CHAPTER - 7

COMPACTIFICATION OF SOFT FUZZY PRODUCT \( \mathcal{C} \)-SPACES

The concept of fuzzy product topological spaces was introduced by K.K.Azad [2]. The notion of strong generalized topological spaces was defined by A.Csaszar[17]. G.Palanichetty [57] introduced and developed the notion of fuzzy generalized topological spaces. In this chapter, the concepts of the soft fuzzy product \( \mathcal{C} \)-structure is introduced. Some interesting properties of the associated product functions on soft fuzzy product \( \mathcal{C} \)-spaces are studied. Further, compactification of the soft fuzzy product \( \mathcal{C} \)-space is established.

7.1 PROPERTIES OF THE ASSOCIATED PRODUCT FUNCTIONS

In this section, a new structure called soft fuzzy product \( \mathcal{C} \)-structure is introduced. An associated product function is introduced and some of its properties are studied.

**Definition 7.1.1.** Let \((X_1 \times X_2, \tau_1 \times \tau_2)\) be a soft fuzzy product topological space. A **soft fuzzy product \( \mathcal{C} \)-structure** on a non-empty set \(X_1 \times X_2\) is a
family \( \text{st}(r_1 \times r_2) \) of soft fuzzy \( C \)-open sets in \( (X_1 \times X_2, r_1 \times r_2) \) satisfying the following axioms:

1. \((0, \varphi), (1, X) \in \text{st}(r_1 \times r_2)\).

2. For any finite number of soft fuzzy \( C \)-open sets \( (\lambda_j, N_j) \in \text{st}(\tau_1 \times \tau_2), j = 1, 2, 3, ... n \), \( H_n(\lambda_j, N_j) \in \text{st}(\tau_1 \times \tau_2) \).

Then, the pair \((X_1 \times X_2, \text{st}(r_1 \times r_2))\) is called a \textbf{soft fuzzy product} \( C \)-space.

Definition 7.1.2. Let \( X = (X_1 \times X_2, \text{st}(r_1 \times r_2)) \) and \( Y = (Y_1 \times Y_2, \text{st}(\sigma_1 \times \sigma_2)) \) be any two soft fuzzy product \( C \)-spaces. A function \( f : X \to Y \) is said to be a \textbf{soft fuzzy} \( C^I \)-\textbf{continuous product function}, if the inverse image of every soft fuzzy \( C \)-open set in \( (Y_1 \times Y_2, \text{st}(\sigma_1 \times \sigma_2)) \) is a soft fuzzy \( C \)-open set in \( (X_1 \times X_2, \text{st}(r_1 \times r_2)) \).

Definition 7.1.3. Let \( (X_1 \times X_2, T_1 \times T_2) \) be a product topological space and \( I = [0, 1] \) be equipped with the usual topology. A pair \( (\mu, M) \) is said to be a \textbf{lower semi} \( C^I \)-\textbf{set}, where \( \mu : X_1 \times X_2 \to I \) with a \( C \)-set \( \mu^{-1}(\alpha, 1) \) in \( (X_1 \times X_2, T_1 \times T_2) \) and \( M \subseteq X_1 \times X_2 \) is also a \( C \)-set in \( (X_1 \times X_2, T_1 \times T_2) \), for all \( \alpha \in [0, 1] \).
**Definition 7.1.4.** A soft fuzzy product \( C \)-space \((X_1 \times X_2, \text{st}(r_1 \times r_2))\) is said to be a **weakly induced soft fuzzy product \( C \)-space**, which is the soft fuzzy product \( C \)-space induced by a topological space \((X_1 \times X_2, T_1 \times T_2)\) if the following conditions hold:

1. \( T_1 \times T_2 = \{ A \subset X_1 \times X_2 : A \text{ is a } C \text{-set of } X_1 \times X_2 \text{ and } (\chi_A, A) \in \text{st}(r_1 \times r_2) \} \)
2. Every \((\mu, M) \in \text{st}(r_1 \times r_2)\) is a lower semi \( C^\# \)-set.

**Definition 7.1.5.** Let \( PrTop \) be the category of all the product topological spaces and the continuous product functions. Let \( SFPPrCst \) be the category of all the soft fuzzy product \( C \)-space and the soft fuzzy \( C^\# \)-continuous product functions. Define a **functor**, \( \omega : PrTop \rightarrow SFPPrCst \) which associates any product topological space \((X_1 \times X_2, T_1 \times T_2)\) to the soft fuzzy product \( C \)-space \((X_1 \times X_2, \omega(T_1 \times T_2))\), where \( \omega(T_1 \times T_2) \) is the totality of all lower semi \( C^\# \)-set. Then \((X_1 \times X_2, \omega(T_1 \times T_2))\) is called the weakly induced soft fuzzy product \( C \)-space.

**Proposition 7.1.1.** Let \( X_i, Y_i \) be the non-empty sets and \( f_i : X_i \rightarrow Y_i \) be the functions, for \( (i = 1, 2) \) respectively. Let \((\lambda_i, N_i)\), \((\mu_i, M_i)\) be the soft fuzzy sets of \( Y_i \) and \( X_i \), for \( (i = 1, 2) \). Then the following are valid:

1. \((f_1 \times f_2)^{-1}(\lambda_1 \times \lambda_2, N_1 \times N_2) = f_1^{-1}(\lambda_1, N_1) \times f_2^{-1}(\lambda_2, N_2)\).
2. \((f_1 \times f_2)(\mu_1 \times \mu_2, m_1 \times M_2) = f_1(\mu_1, M_1) \times f_2(\mu_2, M_2)\).
**Proof.** Proof is clear. \( \square \)

**Definition 7.1.6.** Let \( X_1 \times X_2 \) and \( Y_1 \times Y_2 \) be two non-empty product sets. Let \( f_1 : X_1 \to Y_1 \) and \( f_2 : X_2 \to Y_2 \) be any two functions. Let \( f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2 \) be a product function. Then the associated product function, \( \overrightarrow{f_1 \times f_2} \) is defined as \( \overrightarrow{f_1 \times f_2}((x_1, x_2)_\lambda, \{(x_1, x_2)_\lambda\}) = (f_1 \times f_2)((x_1, x_2)_\lambda, \{(x_1, x_2)_\lambda\}) \), for each soft fuzzy point \( ((x_1, x_2)_\lambda, \{(x_1, x_2)_\lambda\}) \) of \( X_1 \times X_2 \), \( 0 < \lambda \leq 1 \).

