in q(F)$ is denoted by $F \xrightarrow{q} x$ and is interpreted as "$F$ $q$-converges to $x$".
A map \( f : (X, q) \to (Y, p) \) between two convergence spaces is said to be **continuous**, when \( F \xrightarrow{q} x \Rightarrow fF \xrightarrow{p} f(x) \) for each filter \( F \) on \( X \) and \( x \in X \).

The category whose objects consists of all convergence spaces and whose morphisms are all the continuous maps between objects is denoted by \( \text{CONV} \).

**Definition 1.4.40.** [36] A convergence space \( (X, q) \) is called **regular**, when \( F \xrightarrow{q} x \) implies \( \overline{F} \xrightarrow{q} x \), where \( \overline{F} \) is the filter with base \( \{ \overline{F} : F \in F \} \) and \( \overline{F} = \{ x \in X : G \xrightarrow{q} x \text{ and } F \in G, \text{ for some } G \in F(X) \} \).

**Definition 1.4.41.** [33] Let \( F_X(\lambda) = \{ \mu \in I_X : \mu \leq \lambda \} \). A non-empty collection \( F \subset F_X(\lambda) \) of fuzzy sets is called a **fuzzy filter** on \( \lambda \) if and only if

1. \( 0 \notin F \).
2. \( \delta, \gamma \in F \Rightarrow \delta \land \gamma \in F \).
3. \( \gamma \in F, \lambda > \mu > \gamma \Rightarrow \mu \in F \).

**Definition 1.4.42.** [33] A fuzzy filter \( F \) is called a **fuzzy prime filter** if and only if \( \delta \lor \gamma \in F \Rightarrow \delta \in F \text{ or } \gamma \in F \).

**Definition 1.4.43.** [33] A non-empty collection \( B \subseteq F_X(\lambda) \) is called a **fuzzy filter basis** on \( \lambda \) if and only if

1. \( 0 \notin B \).
2. \( \gamma_1, \gamma_2 \in B \Rightarrow \text{there exists } \gamma_3 \in B \text{ such that } \gamma_3 \leq \gamma_1 \land \gamma_2 \).
**Definition 1.4.44.** [45] A filter $F$ on $X$ and a fuzzy filter $F$ are said to be **compatible** if and only if for all $F \in F$ and $\mu \in F$, $\mu$ does not vanish everywhere on $F$.

**Definition 1.4.45.** [42] For any fuzzy filter $F$ on $X$, the characteristic value is defined as $c(F) = \inf_{\lambda \in F} \sup_{x \in X} \lambda(x)$.

**Definition 1.4.46.** [59] Let $F(X)$ and $P(X)$ be the family of fuzzy filters and fuzzy prime filters on $X$. Given a set $X$, the pair $(X, \text{lim})$ is called a fuzzy convergence space, where $\text{lim} : F(X) \to I^X$ provided:

(PST) For all $F \in F(X)$, $\text{lim} F = \inf_{G \in P_m(F)} \text{lim} G$.

(F1p) For all $F \in P(X)$, $\text{lim} F \leq c(F)$.

(F2p) For all $F, G \in P(X)$, when $F \subseteq G$, $\text{lim} F \geq \text{lim} G$.

(C1) For all $x \in X$, $0 < \alpha \leq 1$, $\text{lim} \alpha F \geq \alpha F$.

A map $f : (X, \text{lim}_X) \to (Y, \text{lim}_Y)$ between two fuzzy convergence spaces is said to be **continuous**, when $\text{lim}_X F(x) \geq \text{lim}_Y fF(f(x))$ for each $F \in F(X)$ and $x \in X$.

The category whose objects consists of all fuzzy convergence spaces and whose morphisms are all the continuous maps between objects is denoted by $\text{FCS}$. 

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Definition 1.4.47. [53] A fuzzy convergence space \((X, \text{lim})\) is said to be a fuzzy regular when \(\text{lim}F \geq \text{lim}F\) for each \(F \in F(X)\), where \(\bar{F} = \{ \text{cl}(\lambda) : \lambda \in F \}\) and \(\text{cl}(\lambda) = \sup\{ \text{lim}G : \lambda \in G, G\) is a fuzzy filter of \(X\}\).

Definition 1.4.48. [40] A fuzzy convergence space \((W, e)\) is said to be a fuzzy \(T_0\) space if either \(w\) fails to e-converge to \(t\) or \(t\) fails to e-converge to \(w\) for each fuzzy points \(w\) and \(t\) in \(W\) with \(x_w = x_t\).

Definition 1.4.49. [40] A fuzzy convergence space \((W, e)\) is said to be a fuzzy \(T_1\) space if \(w\) fails to e-converge to \(t\) for each fuzzy points \(w\) and \(t\) in \(W\) with \(x_w = x_t\).

