The finite element method (FEM) is a powerful numerical technique that uses variational and interpolation methods for modeling and solving boundary value problems. The method is also tremendously useful for complex mechanism and structures with unusual geometric shapes. This method is very systematic and modular.

The FEM approximates structure in two separate ways. The first approximation made in finite element modeling is to divide the structure into a number of small simple parts. These small parts are called finite element. Each element is generally simple, such as a beam, bar, or plate. All elements have an equation of motion that can be easily solved by analytical methods or computer programming. The end point of element is called nodes. The element can be connected by node. The collection of finite elements and nodes is called finite element mesh or finite element grid.

The equation of motion for each individual finite element is then formed and solved. This forms the second level of approximation in FEM. The solutions of the element equations are approximated by a linear combination of low order polynomials. Each of these polynomial solutions is made companionable with the adjoining solution at nodes common to two elements. These solutions are brought together in an assembly procedure, resulting in global mass and stiffness matrices. This represents the analysis of structure as a whole.

The phrase finite element method is often abbreviated ‘FEM’. This abbreviation can also denote the phrase finite element model. Another commonly used abbreviation is ‘FEA’, which means finite element analysis. Sometimes the abbreviation ‘FE’ is used to abbreviate finite elements.

In FEM, to improve the accuracy of the computed solution of eigenvalue problem (EVP), two approaches are possible: (i) the mesh could be refined by
reducing the size of the element and (ii) the degree of the polynomials over the elements could be increased. The first procedure is known as the h-version of the FEM and second as the p-version. The p-version provides a higher rate of convergence than the h-version.

Advantages of FEM include the ability to deal with structures with arbitrary loading, including support conditions, and also the ability to model structures of arbitrary geometry. A further advantage of this method is the possibility of modeling composite structures comprising different structural components. However, for the purpose of dynamic analysis, an alternative is to use the exact displacement functions arising from the solution of the governing differential equations for beam vibration. This approach offers certain advantages but has the disadvantage of leading to a non-linear eigenvalue problem when computing the natural frequencies.

When the FEM is employed, two stages must be considered. The first requires study of the individual elements into which the system is divided, while the second involves the study of the assemblage of elements which represent the entire system. Thus, the outline of the FE process may be summarized into five essential steps which are as follows:

- **Definition of the finite element mesh**: The first step involves the process of discretising the structure into appropriate elements.

- **Selection of displacement models (shape function)**: In this process, a suitable displacement function must be selected for a typical element which would lead to a finite number of DOF and would satisfy the boundary conditions of the system. In order to retain the bounding and convergence properties inherent in the Ritz procedure, it is necessary that the element interpolation functions should include the rigid body displacements and uniform strain states, and that they maintain displacement compatibility along the inter element and exterior boundaries.

- **Formulation the equation of motion**: The strains at any point within the element may be expressed in terms of the element nodal displacements. The
static equations of equilibrium can be obtained by using the principle of stationary total potential energy whereas the dynamic equations of motion are obtained by using Hamilton’s principle.

- Solution of the equations of motion: The solution of stiffness equations lead to a set of simultaneous equations whereas the equations of motion in the free vibration case lead to an EVP.

- Determination of the desired properties: Once the nodal displacements have been determined, strain and stress can be calculated from the strain-displacement relationship and by Hooke's law, respectively.

### 4.1 Formulation of stiffness and mass matrices of element

In dynamic considerations the loads are suddenly applied, or when the loads are of a variable nature, the mass and acceleration effect come into the account. When the mechanism vibrates mainly in one plane, two types of vibration must be considered: (i) the flexural vibration and (ii) longitudinal vibration. In mechanism the links are flexible in the plane of motion and relatively stiffer in the plane perpendicular to the motion. Here the torsion effect is neglected and hence it is possible to construct stiffness and mass for link by combination of simple bar element and beam element.

The Lagrange’s equation writes as below:

\[ L = T - \Pi \]  \hspace{1cm} (4.1)

where, \( T \) is the kinetic energy and \( \Pi \) is the potential energy.

The potential energy can be defined as:

\[ \Pi = U - W \]  \hspace{1cm} (4.2)

where, \( U \) is internal strain energy and \( W \) is work done by external forces.

Hamilton’s principal for an arbitrary time interval from \( t_1 \) to \( t_2 \), the state of motion of a body extremizes the functional
\[ I = \int_{t_i}^{t_f} L dt \] (4.3)

The kinetic energy, potential energy and the work done by external forces for a distributed system are function of the axial coordinate x.

\[ \delta \int_{t_i}^{t_f} (U - T - W) dt = 0 \] (4.4)

Let \( q \) represent the generalized coordinate, then

\[ \delta \int_{t_i}^{t_f} \left[ U(q) - T(q, \dot{q}) - W \right] dx dt = 0 \] (4.5)

\[ W = F \times q \] (4.6)

where, \( F \) is the distributed force.