**Proposition 7.1.2.** Let \( X_1 \times X_2 \) and \( Y_1 \times Y_2 \) be two non-empty product sets. Let \( f_1 : X_1 \to Y_1 \) and \( f_2 : X_2 \to Y_2 \) be any two onto functions. If \( f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2 \) is an onto product function, then for each soft fuzzy point \( ((x_1, x_2)_\lambda, \{(x_1, x_2)_\lambda\}) \) of \( X_1 \times X_2 \), \( \overrightarrow{f_1 \times f_2}((x_1, x_2)_\lambda, \{(x_1, x_2)_\lambda\}) \) is the soft fuzzy point of \( Y_1 \times Y_2 \) that takes the value \( \lambda \) in \( (f_1 \times f_2)((x_1, x_2)_\lambda) \).

**Proof.** For \( 0 < \lambda \leq 1 \),

\[
\overrightarrow{f_1 \times f_2}((x_1, x_2)_\lambda, \{(x_1, x_2)_\lambda\}) = (f_1 \times f_2)((x_1, x_2)_\lambda, \{(x_1, x_2)_\lambda\})
\]

\[
= ((f_1 \times f_2)((x_1, x_2)_\lambda), (f_1 \times f_2)((x_1, x_2)_\lambda))
\]

\[
= ((f_1 \times f_2)(x_1, x_2)_\lambda, (f_1(x_1), f_2(x_2))_\lambda)
\]

\[
= ((f_1(x_1), f_2(x_2))_\lambda, (f_1(x_1), f_2(x_2))_\lambda)
\]

Thus \( \overrightarrow{f_1 \times f_2}((x_1, x_2)_\lambda, \{(x_1, x_2)_\lambda\}) \) is the soft fuzzy point of \( Y_1 \times Y_2 \) that takes the value \( \lambda \) in \( (f_1(x_1), f_2(x_2)) = (f_1 \times f_2)((x_1, x_2)_\lambda) \). \( \square \)

**Proposition 7.1.3.** Let \( X_1 \times X_2 \), \( Y_1 \times Y_2 \) and \( Z_1 \times Z_2 \) be the non-empty product sets. Let \( f_1 : X_1 \to Y_1 \), \( f_2 : X_2 \to Y_2 \), \( g_1 : Y_1 \to Z_1 \) and \( g_2 : Y_2 \to Z_2 \) be the
functions. Let \( f^1 = f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2 \) and \( g^1 = g_1 \times g_2 : Y_1 \times Y_2 \to Z_1 \times Z_2 \) be any two onto product functions. Then \( g^1 \circ f^1 = (g_1 \times g_2) \circ (f_1 \times f_2). \)

**Proof.** By using Proposition 7.1.2., for each soft fuzzy point \((x_1, x_2)\), \((x_1, x_2)\) of \( X_1 \times X_2 \) and \( 0 < \lambda \leq 1 \), we have
\[
\overline{g^1 \circ f^1}(x_1, x_2) = (g_1 \times g_2)(f_1 \times f_2)(x_1, x_2) \Rightarrow (x_1, x_2) = (g_1 \times g_2)(f_1 \times f_2)(x_1, x_2).
\]

Hence, \( g^1 \circ f^1 = (g_1 \times g_2) \circ (f_1 \times f_2). \)

**Proposition 7.1.4.** Let \( X_1 \times X_2 \) and \( Y_1 \times Y_2 \) be any two non-empty product sets. Let \( f_1 : X_1 \to Y_1 \) and \( f_2 : X_2 \to Y_2 \) be any two onto functions. Let \( f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2 \) be an onto product function. Then \( f_1 \times f_2 \) is an identity function if and only if \( f_1 \times f_2 \) is also an identity function.

**Proof.** Since \( f_1 \times f_2 \) is an identity function, \( f_1 \times f_2 \) \((x_1, x_2), \{(x_1, x_2)\}\) = \((x_1, x_2), \{(x_1, x_2)\}\), for each soft fuzzy point \((x_1, x_2), \{(x_1, x_2)\}\) of \( X_1 \times X_2 \) and \( 0 < \lambda \leq 1 \). Now, \((f_1 \times f_2)(x_1, x_2), \{(f_1 \times f_2)(x_1, x_2)\}\) = \((f_1 \times f_2)(x_1, x_2), \{(x_1, x_2)\}\), \((f_1 \times f_2)(x_1, x_2), \{(x_1, x_2)\}\) = \((f_1 \times f_2)(x_1, x_2), \{(x_1, x_2)\}\), \((f_1 \times f_2)(x_1, x_2), \{(x_1, x_2)\}\), for each soft fuzzy point \((x_1, x_2), \{(x_1, x_2)\}\) of \( X_1 \times X_2 \).