Definition 1.4.50. [40] A fuzzy convergence space \((W, e)\) is said to be a fuzzy \(T_2\) space if fuzzy filter \(F\) e-converge to \(t\) and \(w\), then \(x_w = x_t\).

Definition 1.4.51. [2] The product \(\lambda \times \mu\) of a fuzzy set \(\lambda\) of \(X\) and a fuzzy set \(\mu\) of \(Y\) is a fuzzy set of \(X \times Y\), defined by \((\lambda \times \mu)(x, y) = \min(\lambda(x), \mu(y))\), for each \((x, y) \in X \times Y\).

Definition 1.4.52. [2] Let \((X_1, \tau_1)\) and \((X_2, \tau_2)\) be the fuzzy topological spaces. The fuzzy product topological space of \(X_1\) and \(X_2\) is the cartesian product \(X_1 \times X_2\) of sets \(X_1\) and \(X_2\) together with the fuzzy topology \(\tau_1 \times \tau_2\) generated by the family \(\{ P_1^{-1}(\lambda_1), P_2^{-1}(\lambda_2) : \lambda_1 \in \tau_1, \lambda_2 \in \tau_2 \}\), where \(P_1\) and \(P_2\) are the projections of \(X_1 \times X_2\) onto \(X_1\) and \(X_2\) respectively.

Definition 1.4.53. [2] Let \((X_1, \tau_1)\) and \((X_2, \tau_2)\) be the fuzzy topological spaces.
Let $\lambda_1 \times \lambda_2$ be the fuzzy set of $X_1 \times X_2$. Then the collection $B = \{\lambda_1 \times \lambda_2 : \lambda_1 \in \tau_1, \lambda_2 \in \tau_2\}$ forms a base for the fuzzy product topology $\tau_1 \times \tau_2$ on $X_1 \times X_2$.

**Definition 1.4.54.** [2] The product $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ of mappings $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$, defined by $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$, for each $(x_1, x_2) \in X_1 \times X_2$.

**Definition 1.4.55.** [8] A fuzzy set $\lambda \in l^X$ is said to be a precompact set (resp. weakly precompact set) if $\lambda \in F$ implies that $\text{sup}_{x \in X} \text{lim}(F(x) = c(F))$ (resp. $\text{sup}_{x \in X} \text{lim}_{\omega \in \tau} F(x) = 1$), for each fuzzy prime filter $F$. Further, fuzzy convergence space $(X, \text{lim})$ is called locally precompact (resp. locally weakly precompact) provided that each fuzzy prime filter $F$ contains a precompact (resp. weakly precompact) element when $\text{lim}F = 1_{\varphi}$.

**Definition 1.4.56.** [51] A fuzzy topological space $(X, \tau)$ is weakly induced by a topological space $(X, T)$ if :

1. $T = [\tau]$ where $[\tau] = \{A \subseteq X : \chi_A \in \tau\}$

2. Every $\mu \in \tau$ is a lower semicontinuous function from $(X, T)$ into $[0, 1]$.

**Definition 1.4.57.** [47] Let $\text{Top}$ be the category of all the topological spaces and the continuous functions. Let $\text{CFT}$ be the category of all the fuzzy topological spaces in the Chang’s sense and the $F$-continuous functions. We will denote $\omega : \text{Top} \to \text{CFT}$ to be the functor which associates to any topological space $(X, T)$, the fuzzy topological space $(X, \omega(T))$, where $\omega(T)$
is the totality of all lower semicontinuous functions of \((X, T)\) to the unit interval. It is called the weakly induced fuzzy topological space by \((X, T)\).

**Definition 1.4.58.** [47] Let \(X\) and \(Y\) be any two non-empty sets and \(f : X \to Y\) be a function. We denote \(\tilde{f}\) the associate map given by \(\tilde{f}(x_\lambda) = f(x_\lambda)\) for each fuzzy point \(x_\lambda\) in \(X\).

**Definition 1.4.59.** [17] A non-empty family \(\mu\) of subsets of a set \(X\) is said to be a **generalized topology** if \(\varnothing \in \mu\) and an arbitrary union of elements of \(\mu\) is again in \(\mu\). The pair \((X, \mu)\) is called a **generalized topological space** and the elements of \(\mu\) are called \(\mu\)-open sets. If \(X \in \mu\), then the pair \((X, \mu)\) is called a **strong generalized topological space**.

**Definition 1.4.60.** [57] A non-empty family \(G\) of fuzzy sets of \(X\) is said to be a **generalized fuzzy topology** if

1. \(0 \in G\).
2. For \(i \in J, \lambda_i \in G \Rightarrow \bigvee_{i \in J} \lambda_i \in G\).

Then the pair \((X, G)\) is called a **generalized fuzzy topological space**.