Taking the variations of different terms in the Eq. (4.5),

\[ \delta U = \frac{\partial U}{\partial q} \delta q \] (4.7)

\[ \delta T = \frac{\partial T}{\partial q} \delta q + \frac{\partial T}{\partial \dot{q}} \delta \dot{q} \] (4.8)

\[ \delta W = F \delta q \] (4.9)

Substituting the value from Eqs. (4.7-4.9) in Eq. (4.5), we get

\[ \int_{t_i}^{t_f} \int_{t_i}^{t_f} \left[ \frac{\partial U}{\partial q} \delta q - \left( \frac{\partial T}{\partial \dot{q}} \delta \dot{q} + \frac{\partial T}{\partial \dot{q}} \delta \dot{q} \right) - F \delta q \right] dx dt = 0 \] (4.10)

Integrating \( \dot{q} \) term by parts with respect to time, as

\[ \int_{t_i}^{t_f} \frac{\partial T}{\partial \dot{q}} \delta \dot{q} dt = \left[ \frac{\partial T}{\partial \dot{q}} \delta \dot{q} \right]_{t_i}^{t_f} - \int_{t_i}^{t_f} \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}} \right) \delta \dot{q} dt \] (4.11)
\[
\int_{t_i}^{t_f} \frac{\partial T}{\partial q} \delta q \, dt = -\int_{t_i}^{t_f} \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}} \right) \delta q \, dt
\]  
(4.12)

Substituting the values of Eq. (4.12) in Eq. (4.10), we get

\[
\int_{t_i}^{t_f} \int_{0}^{L} \left[ \frac{\partial U}{\partial q} - \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}} \right) - \frac{\partial T}{\partial q} - F \right] \delta q \, dx \, dt = 0
\]  
(4.13)

Therefore Lagrangian equation for dynamical system with two independent parameters, time \( t \) and one generalized coordinate \( q \) is

\[
\frac{\partial U}{\partial q} + \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}} \right) - \frac{\partial T}{\partial q} = F
\]  
(4.14)

Eq. (4.14) then can be used for deriving the elemental properties of bar and beam structure.

4.1.1 The bar element [74]

The longitudinal vibration of a bar demonstrates how a finite element model is constructed. The simplest one dimensional element model is considered as shown in Fig. 4.1.

![Finite element model of bar element](image)

To construct a FE model, the following steps are to be considered:

Step 1: Choose the bar element which has two nodes. The nodal displacements are \( u_1 \) and \( u_2 \) as shown in Fig. 4.1.

Step 2: Define shape function matrix or displacement fields, assuming the displacements at the two nodes are to be known. The displacement of any point within
the element is obtained by assuming a shape function or interpolation function. In general, shape function need to satisfy the following:

- First derivatives must be finite within an element.
- Displacements must be continuous across the element boundary.
- Include representation of constant values of significant stress or strains.

For the axial (tensile) displacement of the bar element suitable choice of shape function is a linear polynomial as below:

\[ u = a_1 + a_2 \times \bar{x} \]  \hspace{1cm} (4.15)

where, \( a_1 \) and \( a_2 \) constants to be determined from the boundary conditions of the element.

\[ u = u_1 \text{ at } \bar{x} = 0 \text{ and } u = u_2 \text{ at } \bar{x} = L \]  \hspace{1cm} (4.16)

By putting the value of constants, Eq. (4.15) can be modified as:

\[ u = u_1 + \frac{1}{L} (u_2 - u_1) \times \bar{x} \]

\[ u = \left(1 - \frac{\bar{x}}{L}\right) \times u_1 + \frac{\bar{x}}{L} \times u_2 \]  \hspace{1cm} (4.17)

The Eq. (4.17) in matrix form can be written as:

\[ u = \left[ \begin{array}{c} 1 - \frac{\bar{x}}{L} \\ \frac{\bar{x}}{L} \end{array} \right] \left[ \begin{array}{c} u_1 \\ u_2 \end{array} \right] \]

\[ u = [N_1 \quad N_2] \left[ \begin{array}{c} u_1 \\ u_2 \end{array} \right] \]  \hspace{1cm} (4.18)

where, \( N_1 \) and \( N_2 \) are shape function of the element.

Shape function matrix, \( N = \left[ \begin{array}{c} N_1 \\ N_2 \end{array} \right] \)  \hspace{1cm} (4.19)
Displacement matrix, \( \bar{d} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \)  

(4.20)

Eq. (4.18) can be modified by using Eqs. (4.19) and (4.20) as:

\[ u = [N][\bar{d}] \]  

(4.21)

Step 3: Establish strain displacement relations as per the elementary theory. The normal component of strain in axial direction can be calculated as:

\[
\varepsilon_x = \frac{du}{d\bar{x}} = \begin{bmatrix} -\frac{1}{L} & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} 
\]

\[ = [N_1, \bar{x} \quad N_2, \bar{x}] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \]

\[ = [B] \begin{bmatrix} \bar{d} \end{bmatrix} \]  

(4.22)

Matrix \([B]\) gives relation between the axial strain and the nodal degree of freedom.

