CHAPTER III

Non Null Electromagnetic Fields and the
Compacted Spin Coefficient Formalism

1. Introduction

It is known that the Newman-Penrose formalism [53] can successfully be used in tackling a number of problems in general theory of relativity. An extension to this formalism has been given by Geroch, Held and Penrose (cf. Chapter I). This formalism is more concise and efficient than the widely known NP formalism, however, the GHP formalism failed to develope its full potential to the extent to which the NP formalism has. About twenty five years ago, soon after the appearance of GHP formalism, Held ([34], [35]) proposed a simple procedure for integration within this formalism and applied it successfully to Petrov type D vacuum metrics. The geometrical meanings of the spin coefficients appearing in this formalism have been given by Ah-san and Malik [2] (cf. Chapter II). Recently the GHP formalism has again attracted the attention of several workers and in this connection, Ludwig [47] has given an extension to this formalism by considering the quantities that transform properly under all diagonal transformation of the underlying spin-frame, i.e., not only under boost rotation but also under conformal scaling. The role of commutator relations in this extended formalism has been explored by Edgar [18]. On the other hand, Kolassis and Ludwig [41] have studied the space times which admit a two dimensional group of conformal motions (and in particular homothetic motion). The so called post Bianchi identities, which play a crucial role in search of Petrov type I solutions of Einstein field equations, have been studied by Ludwig [48] through GHP formalism. More recently, a complete procedure for integration within this
formalism has been given by Edgar, Ludwig and Vickers ([17], [22], [23], [24], [25], [49]).

Motivated by the applications of GHP formalism, in this Chapter the non null (non singular) electromagnetic fields have been studied using this formalism. In section 2, the Maxwell’s equations for an arbitrary type electromagnetic field as well as non null and null electromagnetic fields have been translated in the language of GHP formalism. Various properties of the congruences associated with the non null electromagnetic field have been studied here and it is seen that the expansion and twist of the congruences can be coupled together. The behaviour of the modified Lie derivative operator on the electromagnetic field bivector, Ricci tensor and metric tensor has been investigated in section 3 and a discussion of the results of this Chapter has been mentioned in section 4.

2. The Maxwell’s Equations and the Non Null Electromagnetic Fields

Let $M$ be a four dimensional Lorentzian manifold that admits a Lorentzian metric of signature $(- - - +)$. Let $Z^a = \{l^a, n^a, m^a, \bar{m}^a\}$ be the complex null tetrad satisfying the properties (1-3 - I). With these properties of the tetrad, $g_{ij}$ can be written as

$$g_{ij} = 2l_i n_j - 2m_i \bar{m}_j \quad (1)$$

In terms of the complex null tetrad $Z^a$, the electromagnetic bivector $F_{ij}$ has the following form ([13], [27])

$$F_{ij} = -2\Re \phi_1 l_i n_j + 2i \Im \phi_1 m_i \bar{m}_j + \phi_2 l_i m_j + \bar{\phi}_2 l_i \bar{m}_j - \phi_0 n_i \bar{m}_j - \bar{\phi}_0 \bar{n}_i m_j \quad (2)$$

where

$$\phi_0 = 2F_{ij} l^i m^j \ , \ \phi_1 = 2F_{ij} (l^i n^j + \bar{m}^i m^j) \ , \ \phi_2 = 2F_{ij} \bar{m}^i n^j \quad (3)$$

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are the complex Maxwell scalars, $\Re \phi_1$ and $\Im \phi_1$, respectively, denote the real and imaginary parts of $\phi_1$. The scalar $\phi_1$ describes the Coulomb component of the field, while the scalar $\phi_2$ arises from the electric dipole radiation of an accelerated charge. If acceleration is absent then $\phi_2 = 0$.