Therefore, \((f_1 \times f_2)(x_1, x_2) = (x_1, x_2)\). Thus \( f_1 \times f_2 \) is the identity function.
Similarly if \( f_1 \times f_2 \) is an identity function, then \( \overline{f_1 \times f_2} (\{(x_1, x_2)\}, \{(x_1, x_2)\}) = (f_1 \times f_2) (\{(x_1, x_2)\}, \{(x_1, x_2)\}) = ((f_1 \times f_2)(\{(x_1, x_2)\}), (f_1 \times f_2)(\{(x_1, x_2)\})) = (0, 0) \). Therefore \( \overline{f_1 \times f_2} \) is the identity function. Hence the proof. \( \square \)

**Proposition 7.1.5.** Let \( X_1 \times X_2 \) and \( Y_1 \times Y_2 \) be any two non-empty product sets. Let \( f_1 : X_1 \to Y_1 \) and \( f_2 : X_2 \to Y_2 \) be any two functions.

1. If \( f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2 \) is a surjective product function, then \( \overline{f_1 \times f_2} \) is also a surjective product function.

2. If \( f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2 \) is a one-to-one product function, then \( \overline{f_1 \times f_2} \) is also a one-to-one product function.

**Proof.** (1) For each soft fuzzy point \( ((y_1, y_2)_{\alpha}, \{(y_1, y_2)\}) \) of \( Y_1 \times Y_2 \), we have \( (y_1, y_2) \in Y_1 \times Y_2 \). Then there exists atleast \( (x_1, x_2) \in X_1 \times X_2 \) such that \( (f_1 \times f_2)((x_1, x_2)) = (y_1, y_2) \). Now for \( 0 < \alpha \leq 1 \), \( \overline{f_1 \times f_2}((x_1, x_2)_{\alpha}, \{(x_1, x_2)\}) = (f_1 \times f_2)(\{(x_1, x_2)\}) \). Therefore, \( \overline{f_1 \times f_2} (\{(x_1, x_2)\}, \{(x_1, x_2)\}) = ((y_1, y_2)_{\alpha}, \{(y_1, y_2)\}). Hence \( \overline{f_1 \times f_2} \) is a surjective product function.

(2) Let \( ((x_1, x_2)_{\alpha}, \{(x_1, x_2)\}), ((x_1', x_2')_{\beta}, \{(x_1', x_2')\}) \) be the two soft fuzzy points of \( X_1 \times X_2 \) such that \( \overline{f_1 \times f_2} (\{(x_1, x_2)\}) = \overline{f_1 \times f_2} (\{(x_1', x_2')\}) \), where \( 0 < \alpha, \beta \leq 1 \). Now \( (((f_1 \times f_2)(x_1, x_2))_{\alpha}, \{(f_1 \times f_2)(x_1, x_2)\}) = (f_1 \times f_2)(\{(x_1, x_2)\}), \{(x_1, x_2)\}) = (f_1 \times f_2)(\{(x_1', x_2')\}) \). This shows that \( (f_1 \times f_2)(\{(x_1, x_2)\}) = (f_1 \times f_2)(\{(x_1', x_2')\}) \) and \( \alpha = \beta \). Since \( (f_1 \times f_2) \)
is a one-to-one function, we have \((x_1, x_2) = (x'_1, x'_2)\) and \(\alpha = \beta\). Therefore, \((x_1, x_2), \{(x_1, x_2)\} = (\tilde{x}'_1, \tilde{x}'_2), \{(\tilde{x}'_1, \tilde{x}'_2)\}\). Hence \(\tilde{f}_1 \times \tilde{f}_2\) is one-to-one. \(\square\)

**Proposition 7.1.6.** Let \(X_1 \times X_2\) and \(Y_1 \times Y_2\) be any two non-empty product sets. Let \(f_1 : X_1 \to Y_1\) and \(f_2 : X_2 \to Y_2\) be any two functions. If \(f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2\) is a one-to-one product function, then \((f_1 \times f_2)^{-1} = (\tilde{f}_1 \times \tilde{f}_2)^{-1}\).

**Proof.** For each soft fuzzy point \((y_1, y_2), \{(y_1, y_2)\}\) of \(Y_1 \times Y_2\), \(0 < \lambda \leq 1\) and by hypothesis, there exists a unique \((x_1, x_2) \in X_1 \times X_2\) such that \(\tilde{f}_1 \times \tilde{f}_2((x_1, x_2), \{(x_1, x_2)\}) = (y_1, y_2)\). That is, \((x_1, x_2) = (\tilde{f}_1 \times \tilde{f}_2)^{-1}((y_1, y_2))\).

Now, we have to prove that \((f_1 \times f_2)^{-1}((y_1, y_2), \{(y_1, y_2)\}) = ((x_1, x_2), \{(x_1, x_2)\}, \alpha = \lambda\). Now, \(\tilde{f}_1 \times \tilde{f}_2((x_1, x_2), \{(x_1, x_2)\}) = (\tilde{f}_1 \times \tilde{f}_2)^{-1}((y_1, y_2), \{(y_1, y_2)\}) = (\tilde{f}_1 \times \tilde{f}_2)((x_1, x_2), \{(x_1, x_2)\}).\) Since \(\tilde{f}_1 \times \tilde{f}_2\) is a one-to-one function, \(\tilde{f}_1 \times \tilde{f}_2\) is one-to-one. Therefore, \(\tilde{f}_1 \times \tilde{f}_2((x_1, x_2), \{(x_1, x_2)\}) = (\tilde{f}_1 \times \tilde{f}_2)^{-1}((y_1, y_2), \{(y_1, y_2)\}) = (\tilde{f}_1 \times \tilde{f}_2)^{-1}((y_1, y_2), \{(y_1, y_2)\}).\) This is a contradiction. Therefore, \(\tilde{f}_1 \times \tilde{f}_2((x_1, x_2), \{(x_1, x_2)\}) = (\tilde{f}_1 \times \tilde{f}_2)^{-1}((y_1, y_2), \{(y_1, y_2)\}) = (\tilde{f}_1 \times \tilde{f}_2)^{-1}((y_1, y_2), \{(y_1, y_2)\}).\)