Step 4: Determine the strain energy in bar element which is given as:

\[ U = \frac{1}{2} E \times A \int_0^L (\varepsilon_x)^2 d\bar{x} \]  

(4.23)

By using Eqs. (4.21 and 4.22), the Eq. (4.23) can be written as:

\[ U = \frac{1}{2} EA \begin{bmatrix} \bar{d} \end{bmatrix}^T \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} \bar{d} \end{bmatrix} d\bar{x} \]  

(4.24)

Step 5: Construct the elemental stiffness matrix by considering the potential energy term from Lagrangian equation Eq. (4.24) can be written as:

\[
\frac{\partial U}{\partial [\bar{d}]} = EA \begin{bmatrix} B \end{bmatrix}^T \begin{bmatrix} B \end{bmatrix} [\bar{d}] = \{ F \} 
\]

(4.25)

Eq. (4.25) can be written in matrix form as
\[ [K] \{ \ddot{d} \} = \{ F \} \]  \hspace{1cm} (4.26)

where, \([K]\) is the stiffness matrix and is given as:

\[ [K] = \frac{EA}{L} \int_{0}^{L} \begin{bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{bmatrix} \begin{bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{bmatrix} d\xi \]  \hspace{1cm} (4.27)

With the help of strain equation as mention in Eq. (4.22), can be written as:

\[ [K] = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \]  \hspace{1cm} (4.28)

By integration the Eq. (4.28), the stiffness matrix for bar element is written as:

\[ [K] = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \]  \hspace{1cm} (4.29)

Step 6: Construct the elemental mass matrix as follow:

The total mass of the bar element is \( \rho AL \), where, \( \rho \) is the density of the bar element, \( A \) is the cross section of area and \( L \) is the length of the bar element.

The kinetic energy of bar element is given as:

\[ T = \frac{1}{2} \rho A \int_{0}^{L} \dot{\bar{u}}^2 \, d\xi \]  \hspace{1cm} (4.30)

\[ T = \frac{1}{2} \rho A \frac{L}{T} \int_{0}^{L} \begin{bmatrix} \dot{u} \end{bmatrix} \begin{bmatrix} \dot{u} \end{bmatrix} \begin{bmatrix} N \end{bmatrix} \begin{bmatrix} \ddot{\bar{u}} \end{bmatrix} \, d\xi \]  \hspace{1cm} (4.31)

From Eq. (4.14), considering the potential energy term only,

\[ \frac{d}{dt} \frac{\partial T}{\partial \{ \dot{u} \}} = \{ F \} \]  \hspace{1cm} (4.32)

From Eqs. (4.31) and (4.32), we get

\[ \frac{\partial T}{\partial \{ \dot{u} \}} = \rho A \frac{L}{T} \int_{0}^{L} \begin{bmatrix} \dot{N} \end{bmatrix} \begin{bmatrix} \dot{N} \end{bmatrix} \begin{bmatrix} \ddot{\bar{u}} \end{bmatrix} \, d\xi \]
\[
\frac{d}{dt} \frac{\partial T}{\partial u} = \rho A \int_0^L [N]^T [N] \left( \frac{\partial }{\partial } \right) d\bar{x}
\] (4.33)

From Eqs. (4.32) and (4.33),
\[
\rho A \int [N]^T [N] \left( \frac{\partial }{\partial } \right) dx = \{F\}
\]
\[
[M] \left( \frac{\partial }{\partial } \right) = \{F\}
\] (4.34)

where, \([M]\) is the mass matrix of bar element as:
\[
[M] = \rho A \int_0^L [N]^T [N] d\bar{x}
\] (4.35)

Substitute the value of shape function from Eq. (4.21) in Eq. (4.35), we get,
\[
[M] = \rho A \int_0^L \left[ \frac{l - \bar{x}}{L} \right] \left[ \frac{\bar{x}}{L} \right] d\bar{x}
\] (4.36)

By integration the above equation, we get
\[
[M] = \frac{1}{6} \rho AL \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}
\] (4.37)

The above mass matrix used for the distribution mass, is called consistent mass matrix of the bar element.

**4.1.2 The beam element [73,74]**

A finite element model of the beam element is shown in Fig. 4.2. The two nodes 1 and 2 are presented in elemental coordinate system.

To construct a FE model, the following steps are to be considered:
Step 1: Choose the element as shown in Fig. 4.2 the element has two nodes 1 and 2. The nodal deflections are \(u_1\) and \(u_2\) measured positive in the direction of y axis. The nodal slopes are \(u_3\) and \(u_4\) positive in anticlockwise direction.
Step 2: Define shape function Matrix, the displacement and slopes at both nodes 1 and 2 of the element is known. The shape function for the transverse displacement $u(x)$ is assumed to be cubic polynomial.

$$u(x) = a_0 x^3 + a_1 x^2 + a_2 x + a_3$$ \hspace{1cm} (4.38)

The above Eq. (4.38) satisfies the governing differential equation of a beam.

$$EI \frac{d^4 y}{dx^4} = 0$$ \hspace{1cm} (4.39)

The continuity condition of both the displacement and the slope at the nodes is satisfies the cubic polynomial shape function. By using the boundary condition of the element at the nodes calculate the value of four constant.

