Chapter – 3  Rigid Dynamic Analysis of Mechanism

The study of mechanical systems has two distinct aspects: synthesis and analysis. The synthesis involves the prescription of sizes, shapes, materials, etc. so that the mechanism performs the functions for which it was designed. The analysis is the collection of scientific tools at the designer’s disposal to analyze the suitability of the design. The geometrical data of the mechanism are assumed to be known a priori.

In the analysis and design of mechanisms, kinematic quantities such as displacements, velocities and accelerations are of great engineering importance. Displacements and velocities give an insight into the functional behavior of the mechanism. The acceleration, on the other hand, are related to forces by Newton’s principle which themselves are related to stresses and deformations in the mechanism’s components. In the kinematic analysis, the mechanism is assumed to be made up of rigid bodies.

Actually the machine member is moving and the applied forces are not always constant for different configurations of the machine, as in case of four bar planar mechanisms. However, these forces can be considered as constant forces acting on the respective links for a particular configuration.

The force analysis is generally carried out with external forces acting on the machine members. However, since the machine members are dynamic, the additional forces due to their motion are to be taken into account. According to Newton’s second law of motion if a body has linear acceleration, there must be a force or couple acting on the body to cause the corresponding acceleration. In other words, the machine which drives its several members having a cyclic variation of acceleration required to supply more power than required for its purpose, the inertia forces become more prominent for higher speed of machine members.
3.1 Kinematic analysis

Encyclopedia Britannica (1980) defines kinematics as: “Branch of physics and subdivision of classical mechanics, concerns the description of motion of objects without considering the forces that cause or result from the motion. It is an abstract study of motion that aims to provide a description of the spatial position of points in moving (velocity), and the rate at which their velocity is changing (acceleration). When the causative forces are disregarded, motion descriptions are possible only for points having constrained motion; i.e., moving on pre-determine paths. In unconstrained or free motion the forces determine the shape of the main path”.

3.1.1 Position analysis of mechanism

Four bar planar mechanism have only one DOF so that only one parameter is needed to completely define the position of all the links. The parameter generally selected to define the position is the angle of the input link. \( \theta_2 \) as mention in Fig. 3.1 and aim is to calculate the angle \( \theta_3 \) and \( \theta_4 \). The length of all links is known by synthesis. For four bar planar mechanism, notation 1, 2, 3 and 4 are used for ground link, driver link (crank), coupler link and lever (rocker) respectively. Position analysis can be done by two method (i) Graphical and, (ii) Analytical. The graphical analysis of four bar planar mechanism is very simple and easy. All link angles are measured from a positive X axis as shown in Fig. 3.1.

The accuracy of graphical results will depend upon the accuracy of drawing and measuring. This solution is only valid for the given value of \( \theta_2 \). For other value of \( \theta_2 \) it is required to redraw the mechanism. This can become burden and boring task if we require a complete analysis at every 5° or 10° of interval of \( \theta_2 \) is required. In this type of situation it will be better to derive an analytical solution for \( \theta_3 \) and \( \theta_4 \) which can be easily solved with the help of computer. For analytical solution the links are to be represented as position vectors.
The equation for a four bar mechanism can be derived by complex-algebra. Fig. 3.2 shows the same four bar mechanism as shown in Fig. 3.1, but the links are to be drawn as position vectors which create a vector loop and also this loop closes on
itself. Due to this reason the sum of the vectors in the loop must be zero. The link length to be considered as the length of vector and are known. The instant mechanism position shown is defined by only input angle \( \theta_2 \) because it is a single DOF mechanism.