Next, \((\tilde{f}_1 \times \tilde{f}_2)^{-1}((y_1, y_2), \{(y_1, y_2)\}) = (f_1 \times f_2)^{-1}((y_1, y_2), \{(y_1, y_2)\}) = (f_1 \times f_2)^{-1}((y_1, y_2), \{(y_1, y_2)\}).\) Hence, \((f_1 \times f_2)^{-1} = (\tilde{f}_1 \times \tilde{f}_2)^{-1} = ((f_1 \times f_2)^{-1})^{-1}.\) \(\square\)

**Proposition 7.1.7.** Let \(X_1 \times X_2\) and \(Y_1 \times Y_2\) be any two non-empty product sets. Let \(f_1 : X_1 \to Y_1\) and \(f_2 : X_2 \to Y_2\) be any two functions. Let \(f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2\) be a product function.
(1) If $f_1 \times f_2$ is onto, then $f_1 \times f_2$ is also onto.

(2) If $f_1 \times f_2$ is one-to-one, then $f_1 \times f_2$ is also one-to-one.

**Proof.** (1) For each $(y_1, y_2) \in Y_1 \times Y_2$, $(y_1, y_2)_\beta = (y_1, y_2)_\beta$ is the soft fuzzy point of $Y_1 \times Y_2$ and $0 < \beta \leq 1$. By hypothesis there exist at least $((x_1, x_2)_\alpha , (x_1, x_2)_\beta)$ such that $f_1 \times f_2 ((x_1, x_2)_\alpha , (x_1, x_2)_\beta) = (y_1, y_2)_\beta$. Then $(f_1 \times f_2)((x_1, x_2)_\alpha , (x_1, x_2)_\beta) = (y_1, y_2)_\beta$ and $(f_1 \times f_2)^{-1}(y_1, y_2)_\beta = \emptyset$. Hence $f_1 \times f_2$ is an onto function.

(2) Let $(x_1, x_2), (x'_1, x'_2) \in X_1 \times X_2$ with $(f_1 \times f_2)((x_1, x_2)) = (f_1 \times f_2)((x'_1, x'_2))$. Now, $f_1 \times f_2((x_1, x_2)_\alpha , (x_1, x_2)_\beta) = (f_1 \times f_2)((x_1, x_2)_\alpha , (x_1, x_2)_\beta) = (((f_1 \times f_2)(x_1, x_2))_\alpha , (f_1 \times f_2)(x_1, x_2))_\beta) = (((f_1 \times f_2)(x'_1, x'_2))_\alpha , (f_1 \times f_2)(x'_1, x'_2))_\beta) = (x'_1, x'_2)$. Since $f'_1 \times f'_2$ is a one-to-one function, $(x'_1, x'_2)_\alpha , (x'_1, x'_2)_\beta) = (x_1, x_2)_\alpha , (x_1, x_2)_\beta)$. Therefore $(x_1, x_2) = (x'_1, x'_2)$. Thus $f_1 \times f_2$ is a one-to-one function.

**Definition 7.1.7.** Let $(X_1 \times X_2, T_1 \times T_2)$ and $(Y_1 \times Y_2, S_1 \times S_2)$ be any two product topological spaces. A function $f : X_1 \times X_2 \to Y_1 \times Y_2$ is said to be a **C-irresolute product function**, if the inverse image of every C-set in $(Y_1 \times Y_2, S_1 \times S_2)$ is a C-set in $(X_1 \times X_2, T_1 \times T_2)$.

**Proposition 7.1.8.** Let $(X_1 \times X_2, T_1 \times T_2)$ and $(Y_1 \times Y_2, S_1 \times S_2)$ be any two product topological spaces. Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be any two
functions. Then \( f_1 \times f_2 : (X_1 \times X_2, T_1 \times T_2) \rightarrow (Y_1 \times Y_2, S_1 \times S_2) \) is a C-irresolute product function if and only if \( \overline{f_1 \times f_2} : (X_1 \times X_2, \omega(T_1 \times T_2)) \rightarrow (Y_1 \times Y_2, \omega(S_1 \times S_2)) \) is a soft fuzzy \( \mathcal{C}^q \)-continuous product function.

**Proof.** Assume that \( f_1 \times f_2 \) is a C-irresolute product function. For each soft fuzzy \( \mathcal{C} \)-open set \( (\mu, M) \) in \( (Y_1 \times Y_2, \omega(S_1 \times S_2)) \), we have \( \mu^{-1}(\alpha, 1) \) is a C-set in \( (Y_1 \times Y_2, S_1 \times S_2) \), for all \( \alpha \in [0, 1] \) and \( M \subseteq X_1 \times X_2 \) is a C-set in \( (Y_1 \times Y_2, S_1 \times S_2) \). By hypothesis \( (f_1 \times f_2)^{-1}(\mu^{-1}(\alpha, 1)) \) is a C-set in \( (X_1 \times X_2, T_1 \times T_2) \) and also \( (f_1 \times f_2)^{-1}(M) \subseteq X_1 \times X_2 \) is a C-set in \( (X_1 \times X_2, T_1 \times T_2) \). Therefore, \( ((\mu \circ (f_1 \times f_2)), (f_1 \times f_2)^{-1}(M)) \) is a soft fuzzy \( \mathcal{C} \)-open set in \( (X_1 \times X_2, \omega(T_1 \times T_2)) \).