$$u(0) = u_1 = a_4$$

$$\frac{du(0)}{dx} = u_3 = a_3$$

$$u(L) = u_2 = a_1 L^3 + a_2 L^2 + a_3 L + a_4$$

$$\frac{du(L)}{dx} = u_4 = 3a_1 L^2 + 2a_2 L + a_3$$ \hspace{1cm} (4.40)

By solving the above Eq. (4.40) got the value of constant and substituting in Eq. (4.39),
\[ u(\bar{x}) = \left[ \frac{2}{L} (u_1 - u_2) + \frac{1}{L^2} (u_3 - u_4) \right] \bar{x}^3 \]
\[ - \left[ \frac{3}{L^3} (u_1 - u_2) + \frac{1}{L} (2u_3 + u_4) \right] \bar{x}^2 + u_3 \bar{x} + u_1 \]  \hspace{1cm} (4.41)

Collecting terms of nodal degree of freedom and writing in matrix form

\[ u = [N]^T \bar{d} \]  \hspace{1cm} (4.42)

where, \([N] = [N_1 \quad N_2 \quad N_3 \quad N_4]\) and

\[ [N_1] = \frac{1}{L} \left(2\bar{x}^3 - 3\bar{x}^2 L + L^3\right) \]

\[ [N_2] = \frac{1}{L^3} \left(\bar{x}^3 L - 2\bar{x}^2 L^2 + \bar{x} L^3\right) \]

\[ [N_3] = \frac{1}{L^3} \left(-2\bar{x}^3 + 3\bar{x}^2 L\right) \]

\[ [N_4] = \frac{1}{L^3} \left(\bar{x}^3 L - \bar{x}^2 L^2\right) \]  \hspace{1cm} (4.43)

Step 3: Establish the relations for bending moment displacement, the bending moment in the element is given by

\[ M_z(\bar{x}) = EI_{zz} \frac{d^2 u}{d\bar{x}^2} \]  \hspace{1cm} (4.44)

Using Eq. (4.42), Eq. (4.44) can be modified as:

\[ M_z(\bar{x}) = EI_{zz} [B]^T \bar{d} \]  \hspace{1cm} (4.45)

where, \([B] = [N_{1,zz} \quad N_{2,zz} \quad N_{3,zz} \quad N_{4,zz}]\)  \hspace{1cm} (4.46)

and\[ [N_{1,zz}] = \frac{1}{L^3} \left(12\bar{x} - 6L\right) \]

\[ [N_{2,zz}] = \frac{1}{L^3} \left(6\bar{x} L - 4L^2\right) \]


\[
[N_{3,\pi}] = \frac{1}{L^3} (-12\overline{\xi} + 6L)
\]

\[
[N_{4,\pi}] = \frac{1}{L^3} (6\overline{\xi}L - 2L^2)
\]

and \(\{d\} = \begin{bmatrix} u_1 \\ u_3 \\ u_2 \\ u_4 \end{bmatrix}\) \hspace{1cm} (4.47)

Step 4: Determine the Strain energy in beam element which is given as:

\[
U = \frac{1}{2} EI \int_0^L (\dot{u}_{\xi,\xi})^2 d\overline{\xi}
\]

\[
= \frac{1}{2} EI \int_0^L \{d\}^T \begin{bmatrix} B \end{bmatrix}^T \begin{bmatrix} B \end{bmatrix} \{d\} d\overline{\xi} \hspace{1cm} (4.48)
\]

Step 5: Formation of elemental stiffness matrix by considering the potential energy term only from Eq. (4.14), rewrite the Eq. (4.48) as:

\[
\frac{\partial U}{\partial \{d\}} = EI \int_0^L \begin{bmatrix} B \end{bmatrix}^T \begin{bmatrix} B \end{bmatrix} d\overline{\xi} = \{F\} \hspace{1cm} (4.49)
\]

With the help of Eqs. (4.47) and (4.49) the stiffness can be written as:

\[
[K] = EI \int_0^L \begin{bmatrix} N_{1,\pi} \\ N_{2,\pi} \\ N_{3,\pi} \\ N_{4,\pi} \end{bmatrix} \begin{bmatrix} N_{1,\pi} & N_{2,\pi} & N_{3,\pi} & N_{4,\pi} \end{bmatrix} d\overline{\xi} \hspace{1cm} (4.50)
\]

Substituting for the derivatives of shape function in above equation and integration, the stiffness matrix as:
\[
[K] = \frac{EI}{L^3} \begin{bmatrix}
12 & 6L & -12 & 6L \\
6L & 4L^2 & -6L & 2L^2 \\
-12 & -6L & 12 & -6L \\
6L & 2L^2 & -6L & 4L^2
\end{bmatrix}
\] (4.51)

Step 6: Construct the elemental mass matrix as follow:

The kinetic energy of beam element is

\[
T = \frac{1}{2} \rho A \int_0^L u^2 \, d\bar{x}
\] (4.52)