The final resulting equation can be written as:

\[
A \times \tan^2\left(\frac{\theta_2}{2}\right) + B \times \tan\left(\frac{\theta_2}{2}\right) + C = 0
\]  

(3.1)

Where, \( A = \cos \theta_2 - K_1 - K_2 \times \cos \theta_2 + K_3 \)

\[
B = -2 \times \sin \theta_2
\]

\[
C = K_1 - (K_2 + 1) \times \cos \theta_2 + K_3
\]

The constants \( K_1, K_2 \) and \( K_3 \) are defined in terms of the constant link lengths. The \( a, b, c \) and \( d \) are the length of links 1, 2, 3 and 4 respectively.

\[
K_1 = \frac{d}{a}, \quad K_2 = \frac{d}{c}, \quad K_3 = \frac{a^2 - b^2 + c^2 + d^2}{2 \times a \times c}, \quad K_4 = \frac{d}{b}, \quad K_5 = \frac{c^2 - d^2 - a^2 - b^2}{2 \times a \times b}
\]

The Eq. (3.1) is quadratic in form, and the solution is:

\[
\tan\left(\frac{\theta_2}{2}\right) = -\frac{B \pm \sqrt{B^2 - 4 \times A \times C}}{2 \times A}
\]  

(3.2)

\[
\theta_{4,5} = 2 \times \tan^{-1}\left(\frac{-B \pm \sqrt{B^2 - 4 \times A \times C}}{2 \times A}\right)
\]  

(3.3)

Eq. (3.2) has two solutions, obtained from the \( \pm \) conditions on the radical. If the radical is negative, then the solution is complex conjugate, which simply means that the link lengths chosen are not suitable to make connection for the given value of the input angle (crank angle) \( \theta_2 \). This can be possible either when the length of links is completely incapable to make connection in any position or, in a non-Grashof mechanism, when the crank angle is beyond a toggle limit position. Thus, there is no real solution for that value of crank angle \( \theta_2 \). Apart from this solution, the solution
will generally real and unequal, means there are two values of $\theta_4$ corresponding to any one value of $\theta_2$. These are considered as crossed and open configurations of mechanism and also as the two branches of the mechanism. The positive value gives $\theta_4$ for the crossed and minus value gives $\theta_4$ for the opened configuration in case of Grashof four bar planar mechanism as shown in Fig. 3.3.

The solution for angle $\theta_3$ is essentially similar to that for $\theta_4$ and can be written as:

The quadratic equation for angle $\theta_3$ as bellow:

$$D \times \tan^2 \left( \frac{\theta_3}{2} \right) + E \times \tan \left( \frac{\theta_3}{2} \right) + F = 0$$  \hfill (3.4)

where, $D = \cos \theta_2 - K_1 + K_4 \times \cos \theta_2 + K_5$, $E = -2 \times \sin \theta_2$

$$F = K_1 + (K_4 - 1) \times \cos \theta_2 + K_5$$

Fig. 3.3 Open and crossed configurations of the four bar mechanism

The solution of Eq. (3.4) is state in Eq. (3.5)
\[ \theta_{3,2} = 2 \times \tan^{-1} \left( \frac{-E \pm \sqrt{E^2 - 4 \times D \times F}}{2 \times D} \right) \]  \hspace{1cm} (3.5)

Similar to the angle \( \theta_4 \), the angle \( \theta_3 \) has two values, corresponding to the crossed and open configurations of mechanism, as shown in Fig. 3.3.

Position analysis of six bar mechanism (Watt’s mechanism) is to be carried by considering the two four bar mechanism in series, as shown in Fig. 3.4(a). Two vector

![Fig. 3.4 Six bar mechanism](image)

(a) Watt’s mechanism  \hspace{1cm} (b) Watt’s mechanism with vector loops
loops are considered for a six-bar mechanism as shown in Fig. 3.4(b). These vector loop equations can be solved in succession with the results of the first loop to be considered as input for the second loop. The solutions of four-bar planar mechanism as given by Eqs. (3.2) and (3.4) are applied two times in the analysis of a six-bar mechanism.

### 3.1.2 Transmission angle

The transmission angle ($\mu$) is defined as the acute angle between the rocker and the coupler in a mechanism. It is shown in Fig. 3.5. For a four-bar planar mechanism, it will be the difference of angle $\theta_3$ and angle $\theta_4$.