Now it is clear that,

\[
\overline{f_1 \times f_2}^{-1}(\mu, M) = (f_1 \times f_2)^{-1}(\mu, M) = ((f_1 \times f_2)^{-1}(\mu), (f_1 \times f_2)^{-1}(M)) = ((\mu \circ (f_1 \times f_2)), (f_1 \times f_2)^{-1}(M))
\]

Thus, \( \overline{f_1 \times f_2}^{-1}(\mu, M) \) is a soft fuzzy \( \mathcal{C} \)-open set in \( (X_1 \times X_2, \omega(T_1 \times T_2)) \). Hence \( \overline{f_1 \times f_2} \) is a soft fuzzy \( \mathcal{C}^q \)-continuous product function.

Conversely assume that \( \overline{f_1 \times f_2} \) is a soft fuzzy \( \mathcal{C}^q \)-continuous product function. Let \( A \) be a C-set in \( (Y_1 \times Y_2, S_1 \times S_2) \). Then \( (\chi_A, A) \) is a soft fuzzy \( \mathcal{C} \)-open set in \( (Y_1 \times Y_2, \omega(S_1 \times S_2)) \). Now \( \overline{f_1 \times f_2}^{-1}(\chi_A, A) = (f_1 \times f_2)^{-1}(\chi_A, A) = (\chi_{(f_1 \times f_2)^{-1}(A)}), (f_1 \times f_2)^{-1}(A)) \). That is, \( \overline{f_1 \times f_2}^{-1}(\chi_A, A) \) is a soft fuzzy \( \mathcal{C} \)-open set in \( (X_1 \times X_2, \omega(T_1 \times T_2)) \). Therefore, \( (f_1 \times f_2)^{-1}(A) \) is a C-set in \( T_1 \times T_2 \).
Hence \( f_1 \times f_2 \) is a C-irresolute product function. \( \square \)

**Definition 7.1.8.** Let \((X_1 \times X_2, T_1 \times T_2)\) and \((Y_1 \times Y_2, S_1 \times S_2)\) be any two product topological spaces. Let \((X_1 \times X_2, \omega(T_1 \times T_2))\) and \((Y_1 \times Y_2, \omega(S_1 \times S_2))\) be any two soft fuzzy product \(C\)-spaces and \(f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2\) be a product function. Then, \(f_1 \times f_2\) is said to be a **soft fuzzy \(C\)-homeomorphism**, if the following conditions are satisfied:

1. \(f_1 \times f_2\) is a soft fuzzy \(C\)-continuous product function.
2. \(f_1 \times f_2\) is a soft fuzzy bijective product function.
3. \((f_1 \times f_2)^{-1}\) is also a soft fuzzy \(C\)-continuous product function.

**Proposition 7.1.9.** Let \((X_1 \times X_2, T_1 \times T_2)\) and \((Y_1 \times Y_2, S_1 \times S_2)\) be any two product topological spaces. Let \((X_1 \times X_2, \omega(T_1 \times T_2))\) and \((Y_1 \times Y_2, \omega(S_1 \times S_2))\) be any two weakly induced soft fuzzy product \(C\)-spaces and \(\tilde{f}_1 \times \tilde{f}_2\) be a soft fuzzy \(C\)-continuous product function from \((X_1 \times X_2, \omega(T_1 \times T_2))\) onto \((Y_1 \times Y_2, \omega(S_1 \times S_2))\). Let \((X_1 \times X_2)/R\) be the quotient set on \(X_1 \times X_2\) with an equivalence relation \(R\). If there exists a soft fuzzy \(C\)-continuous product function \(\overline{g_1 \times g_2}\) from \((Y_1 \times Y_2, \omega(S_1 \times S_2))\) to \((X_1 \times X_2, \omega(T_1 \times T_2))\) such that \(\tilde{f}_1 \times \tilde{f}_2 \circ \overline{g_1 \times g_2} = id_{Y_1 \times Y_2}\), where \(id_{Y_1 \times Y_2}\) is the identity function, then \((Y_1 \times Y_2, \omega(S_1 \times S_2))\) is soft fuzzy \(C\)-homeomorphic with \((X_1 \times X_2)/R\).

**Proof.** Since \((\tilde{f}_1 \times \tilde{f}_2) \circ (\overline{g_1 \times g_2}) = id_{Y_1 \times Y_2}\), by Definition 7.1.7, we have \((f_1 \times f_2) \circ (g_1 \times g_2) = id_{Y_1 \times Y_2}\). Since \(\tilde{f}_1 \times \tilde{f}_2\) and \(\overline{g_1 \times g_2}\) is a soft fuzzy \(C\)-
continuous product functions, by Proposition 7.1.9., \( f_1 \times f_2 \) and \( g_1 \times g_2 = (f_1 \times f_2)^{-1} \) are the C-irresolute product functions. Then the function \( h_1 \times h_2 : (X_1 \times X_2) / R \to Y_1 \times Y_2 \) induced by \( f_1 \times f_2 \) is a C-irresolute product function and \( (h_1 \times h_2)^{-1} \) is also a C-irresolute product function. Now by Proposition 7.1.7., \( g_1 \times g_2 = (f_1 \times f_2)^{-1} = (f_1 \times f_2)^{-1} \). Thus by Proposition 7.1.6. and Proposition 7.1.9., \( h_1 \times h_2 \) is a soft fuzzy \( C^a \)-continuous bijective product function. Hence \( h_1 \times h_2 \) is a soft fuzzy \( C^a \)-homeomorphism. \( \square \)

### 7.2 COMPACTIFICATION OF SOFT FUZZY PRODUCT \( C \)-SPACES

In this section, compactification of soft fuzzy product \( C \)-spaces through the soft fuzzy product strong generalized topological space is established.