By same method used in section 4.1.1 for bar element, the mass matrix for element same as Eq. (4.35) as below:

\[
[M] = \frac{\rho A}{L} \int_0^L \begin{bmatrix}
\left(2\bar{x}^3 - 3\bar{x}^2 L + L^3\right) \\
\left(\bar{x}^3 L - 2\bar{x}^2 L^2 + \bar{x}L^3\right) \\
\left(-2\bar{x}^3 + 3\bar{x}^2 L\right) \\
\left(\bar{x}^3 L - \bar{x}^2 L^2\right)
\end{bmatrix} \begin{bmatrix}
\left(2\bar{x}^3 - 3\bar{x}^2 L + L^3\right) \\
\left(\bar{x}^3 L - 2\bar{x}^2 L^2 + \bar{x}L^3\right) \\
\left(-2\bar{x}^3 + 3\bar{x}^2 L\right) \\
\left(\bar{x}^3 L - \bar{x}^2 L^2\right)
\end{bmatrix} \, d\bar{x}
\]

By integration the above equation, the mass matrix for the beam element is as follow:

\[
[M] = \frac{\rho A L}{420} \begin{bmatrix}
156 & 22L & 54 & -13L \\
22L & 4L^2 & 13L & -3L^2 \\
54 & 13L & 156 & -22L \\
-13L & -3L^2 & -22L & 4L^2
\end{bmatrix}
\] (4.53)

4.1.3 The beam element with six DOF

The displacement model, the stiffness matrixes and the mass matrixes considered with the axial and bending loading of an element, have been derived separately in section 4.1.1 and 4.1.2. In plane motion analysis, the beam element has
been six DOF as shown in Fig. 4.3. To derive the stiffness matrix and mass matrix for the beam having six DOF assembling the matrixes of bar and beam element.

The stiffness matrix can be written as:

\[
[K] = \begin{bmatrix}
K_1 & K_2 \\
K_1 & K_2
\end{bmatrix}
\]  

(4.54)

where, \([K_1] = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\) and \([K_2] = \frac{EI}{L^3} \begin{bmatrix}
12 & 6L & -12 & 6L \\
6L & 4L^2 & -6L & 2L^2 \\
-12 & -6L & 12 & -6L \\
6L & 2L^2 & -6L & 4L^2
\end{bmatrix}\)

The mass matrix can be written as:

\[
[M] = \begin{bmatrix}
M_1 & M_2 \\
M_1 & M_2
\end{bmatrix}
\]  

(4.55)

where, \([M_1] = \frac{1}{6} \rho AL \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\) and \([M_2] = \frac{\rho AL}{420} \begin{bmatrix}
156 & 22L & 54 & -13L \\
22L & 4L^2 & 13L & -3L^2 \\
54 & 13L & 156 & -22L \\
-13L & -3L^2 & -22L & 4L^2
\end{bmatrix}\)

Beam and bar, both elements follow the classical beam theory, meaning plane section remain plane and are capable of including shear deflection using shear area coefficients (more important for short stubby beams or bars)

The beam element is capable of doing more compare to bar element as listed below:

---

4.14
• Beam elements can have tapered sections, meaning one end can be smaller/larger/wider/narrower/thinner/thicker than the other, but the shape cannot be totally different.

• Beam elements are capable of accounting for large deflections and differential stiffness due to large deflections.

• Beam elements can have three different offsets. One for shear center, one for the neutral axis and one for the nonstructural mass axis. Whereas bar elements have only one axis, all three are the same neutral axis.

• For a bar element the grid points are located at the section centroidal neutral axis. For beam elements they are always at the shear center axis and the neutral axis is offset from the shear center axis.

• Bar elements are best for doubly symmetrical sections with load applied along centroidal planes, as they are not capable of accounting for bending or twisting or warping of the sections due to axial or transverse loads. This is only possible with beam elements.

4.2 FEM analysis of four bar mechanism

The four bar planar mechanism with elastic links is modeled by connecting a series of beam element as shown in Fig. 4.4. The rigid body motion of mechanism is mention by solid lines (ABCD) and elastic displacement mechanism by dotted line.

![Fig. 4.4 Four bar mechanism with elastic links displaced](image-url)
(APQD) as shown in Fig. 4.4. They are connected in such a way that it allows the model for deviation in the mechanism geometry. A complicated link is to be considered as beam element with uniform cross section throughout its length. Each link is to be as one beam element for simplicity of in modeling.

### 4.2.1 Elastic beam in plane motion

The link of mechanism is represented as a beam element in two reference frames as shown in Fig. 4.5. The frame (OXY) is the fixed frame where, the frame (Oxy) is the rotated frame with element.