![Fig. 3.5 Mechanism transmission angle and static force at a pin joint](image)

\[
\mu = |\theta_3 - \theta_4| \quad (3.6)
\]

If $\mu > \frac{\pi}{2}$

then, $\mu = \pi - \mu$

A torque $T_2$ applied to the crank as shown in Fig. 3.5. Before any motion occurs, this creates a static collinear joint force $F_{34}$ to be developed at a junction of links 3 and 4 (point D). This force is resolved into radial ($F_{34}^r$) and tangential ($F_{34}^t$) components. Radial component are parallel and tangential component perpendicular to the rocker. The tangential component develops torque (bending) on the rocker and radial component...
develops compressive and tensile stresses in rocker. Therefore, the optimum value for the transmission angle is 90°. When transmission angle is less than 45°, the radial component will be greater than the tangential component. Due to this reason, the designer’s aim is to keep the minimum transmission angle (about 35°) to make smooth running and best force transmission.

3.1.3 Toggle positions

Designer has to verify that the mechanism can in fact reach all of the specified design positions without facing a limit or toggle position. A Grashof four bar planar mechanism is considered for two toggle positions as shown in Fig. 3.6. Which also depicts the crank (link 2) is collinear with the coupler (link 3), either extended collinear (APQD) or overlapping collinear (ABCD). These toggle positions also specify the limits of motion of rocker (link 4).

3.1.4 Velocity analysis

Once a position analysis is over, the next step is to determine the velocities of all links and points of interest in the mechanism. There are different methods and approaches are available to determine the velocities of each link of mechanism. Few of them are mention here:

- Relative velocity method (Graphical)
Instant centers method (Graphical)

Analytical method

To calculate the velocities of links at different interval of crank rotation, analytical method can be more suitable for computer programming. Fig. 3.7 shows the position vectors loop for four bar mechanism with velocity vectors. An input angular velocity $\omega_2$ applied to crank (link 2) and this can be time varying input velocity. The solution of velocity vector loop equations give the angular velocity of links 3 and 4, i.e. coupler and rocker. The angular velocities $\omega_3$ and $\omega_4$ of the coupler and rocker are obtained by Eqs. (3.7) and (3.8) are given below:

\[
\omega_3 = \frac{a \times \omega_2 \times \sin(\theta_4 - \theta_2)}{b \times \sin(\theta_3 - \theta_4)} 
\]

\[
\omega_4 = \frac{a \times \omega_2 \times \sin(\theta_2 - \theta_3)}{c \times \sin(\theta_4 - \theta_3)} 
\]

Fig. 3.7 Four bar mechanism with velocity vectors

The angular velocities of all the links are calculated than it is easy to calculate the velocity of any point on any link for any position of mechanism.

Fig. 3.8 shows the four bar mechanism to contain a point $P$, on coupler, point $S$ on crank and point $U$ on rocker.
The algebraic equations for the velocities of these points on links can be written as:

\[ v_s = s \times \omega_2 [-\sin(\theta_2 + \delta_2) + j \times \cos(\theta_2 + \delta_2)] \]  
\[ v_u = u \times \omega_4 [-\sin(\theta_4 + \delta_4) + j \times \cos(\theta_4 + \delta_4)] \]  
\[ v_p = v_A + v_{PA} \]  
where, \[ v_A = a \times \omega_2 (-\sin \theta_2 + j \times \cos \theta_2) \]  
\[ v_{PA} = p \times \omega_3 [-\sin(\theta_3 + \delta_3) + j \times \cos(\theta_3 + \delta_3)] \]

3.1.5 Acceleration analysis

As acceleration plays a role to induce the dynamic force, it is necessary to determine the linear/angular acceleration of all the links and points of interest in the mechanism. The analytical solution for acceleration in four bar mechanism can be derived by solving the vector loop equation of acceleration. The input angular acceleration \( \alpha_2 \) applied to link 2 (crank) as shown in Fig. 3.9.