**Definition 7.2.1.** Let \( X_1 \times X_2 \) be a product set. Let \( (X_1 \times X_2) / R \) be a quotient set of \( X_1 \times X_2 \) with an equivalence relation \( R \). Then the collection of all quotient sets of \( X_1 \times X_2 \) is denoted by \( Q(X_1 \times X_2) \).

**Note 7.2.1.** Let \( R_{(x_1, x_2)} \) be an equivalence relation. Then, \( (X_1 \times X_2) / R_{(x_1, x_2)} = \{(x_1, x_2), [y_1, y_2] : (x_1, x_2) \sim (z_1, z_2), (y_1, y_2) \sim (z_1, z_2), \forall (z_1, z_2) \in X_1 \times X_2\} \) is also a quotient set of \( X_1 \times X_2 \).

**Definition 7.2.2.** Let \( (X_1 \times X_2, st(t_1 \times t_2)) \) be a non-compact soft fuzzy product \( C \)-space. Associated with each soft fuzzy product \( C \)-open set \( (\mu, M) \in st(t_1 \times t_2) \), \((\mu, M)^* = (\mu^*, M^*) \) is a soft fuzzy set of \( Q(X_1 \times X_2) \). For each \( (X_1 \times X_2) / \)}
Proposition 7.2.1. Under Definition 7.2.3., the following identities hold:

1. \((0, \varphi)\) = \((0, \varphi)\).
2. \((1_{X_1 \times X_2}, X_1 \times X_2)\) = \((1_{Q(X_1 \times X_2)}, Q(X_1 \times X_2))\).

Proof. Proof is simple. \(\square\)

Definition 7.2.3. A soft fuzzy strong generalized product topology on a non-empty product set \(X\) is a family \(G\) of soft fuzzy product sets of \(X\) satisfying the following axioms:

1. \((0, \varphi), (1, X) \in G\).
2. For any family of soft fuzzy sets \((A_j, N_j) \in G, j \in J\) \(\Rightarrow H_j \in G\) if \((A_j, N_j) \in G\).

Then the pair \((X, G)\) is called a soft fuzzy strong generalized product topological space (in short, SFsGPTS). Any soft fuzzy product set in \(G\) is said to be a soft fuzzy strong generalized open set (in short, SFsGOS) of \(X\). The
complement of soft fuzzy strong generalized open set is **soft fuzzy strong generalized closed**, denoted **SFsGCS**.

**Definition 7.2.4.** Let \((X_1 \times X_2, G)\) be a soft fuzzy strong generalized product topological space and \((\lambda, N)\) be a soft fuzzy product set. Then the **soft fuzzy strong generalized interior** and **soft fuzzy strong generalized closure** of \((\lambda, N)\) are defined by \(SFsGcl(\lambda, N) = H(\mu, M) : (\mu, M)\) is a soft fuzzy strong generalized closed set and \((\lambda, N) \pm (\mu, M)\) and \(SFsGint(\lambda, N) = H(\gamma, L) : (\gamma, L)\) is a soft fuzzy strong generalized open set and \((\lambda, N) \pm (\gamma, L)\).

**Definition 7.2.5.** Let \(G\) be a soft fuzzy strong generalized product topology on \(X_1 \times X_2\). Then the subfamily \(B \subseteq G\) of \(X_1 \times X_2\) is called a **base** for \(G\) if and only if given \((\mu, M) \in G\) and \((x_r, \{x_j\}) \in (\mu, M)\), there exists \((v, N) \in B\) with \((x_r, \{x_j\}) \in (v, N) \pm (\mu, M)\).

**Proposition 7.2.2.** Under the Definition 7.2.2. the collection \(b^* = \{H(\mu, M) : (\mu, M) \in st(\tau_1 \times \tau_2)\}\) is a base for some soft fuzzy product strong generalized topology on \(Q(X_1 \times X_2)\).

**Proof.** For each soft fuzzy \(C\)-open sets \((\mu_i, M_i), i \in J\), we have \((H_{i \in J}(\mu_i, M_i))^* = (\bigvee_{i \in J} \mu_i, \bigcup_{i \in J} M_i)^* = ((\bigvee_{i \in J} \mu_i)^*, (\bigcup_{i \in J} M_i)^*)\). Now for each \((X_1 \times X_2)/R \in Q(X_1 \times X_2)\),

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\((V_{i \in J} \mu_i)'((X_1 \times X_2) \mid R)\),
  \[= V_{i \in J} \mu_i((x_1, x_2)),\]
  if there exists \((x_1, x_2)\) such that
  \((X_1 \times X_2) \mid R = (X_1 \times X_2) \mid R_{(x_1, x_2)};\)

\[V_{i \in J} \mu_i((x_1, x_2)),\]
  \[= V_{(x_1, x_2) \in (X_1 \times X_2) \mid R} V_{i \in J} \mu_i((x_1, x_2)),\]
  otherwise.

\[= V_{i \in J} \mu_i'(X_1 \times X_2) \mid R.\]

Thus \((V_{i \in J} \mu_i)' = V_{i \in J} \mu_i'.\)

Now for \(i \in J\),
\[\varphi,\]
if \(U_{i \in J} M_i = \varphi;\]
\[(\cup_{i \in J} M_i)' = Q(X_1 \times X_2),\]
if \(U_{i \in J} M_i = X_1 \times X_2;\]
\[X_1 \times X_2 / R_{(x_1, x_2)},\]
if \((x_1, x_2) \in U_{i \in J} M_i \subset X_1 \times X_2.\]

\[\varphi,\]
if \(M_i = \varphi\), for some \(i \in J;\]
\[Q(X_1 \times X_2),\]
if \(M_i = X_1 \times X_2\), for some \(i \in J;\]
\[X_1 \times X_2 / R_{(x_1, x_2)},\]
if \((x_1, x_2) \in M_i \subset X_1 \times X_2\), for some \(i \in J.\]
\[= \cup_{i \in J} M_i'.\]