When a source term $J^i$ is present, the Maxwell’s equations

$$\nabla_j F^{ij} = \frac{1}{2} J^i, \quad \nabla_j F^{*ij} = \frac{1}{2} J^i$$

(4)

where $F^{ij}$ is a real bivector and $F^{*ij}$ is its dual, can be expressed as

$$\nabla_j N^{ij} = \frac{1}{2} J^i$$

(5)

where

$$N^{ij} = \frac{1}{2} (F^{ij} + F^{*ij}) = \phi_2 l_0 [l^i m^j] - \phi_1 (l^i n^j - m^i \tilde{m}^j) - \phi_0 n^i \tilde{n}^j$$

(6)

and $J^i$ satisfies the conservation law

$$\nabla_i J^i = 0$$

(7)

Equations (5) and (7) may be referred to as the basic equations of the electromagnetic field. From the properties of the complex null tetrad $Z^a_\mu$, we have

$$J^i = Z^{ia} J_a = Z^a_\mu \eta^{ab} J_b$$

$$= l^a J_2 + n^a J_1 - m^a \tilde{J}_4 - \tilde{m}^a J_3$$

$$= n^a I_0 + l^a I_2 - m^a I_1 - \tilde{m}^a I_1$$

(8)
where \( I_0, I_1, \bar{I}_1, I_2 \) are the source scalars. \( I_0, I_2 \) are real and \( I_1, \bar{I}_1 \) are the complex conjugates. Also from equation (8)

\[
J_iJ^i = J^2 = 2(I_0I_2 - I_1\bar{I}_1)
\] (9)

It is known that [44] the electromagnetic field admits two real invariants and they can be combined into a single complex invariant

\[
K = \frac{1}{2}(F_{ij}F^{ij} + iF_{ij}F^{*\ ij})
\] (10)

which is also equivalent to \( K = N_{ij}N^{ij} \). By employing equation (10) and the Maxwell's scalars (cf. eqn. (3)), \( K \) reduces to

\[
K = 2(\phi_0\phi_2 - \phi_1^2)
\] (11)

Recall that an electromagnetic field is non null (non singular, or, algebraically general) if \( K \neq 0 \) and null (singular, or, algebraically special) if \( K = 0 \). Depending upon the vanishing of the Maxwell’s scalars (3), the electromagnetic field can be classified as ([13])

Type A : non null \( (K \neq 0) \) \( \phi_0 = \phi_2 = 0, \phi_1 \neq 0 \)

Type B : null \( (K = 0) \) \( \phi_0 = \phi_1 = 0, \phi_2 \neq 0 \) (12)

Type C : null \( (K = 0) \) \( \phi_1 = \phi_2 = 0, \phi_0 \neq 0 \)

It may be noted that, in fact, there are just two types (types A and B). Types B and C can be transformed into each other by switching \( l^a \) and \( n^a \)
in the null basis \( \{ l^a, n^a, m^a, \bar{m}^a \} \).\(^1\) For the sake of completeness, we have mentioned here all the three types.

Equation (12) now leads to the following forms of \( F_{ij} \) (as defined by equation (2)) for non null and null electromagnetic fields, respectively.

**Type A:** \[ F_{ij} = -2 \Re \phi_1 l_{[i} n_{j]} + 2i \Im \phi_1 m_{[i} \bar{m}_{j]} \] (13)

**Type B:** \[ F_{ij} = \phi_2 l_{[i} m_{j]} + \bar{\phi}_2 l_{[i} \bar{m}_{j]} \] (14)

From equations (2), (3), (5), (6) and (19 b - I), the GHP version of the Maxwell’s equations with a source term is given by

**Lemma 1.** In the presence of a source the Maxwell’s equations for an electromagnetic field of arbitrary type are given by

\[
\begin{align*}
\mathcal{P} \phi_1 - \nabla' \phi_0 & = 2 \rho \phi_1 - \tau' \phi_0 - \kappa \phi_2 + I_0 \\
\mathcal{D} \phi_1 - \mathcal{P}' \phi_0 & = 2 \tau \phi_1 - \rho' \phi_0 - \sigma \phi_2 - I_1 \\
\mathcal{P} \phi_2 - \mathcal{D}' \phi_1 & = \rho \phi_2 - 2 \tau' \phi_1 - \sigma' \phi_0 - \bar{I}_1 \\
\mathcal{D} \phi_2 - \mathcal{P}' \phi_1 & = \tau \phi_2 - 2 \rho' \phi_1 + \kappa' \phi_0 + I_2
\end{align*}
\] (15)

\[
\mathcal{P}' I_0 + \mathcal{P} I_2 - \nabla' I_1 - \nabla' I_2 = \left( \rho' + \tilde{\rho}' \right) I_0 - \left( \tau' + \tilde{\tau}' \right) I_1 - \left( \sigma' + \tilde{\sigma}' \right) I_2 + \left( \rho + \tilde{\rho} \right) I_2
\] (16)