![Fig. 4.5 Rigid and elastic body with coordinate systems](image)

The rigid body position of beam is shown with solid line and elastically deformed position of beam is shown with dotted line as shown Fig. 4.5. The ‘x’ axis of the rotated frame is forever parallel to the rigid body position of the beam element during its motion. The elastic deflection of the beam element may be specified by six generalized nodal displacement coordinates $u_1$ to $u_6$. Following relationships can be developed from Fig. 4.5.
To derive the velocity of node $A$ in fixed frame, set of Eq. (4.56) is differentiated with respect to time and is given as:

\[ \begin{align*} 
\dot{X}_A &= \dot{X}_A + \dot{u}_1 \cos \theta - \dot{u}_2 \sin \theta - u_2 \dot{\theta} \cos \theta \\
\dot{Y}_A &= \dot{Y}_A + \dot{u}_1 \sin \theta + u_2 \dot{\theta} \cos \theta - u_2 \dot{\theta} \sin \theta \\
\dot{\theta}_A &= \dot{\theta} + \ddot{u}_3 
\end{align*} \]  

(4.57)

Derivatives of Eq. (4.57) with respect time give the acceleration of node $A$ in fixed frame.

\[ \begin{align*} 
\ddot{X}_A &= \ddot{X}_A + \dddot{u}_1 \cos \theta - 2\dddot{u}_1 \dot{\theta} \sin \theta - u_2 \dddot{\theta} \cos \theta - u_1 \dddot{\theta} \sin \theta \\
&\quad - \dddot{u}_2 \sin \theta - 2\dddot{u}_2 \dot{\theta} \cos \theta + u_2 \dddot{\theta} \sin \theta - u_2 \dddot{\theta} \cos \theta \\
\ddot{Y}_A &= \ddot{Y}_A + \dddot{u}_1 \sin \theta + 2\dddot{u}_1 \dot{\theta} \cos \theta - u_1 \dddot{\theta} \sin \theta + u_1 \dddot{\theta} \cos \theta \\
&\quad + \dddot{u}_2 \cos \theta - 2\dddot{u}_2 \dot{\theta} \sin \theta - u_2 \dddot{\theta} \sin \theta - u_2 \dddot{\theta} \cos \theta \\
\ddot{\theta}_A &= \ddot{\theta} + \dddot{u}_3 
\end{align*} \]  

(4.58)

Above Eq. (4.58) can be written with respect to rotating frame with help of following transformation:

\[ \begin{align*} 
\dddot{x}_A &= \dddot{X}_A \cos \theta + \dddot{Y}_A \sin \theta \\
\dddot{y}_A &= -\dddot{X}_A \sin \theta + \dddot{Y}_A \cos \theta \\
\dddot{\theta}_A &= \dddot{\theta} + \dddot{u}_3 
\end{align*} \]  

(4.59)

By combined and simplified the Eqs. (4.58) and (4.59), we get the acceleration of node $A$ in rotated frame:
\[
\begin{align*}
\ddot{x}' &= \ddot{x}_A + \ddot{u}_1 - u_t \dddot{\theta}^2 - 2\ddot{u}_2 \dot{\theta} - u_z \dddot{\theta} \\
\ddot{y}' &= \ddot{y}_A + \ddot{u}_2 - u_t \dddot{\theta}^2 + 2\ddot{u}_1 \dot{\theta} + u_1 \dddot{\theta} \\
\dddot{\theta} &= \dddot{\theta} + \ddot{u}_3
\end{align*}
\] (4.60)

Similarly, the acceleration of node B in rotated frame can be written as:

\[
\begin{align*}
\ddot{x}_B' &= \ddot{x}_B + \ddot{u}_4 - u_t \dddot{\theta}^2 - 2\ddot{u}_5 \dot{\theta} - u_5 \dddot{\theta} \\
\ddot{y}_B' &= \ddot{y}_B + \ddot{u}_5 - u_t \dddot{\theta}^2 + 2\ddot{u}_4 \dot{\theta} + u_4 \dddot{\theta} \\
\dddot{\theta}_B &= \dddot{\theta} + \ddot{u}_6
\end{align*}
\] (4.61)

where, \( \dddot{x}_A, \dddot{y}_A, \dddot{x}_B, \dddot{y}_B, \dddot{\theta} \) and \( \dddot{\theta} \) are the kinematic terms of the rigid body motion of the element.

Now defining the following column vectors as:

\[
\{u_{ai}\} = \begin{bmatrix} \ddot{x}_A \\ \ddot{y}_A \\ \dddot{\theta} \\ \ddot{x}_B \\ \ddot{y}_B \\ \dddot{\theta}_B \end{bmatrix} \quad \text{and} \quad \{u_{ai}\} = \begin{bmatrix} \ddot{x}_A \\ \ddot{y}_A \\ \dddot{\theta} \\ \ddot{x}_B \\ \ddot{y}_B \\ \dddot{\theta}_B \end{bmatrix}, \quad i = 1, 2, \ldots, 6 \quad (4.62)
\]