Therefore \((H_{i \in J}(\mu_i, M_i))' = H_{i \in J}(\mu_i, M_i)'\). Hence, \(b^*\) forms a base on \(Q(X_1 \times X_2)\).
Notation 7.2.1. If \((X_1 \times X_2, \text{st}(r_1 \times r_2))\) is the soft fuzzy product \(C\)-space, then the soft fuzzy strong generalized product topology, generated by the base \(b^*\) is denoted by \((r_1 \times r_2)^* = r_1^* \times r_2^*\). Therefore, \((Q(X_1 \times X_2), (r_1 \times r_2)^*)\) which is also denoted as a soft fuzzy strong generalized topological space.

Definition 7.2.6. Let \(q : X_1 \times X_2 \to Q(X_1 \times X_2)\) be a function defined by \(q(\alpha_1, x_2) = X_1 \times X_2 / R_{(\alpha_1, x_2)}\) for each \((\alpha_1, x_2) \in X_1 \times X_2\).

Proposition 7.2.3. Under Definition 7.2.2., \(q(1_{X_1 \times X_2}, X_1 \times X_2)\) is soft fuzzy strong generalized-dense in \((Q(X_1 \times X_2), (r_1 \times r_2)^*)\), that is \(SFSGc(q(1_{X_1 \times X_2}, X_1 \times X_2) = (1_{Q(X_1 \times X_2)}, Q(X_1 \times X_2))\).

Proof. Given a soft fuzzy \(C\)-open set \((\mu, M)\) in \(SFPCst(X_1 \times X_2)\), we have \(q(\mu, M)\) is a soft fuzzy product set of \(Q(X_1 \times X_2)\). Now \(q(\mu, M) = (q(\mu), q(M))\) where

\[
q(\mu)(X_1 \times X_2 / R) = \begin{cases} 
\{q(\alpha_1, x_2) : (\alpha_1, x_2) \in X_1 \times X_2\} & \text{if } q^{-1}(X_1 \times X_2 / R) = \emptyset; \\
0 & \text{if } q^{-1}(X_1 \times X_2 / R) = \emptyset.
\end{cases}
\]

\[
\mu((\alpha_1, x_2)), \text{ if there exists } (\alpha_1, x_2) \in X_1 \times X_2 \text{ such that } X_1 \times X_2 / R = X_1 \times X_2 / R_{(\alpha_1, x_2)}; \\
0, \text{ otherwise}
\]

\(q(M) = \{q(\alpha_1, x_2) : (\alpha_1, x_2) \in M\}.\)
Let \((\delta, L)\) = \(H_{j\in J}(\mu_j, M_j)\) = SFsGcl\((q((1_{X_1 \times X_2}, X_1 \times X_2)))\). Since \(q(1_{X_1 \times X_2}, X_1 \times X_2) \subset (\delta, L)^*\), we have for each \((x_1, x_2) \in X_1 \times X_2\), \(\delta^*((X_1 \times X_2) / R_{(x_1, x_2)}) \supset q(1_{X_1 \times X_2}, X_1 \times X_2)((X_1 \times X_2) / R_{(x_1, x_2)}) = 1\) and \(L^* \supset q(X_1 \times X_2)\). This implies that, \(\delta^*((X_1 \times X_2) / R_{(x_1, x_2)}) = \bigvee_{i\in J} \mu_i^*((X_1 \times X_2) / R_{(x_1, x_2)}) = 1\)
\(L^* = \bigcup_{i\in J} M_i^* = Q(X_1 \times X_2)\). Now for each \((X_1 \times X_2) / R \in Q(X_1 \times X_2)\) and for \(i \in J\), \(\bigvee_{i\in J} \mu_i^* = 1_{Q(X_1 \times X_2)}\) and \(\bigcup_{i\in J} M_i^* = Q(X_1 \times X_2)\). This implies that \((\delta^*, L^*) = H_{i\in J}(\mu_i, M_i)^* = (H_{i\in J}(\mu_i, M_i))^* = (\bigvee_{i\in J} \mu_i^*, \bigcup_{i\in J} M_i^*) = (1_{Q(X_1 \times X_2)}, Q(X_1 \times X_2))\). Therefore, \(q(1_{X_1 \times X_2}, X_1 \times X_2)\) is a soft fuzzy strong generalized-dense set in \(Q(X_1 \times X_2)\).

**Definition 7.2.7.** Let \(X_1 \times X_2\) be the product set. Let \((X_1 \times X_2, \text{st}(r_1 \times r_2))\) and \((X_1 \times X_2, G)\) be the soft fuzzy product \(C\)-space and soft fuzzy strong generalized product topological space. Then the function \(h : (X_1 \times X_2, \text{st}(r_1 \times r_2)) \rightarrow (X_1 \times X_2, G)\) is said to be **soft fuzzy \(\ast\)-continuous product function** if the inverse image of soft fuzzy strong generalized open set is a soft fuzzy \(C\)-open set.

**Definition 7.2.8.** Let \(X_1 \times X_2\) be the product set. Let \((X_1 \times X_2, \text{st}(r_1 \times r_2))\) and \((X_1 \times X_2, G)\) be the soft fuzzy product \(C\)-space and soft fuzzy strong generalized product topological space. Then the function \(h : (X_1 \times X_2, \text{st}(r_1 \times r_2)) \rightarrow (X_1 \times X_2, G)\) is said to be **soft fuzzy \(\ast\)-open product function** if the image of soft fuzzy \(C\)-open set is a soft fuzzy strong generalized open set.