The angular acceleration of the coupler, \( \alpha_3 \) and angular acceleration of rocker, \( \alpha_4 \) is derived by differentiating their angular velocity with respect to time as follows:
The acceleration of any point on any link for any input position of the linkage can be calculated as given in following equations.

The acceleration of point $S$ on crank is

$$a_s = s \times \alpha_2 \left[ -\sin(\theta_2 + \delta_2) + j \times \cos(\theta_2 + \delta_2) \right]$$
$$- s \times \omega^2 \left[ \cos(\theta_2 + \delta_2) + j \times \sin(\theta_2 + \delta_2) \right] \quad (3.14)$$
The acceleration of point $U$ on rocker or output link is

$$a_u = u \times \alpha_4 \left[- \sin(\theta_4 + \delta_4) + j \times \cos(\theta_4 + \delta_4)\right]$$

$$- u \times \omega_2^2 \left[\cos(\theta_4 + \delta_4) + j \times \sin(\theta_4 + \delta_4)\right]$$

The acceleration of point $P$ on coupler is

$$a_p = a_A + a_{pA}$$

where,

$$a_A = a \times \alpha_2 \left(\sin \theta_2 + j \times \cos \theta_2\right) - a \times \omega_2^2 \left(\cos \theta_2 + j \times \sin \theta_2\right)$$

$$a_{pA} = p \times \alpha_3 \left[- \sin(\theta_3 + \delta_3) + j \times \cos(\theta_3 + \delta_3)\right] - p \times \omega_3^2 \left[\cos(\theta_3 + \delta_3) + j \times \sin(\theta_3 + \delta_3)\right]$$

### 3.2 Dynamic analysis

Dynamic force analysis involves the application of Newton’s three laws of motion. The second law is expressed in terms of rate of change of momentum,

$$M = m \times \ddot{v}$$

$$F = m \times \ddot{a}$$

(3.17)
We can differentiate between two subclasses of dynamics analysis depending upon which quantities are known and which are to be found. The “forward dynamics analysis” is the one in which we know everything about the forces and/or torques being exerted on the system, and we wish to determine the accelerations, velocities, and displacements which result from the application of those forces and torques. Given ‘F’ and ‘m’, solve for ‘a’.

The second subclass of dynamics analysis, called the “inverse dynamics analysis” is one in which we know the (desired) accelerations, velocities, and displacements to be imposed upon our system and wish to solve for the magnitudes and directions of the forces and torques which are necessary to provide the desired motions and which result from them. This inverse dynamics case is sometimes also called kinetostatics. Given ‘a’ and ‘m’, solve for ‘F’ [70].

It is often well-situated in dynamic analysis to create a simplified model of a complicated component. These models are sometimes considered to be a collection of point masses connected by massless rods. For a model of a rigid body to be dynamically equivalent to the original body, the following three conditions must be satisfied [70]:

- The mass of the model must equal that of the original body.
- The center of gravity must be in the same location as that of the original body.
- The mass moment of inertia must equal that of the original body.

In static-dynamic force analysis, mechanisms are assumed to be comprising of rigid links connected by frictionless joint without clearance. The angular acceleration of the moving links can be calculated by the methods discuss in section 1.2.3. The objective of the static dynamic analysis is to determine the forces at pin joints and the required input crank torque at different crank angle. Generally dynamic force analysis can be done by the following two techniques:

- The superposition method
- The linear simultaneous equation solution method
Here, the linear simultaneous equations solution method is used. In this method, all the relevant equations for the mechanism as a set of linear simultaneous equations are represented in matrix form. These equations can then be solved to obtain the results.

Four bar planar mechanism as shown in Fig. 3.11 represents dimensions of all the link, link position, and locations of the links’ center of gravity (CG). The angular velocities and accelerations and linear accelerations of all those CGs of all links at different crank position have been calculated from kinematic analysis discuss in section 1.1.