From Eqs. (4.60) – (4.62), we get:

\[
\begin{bmatrix} \ddot{u}_{a_1} \\ \ddot{u}_{a_2} \\ \ddot{u}_{a_3} \\ \ddot{u}_{a_4} \\ \ddot{u}_{a_5} \\ \ddot{u}_{a_6} \end{bmatrix} = \begin{bmatrix} \ddot{u}_r + \ddot{u}_t - u_t \dddot{\theta}^2 - 2\ddot{u}_s \dot{\theta} - u_s \dddot{\theta} \\ \ddot{u}_r + \ddot{u}_t - u_t \dddot{\theta}^2 + 2\ddot{u}_s \dot{\theta} + u_s \dddot{\theta} \\ \dddot{u}_r + \dddot{u}_t + 0 + 0 + 0 \\ \ddot{u}_r + \ddot{u}_t - u_t \dddot{\theta}^2 - 2\ddot{u}_s \dot{\theta} - u_s \dddot{\theta} \\ \ddot{u}_r + \ddot{u}_t - u_t \dddot{\theta}^2 + 2\ddot{u}_s \dot{\theta} + u_s \dddot{\theta} \\ \dddot{u}_r + \dddot{u}_t + 0 + 0 + 0 \end{bmatrix} \quad (4.63)
\]

The Eq. (4.63) can be rewritten as:

\[
\{u_{ai}\} = \{u_{ri}\} + \{u_{ti}\} + \{a_z\} + \{a_t\} + \{a_r\} \quad (4.64)
\]
Where, \{a_n\}, \{a_c\} and \{a_t\} are the normal, Coriolis and tangential components of acceleration respectively.

As the components \{a_n\}, \{a_c\} and \{a_t\} are very small as compared to terms \{\ddot{u}_r\} and \{\ddot{u}\}. The Eq. (4.64) for acceleration can be modified as:

\[
\{\ddot{u}_a\} = \{\ddot{u}_r\} + \{\ddot{u}\}
\]  \hspace{1cm} (4.65)

Similarly for velocity:

\[
\{\dot{u}_a\} = \{\dot{u}_r\} + \{\dot{u}\}
\]  \hspace{1cm} (4.66)

### 4.2.2 Element mass and stiffness matrixes

The mass and stiffness matrixes of the beam element in a local coordinate frame have been derived in section 4.1.3.

The kinetic energy and strain (potential) energy of one beam element (one link) in matrix form are as below:

\[
T = \frac{1}{2} \{\ddot{u}_a\}^T [\mathbf{m}] \{\ddot{u}_a\} \quad \text{and}
\]

\[
T = \frac{1}{2} \{\dot{u}\}^T [\mathbf{m}] \{\dot{u}\}
\]  \hspace{1cm} (4.67)

where, \([\mathbf{m}]\) is the mass matrix [12], can be given as:

\[
[\mathbf{m}] = \rho A L \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{13}{35} & \frac{1}{105} & 0 \\
0 & \frac{11}{210} L & \frac{1}{105} L^2 & 0 \\
\frac{1}{6} & 0 & 0 & \frac{1}{3} \\
0 & \frac{9}{70} & \frac{13}{420} L & 0 \\
0 & \frac{13}{420} L & \frac{1}{140} & 0 \\
0 & \frac{1}{210} L & 0 & \frac{13}{105} \\
\end{bmatrix}
\]  \hspace{1cm} (4.68)
The stiffness matrix, $\mathbf{K}$ is formed as [12],

$$
\mathbf{K} = \begin{bmatrix}
\frac{EA}{L} & \frac{12EI}{L^3} & \frac{6EI}{L^2} & \frac{4EI}{L} \\
0 & \frac{6EI}{L^2} & \frac{4EI}{L} & \\
-\frac{12EI}{L^3} & -\frac{6EI}{L^2} & -\frac{12EI}{L^3} & \\
0 & \frac{2EI}{L^2} & 0 & \frac{6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix}
$$

From the equation of motion as given by Eq. (4.14) and from Eqs. (4.68) and (4.69), for beam element equation of motion is given as:

$$
[m]\ddot{\mathbf{u}}(t) + [K]\mathbf{u}(t) = \mathbf{Q}
$$

(4.70)

### 4.2.3 Transformation of matrixes to global coordinates

The mass and stiffness matrixes developed in the section 4.2.2 are expressed in local or element coordinate system. In practice, the mechanism or machine are made up of number of elements with different orientations. Therefore, presenting the displacements in a coordinate system particular to each element will create difficulties in matching the displacement at a given node during the assembling process. Thus, while having several local coordinate systems, it is required to use a global coordinate system for a given problem [71].

The general element with two nodes 1 and 2 and two coordinate i.e. local and the global coordinate system is shown in Fig. 4.6.

With the reference of Fig. 4.6, the set of equations for node 1 is given as:

$$
\begin{align*}
\mathbf{u}_1 &= \mathbf{U}_1 \cos \theta + \mathbf{U}_2 \sin \theta \\
\mathbf{u}_2 &= -\mathbf{U}_1 \sin \theta + \mathbf{U}_2 \cos \theta \\
\mathbf{u}_3 &= \mathbf{U}_3
\end{align*}
$$

(4.71)
Fig. 4.6 Transformation of coordinate from local to global coordinate system

Similarly for node 2

\[
\begin{align*}
    u_4 &= U_4 \cos \theta + U_5 \sin \theta \\
    u_5 &= -U_4 \sin \theta + U_5 \cos \theta \\
    u_6 &= U_6
\end{align*}
\]

Eqs. (4.71) and (4.72) may be written in matrix form as:

\[
[R] = \begin{bmatrix}
    \cos \theta & \sin \theta & 0 & 0 & 0 & 0 \\
    -\sin \theta & \cos \theta & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 0 & \cos \theta & \sin \theta & 0 \\
    0 & 0 & 0 & -\sin \theta & \cos \theta & 0 \\
    0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

where, \([R]\) is transformation matrix.
4.3 Finite element model of four bar mechanism

The FEM of four bar planar mechanism with three beam elements is shown in Fig. 4.7. In which each link is modeled by one finite beam element.