**Definition 7.2.9.** Let \(X_1 \times X_2\) be the product set. Let \((X_1 \times X_2, \text{st}(r_1 \times r_2))\)
and \((X_1 \times X_2, G)\) be the soft fuzzy product \(C\)-space and soft fuzzy strong generalized product topological space. Then the function \(h : (X_1 \times X_2, \text{st}(r_1 \times r_2)) \rightarrow (X_1 \times X_2, G)\) is said to be soft fuzzy \(*\)-embedding, if \(h\) is a one to one function, soft fuzzy \(*\)-continuous product function and soft fuzzy \(*\)-open product function.

**Proposition 7.2.4.** The function \(q\) is a soft fuzzy \(*\)-embedding of \(X_1 \times X_2\) into \(Q(X_1 \times X_2)\).

**Proof.** (a) **q is a one to one function:** If \((x_1, x_2) = (y_1, y_2)\), then \(R_{(x_1,x_2)} = R_{(y_1,y_2)}\). Let \((x_1, x_2)\alpha, \{(x_1, x_2)\}\) = \((y_1, y_2)\beta, \{(y_1, y_2)\}\) be any two soft fuzzy points.

**case (i):** If \((x_1, x_2) = (y_1, y_2)\) and \(\alpha = \beta\), then \(q((x_1, x_2)\alpha, \{(x_1, x_2)\}) = (X_1 \times X_2 / R_{(x_1,x_2)}\alpha, \{X_1 \times X_2 / R_{(x_1,x_2)}\}).\) Similarly \(q((y_1, y_2)\beta, \{(y_1, y_2)\}) = (X_1 \times X_2 / R_{(x_1,x_2)}\beta, \{X_1 \times X_2 / R_{(x_1,x_2)}\})\) and it is clear that \(q((x_1, x_2)\alpha, \{(x_1, x_2)\}) = q((y_1, y_2)\beta, \{(y_1, y_2)\}).\) Hence \(q\) is a one to one function.

**case (ii):** If \((x_1, x_2) = (y_1, y_2)\) and \(\alpha = \beta\), then it is clear that \(q((x_1, x_2)\alpha, \{(x_1, x_2)\}) = q((y_1, y_2)\beta, \{(y_1, y_2)\}).\) Hence \(q\) is a one to one function.

(b) **q is a soft fuzzy \(*\)-continuous product function :**

For each \((\mu, M)^{*} \in b^{*} \) and \((x_1, x_2) \in X_1 \times X_2\), we have
\[
q^{-1}(\mu, M)^{*} = q^{-1}(\mu^{*}, M^{*})
\]
\[
= (q^{-1}(\mu^{*}), q^{-1}(M^{*}))
\]
\[
= (\mu^{*} \circ q, q^{-1}(M^{*}))
\]
where
\[ \mu^* \circ q((x_1, x_2)) = \mu^*(q((x_1, x_2))) = \mu^*(X_1 \times X_2 / R(x_1, x_2)) = \mu((x_1, x_2)) \]
and \( q^{-1}(M^*) = M \). Thus \( q^{-1}(\mu, M)^* = (\mu, M) \in \text{st}(\tau_1 \times \tau_2) \). Hence \( q \) is soft fuzzy product \(^*\)-continuous function.

(c) \( q \) is a soft fuzzy \(^*\)-open product function:

It is clear that \( q \) is a soft fuzzy product \(^*\)-open function. Hence \( q \) is a soft fuzzy \(^*\)-embedding of \( X_1 \times X_2 \) into \( Q(X_1 \times X_2) \).

\[ \square \]

Definition 7.2.10. Let \( (X_1 \times X_2, G) \) be a soft fuzzy strong generalized product topological space and \( J \) be an indexed set. \( (X_1 \times X_2, G) \) is said to be a soft fuzzy strong generalized product compact space whenever \( H_{j \in J}(\lambda_j, N_j) = (1, X_1 \times X_2), (\lambda_j, N_j) \in G, j \in J \), there is a finite subset \( F \) of \( J \) with \( H_{j \in F}(\lambda_j, N_j) = (1, X_1 \times X_2) \).

Proposition 7.2.5. The soft fuzzy strong generalized product topological space \( (Q(X_1 \times X_2), (\tau_1 \times \tau_2)^*) \) is a soft fuzzy strong generalized product compact space.

Proof. Let \( K = \{(\lambda^*_i, N^*_i) \in (\tau_1 \times \tau_2)^* : (\lambda_i, N_i) \in \text{st}(\tau_1 \times \tau_2) \} \) for \( i \in J \) be a soft fuzzy strong generalized open sets in \( (\tau_1 \times \tau_2)^* \) and \( J \) be an indexed set. If \( H_{i \in J}(\lambda^*_i, N^*_i) = (1_{Q(X_1 \times X_2)}, Q(X_1 \times X_2)) \), then by Definition 7.2.2., it is clear that \( H_{i \in F}(\lambda^*_i, N^*_i) \pm (1_{Q(X_1 \times X_2)}, Q(X_1 \times X_2)) \) for some finite subset \( F \) of \( J \). Hence, \( (Q(X_1 \times X_2), (\tau_1 \times \tau_2)^*) \) is soft fuzzy strong generalized product.
compact space.

**Proposition 7.2.6.** \((Q(X_1 \times X_2), (τ_1 \times τ_2)')\) is a soft fuzzy strong generalized compactification of the soft fuzzy product \(C\)-space.

**Proof.** Proof is clear from Proposition 7.2.3. to Proposition 7.2.5.