Torque requires for driving the mechanism is mainly depends on mass moment of inertia of each link from the Newton’s second law of motion can be expressed as:

\[ T = I \times \alpha \]  

(3.18)

where, \( T \) is torque, \( \alpha \) is angular acceleration and \( I \) is mass moment of inertia as \( I = m \times k^2 \)  where, \( m \) is the mass of the link and \( k \) is the radius of gyration.
From the above equations it is clear that torque require to drive the mechanism is depending upon the mass and radius of gyration of each link. Forces acting at all pin joints as shown in Figs. 3.11 and 3.12. D’Alembert principal is used in kineto-static analysis of mechanism and are as follows:

\[
\sum F - m \times a = 0
\]

\[
\sum T - I \times \alpha = 0
\]  \hspace{1cm} (3.19)

From Eq. (3.19), for link 2 (crank) can be written as:

\[
F_{12x} + F_{32y} = m_2 \times a_{G2x}
\]  \hspace{1cm} (3.20)

\[
F_{12y} + F_{32y} = m_2 \times a_{G2y}
\]  \hspace{1cm} (3.21)

\[
T_{12} + (R_{12x} \times F_{12y} - R_{12y} \times F_{12x}) + (R_{32x} \times F_{32y} - R_{32y} \times F_{32x}) = I_{G2} \times \alpha_2
\]  \hspace{1cm} (3.22)

Similarly for link 3 (coupler),

\[
F_{43x} + F_{32x} + F_{px} = m_3 \times a_{G3x}
\]  \hspace{1cm} (3.23)

\[
F_{43y} + F_{32y} + F_{py} = m_3 \times a_{G3y}
\]  \hspace{1cm} (3.24)

Fig. 3.12 Free-body diagrams [70]
\[
\left( R_{43x} \times F_{43y} - R_{43y} \times F_{43x} \right) - \left( R_{23x} \times F_{32y} - R_{23y} \times F_{32x} \right) + \left( R_{px} \times F_{py} - R_{py} \times F_{px} \right) = I_{G3} \times \alpha_3
\]  
(3.25)

and for link 4 (rocker),
\[
F_{14x} + F_{43x} = m_4 \times a_{G4x}
\]  
(3.26)
\[
F_{14y} + F_{43y} = m_4 \times a_{G4y}
\]  
(3.27)
\[
\left( R_{14x} \times F_{14y} - R_{14y} \times F_{14x} \right) - \left( R_{34x} \times F_{34y} - R_{34y} \times F_{34x} \right) + T_4 = I_{G4} \times \alpha_4
\]  
(3.28)

There are nine unknowns present in Eqs. (3.20-3.28), and \( F_{12x}, F_{12y}, F_{32x}, F_{32y}, F_{43x}, F_{43y}, F_{14x}, F_{14y} \) and \( T_{12} \). These nine equations are represented in matrix form as (Eq. 3.29):

\[
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-R_{12y} & R_{12x} & -R_{12y} & R_{32x} & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & R_{23y} & -R_{23x} & -R_{43y} & R_{43x} & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & R_{34y} & -R_{34x} & -R_{14y} & R_{14x} & 0 \\
\end{bmatrix} \times \begin{bmatrix}
F_{12x} \\
F_{12y} \\
F_{32x} \\
F_{32y} \\
F_{43x} \\
F_{43y} \\
F_{14x} \\
F_{14y} \\
T_{12}
\end{bmatrix} = \begin{bmatrix}
m_2 a_{G2x} \\
m_2 a_{G2y} \\
I_{G2} \alpha_2 \\
m_3 a_{G3x} - F_{px} \\
m_3 a_{G3y} - F_{py} \\
I_{G3} \alpha_3 - R_{px} F_{py} + R_{py} F_{px} \\
m_4 a_{G4x} \\
m_4 a_{G4y} \\
I_{G4} \alpha_4 - T_4
\end{bmatrix}
\]  
(3.29)

The Eq. (3.29), then can be solved for the above mentioned unknowns.

This kinematic analysis is useful to find the velocity and acceleration of the mechanisms at the different position of the links. Moreover, this kinematic analysis is prerequisite for the dynamic analysis to find the joint forces and torque required for the various positions of manipulators.