![Fig. 4.7 Finite element model of four bar mechanism](image)

4.3.1 Assembly of the system matrixes

Assembly of stiffness and mass matrices and the generalized forces vector of individual elements to form the system (overall) matrixes for the entire mechanism is achieved by ensuring that the geometric compatibility is satisfied at all nodes. Equation of motion given in Eq. (4.70) of the beam element is rewritten in the system coordinate as:

\[
[m]\ddot{U}(t) + [K]U(t) = \{Q\}
\]

where, \([m] = [R]^T [\bar{m}] [R]\), \([k] = [R]^T [K] [R]\) and \(Q = [R]^T [\ddot{Q}]\)

The kinetic energy of link 2 (crank) as beam elements 1 may be written as:

\[
T_i = \frac{1}{2} \{\dot{U}_a\}^T [m] \{\dot{U}_a\}
\]

(4.75)
Eq. (4.75) can be expressed as:

\[
\begin{bmatrix}
\dot{U}_{a1} \\
\dot{U}_{a2} \\
\dot{U}_{a3}
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
m_{11} & m_{12} & m_{13} \\
m_{21} & m_{22} & m_{23} \\
m_{31} & m_{32} & m_{33}
\end{bmatrix} \begin{bmatrix}
\dot{U}_{a1} \\
\dot{U}_{a2} \\
\dot{U}_{a3}
\end{bmatrix}
\]

(4.76)

Hence, total kinetic energy of mechanism is given as:

\[
T_i = T_1 + T_2 + T_3
\]

(4.77)

The Eq. (4.77) is expressed in matrix form as:

\[
T = \frac{1}{2} \begin{bmatrix}
\dot{U}_i
\end{bmatrix}^T [M] \begin{bmatrix}
\dot{U}_i
\end{bmatrix}, \quad i = 1,2,\ldots,9
\]

(4.78)

Hence, matrix \([M]\) is the total system mass matrix. Similarly, from strain energy consideration, the total system stiffness matrix \([K]\) may be derived by superimposing the strain energies of the individual elements as:

\[
U = \frac{1}{2} \begin{bmatrix}
U_i
\end{bmatrix}^T [K] \begin{bmatrix}
U_i
\end{bmatrix}, \quad i = 1,2,\ldots,9
\]

(4.79)

4.3.2 Equation of motion

The equation of motion for mechanism may be written in matrix form as described in Eqs. (4.80 and 4.81) [11-12]:

\[
[M] \ddot{U}_a + [K] U = \{Q\}
\]

(4.80)

If the structural damping matrix for the mechanism is denoted by \([C]\), then by including the damping forces, the equation of motion becomes:

\[
[M] \ddot{U}_a + [C] \dot{U} + [K] U = -[M] \ddot{U}_r
\]

(4.81)

Here, the coefficient matrixes \([M]\), \([C]\) and \([K]\) are the function of the mechanism geometry and change with change in crank angle and also \(\ddot{U}_r\) is represent the rigid body acceleration vector.
4.3.3 Damping in mechanism

In actual mechanism some energy dissipation is always present. Measurement and modeling of the material damping of a system generally proves to be a difficult problem that requires further research. It is therefore necessary to assume an approximate form for the material damping. A proportional viscous damping form is customarily assumed due to the ease in which it can be incorporated into the equation of motion, and also to ensure that the equations of motion can be uncoupled.

4.3.4 Stress calculation

Axial forces within a link are generated due to its own longitudinal vibration, the foreshortening due to its transverse vibrations and the elastic effect of the other links transmitted through the pins at its ends during the internal.

Strain and stress are calculated as bellow:

The axial strains at the neutral axis is
\[
\varepsilon = -\frac{1}{L} u_1(t) + \frac{1}{L} u_4(t) \tag{4.82}
\]

The axial stress is
\[
\sigma = E \left[ -\frac{1}{L} \frac{1}{L} u_1(t) \right] \tag{4.83}
\]

4.3.5 Method of solution

In numerical methods, this continuous motion is replaced by a number of discretized steps. The concept is analogous to finite element theory, where the elastic medium itself is discretized [71]. During each time step, the system parameters (mass, damping, and stiffness) are assumed to remain constant in solving the equation of motion. This produces is only an approximate solution, while the true solution is approached as the step size tends to zero.

Many methods have been suggested by researchers to find out the solution of equation of motion are listed below:

- Direct integration method
• Modal analysis
• Fourier series method
• Newmark method
• Runge-Kutta method