\(^1\) The electromagnetic field tensor \( F_{ab} \) (in spinor language) is determined by a symmetric spinor \( \Phi_{AB} \) and one can write

\[ \Phi_{AB} = \alpha_A \beta_B + \alpha_B \beta_A \]

where \( \alpha \) and \( \beta \) are spinors. If \( \alpha \) and \( \beta \) are linearly independent, electromagnetic field is said to be **algebraically general**, otherwise it is **algebraically special**. According to this terminology, in fact we are studying the algebraically general electromagnetic fields in this Chapter. However, in the literature the terms ‘non null’ and ‘null’ are commonly used for algebraically general and algebraically special electromagnetic fields, respectively.

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The spin and boost types of the Maxwell’s scalars $\phi_0, \phi_1$ and $\phi_2$ and the source scalars $I_0, I_1, I_2$ are given as follows:

$$\phi_0 = -\phi'_2 : \{2,0\}, \ \phi_1 = -\phi'_1 : \{0,0\}, \ \phi_2 = -\phi'_0 : \{-2,0\} \quad (17 \ a)$$

$$I_0 : \{1,1\}, \ I_1 : \{-1,1\}, \ I_2 : \{-1,-1\} \quad (17 \ b)$$

Also, the energy momentum tensor $T^{ab}$ satisfies the equation $\nabla_a T^{ab} = F^a_b J^b$ and hence $\nabla_a T^{ab} = 0$ requires that

$$F^a_b J^b = 0 \quad (18)$$

From Lemma 1, the source-free Maxwell’s equations for an electromagnetic field of arbitrary type are equivalent to

$$\mathcal{P}\phi_1 - \mathcal{D}'\phi_0 = 2\rho\phi_1 - \tau'\phi_0 - \kappa\phi_2 \quad (19 \ a)$$

$$\mathcal{D}\phi_1 - \mathcal{P}'\phi_0 = 2\tau\phi_1 - \rho'\phi_0 - \sigma\phi_2 \quad (19 \ b)$$

$$\mathcal{P}\phi_2 - \mathcal{D}'\phi_1 = \rho\phi_2 - 2\tau'\phi_1 - \sigma'\phi_0 \quad (19 \ c)$$

$$\mathcal{D}\phi_2 - \mathcal{P}'\phi_1 = \tau\phi_2 - 2\rho'\phi_1 + \kappa'\phi_0 \quad (19 \ d)$$

so that from equations (12) and (19), the source-free Maxwell’s equations for a non null electromagnetic field are equivalent to

$$\mathcal{P}\phi_1 = 2\rho\phi_1, \ \mathcal{D}\phi_1 = 2\tau\phi_1, \ \mathcal{D}'\phi_1 = 2\tau'\phi_1, \ \mathcal{P}'\phi_1 = 2\rho'\phi_1 \quad (20)$$
while for null electromagnetic fields of types B and C, the source-free Maxwell’s equations are, respectively, equivalent to

\[ \mathcal{P} \phi_2 = \rho \phi_2 , \quad \mathcal{D} \phi_2 = \tau \phi_2 , \quad \kappa = \sigma = 0 \quad (21) \]
\[ \mathcal{P}' \phi_0 = \rho' \phi_0 , \quad \mathcal{D}' \phi_0 = \tau' \phi_0 , \quad \kappa' = \sigma' = 0 \quad (22) \]

Since \( \phi_2 \neq 0 \) and \( \phi_0 \neq 0 \), the equations (21) and (22) incidently establish the Mariot-Robinson theorem [56], i.e., \( \kappa = \sigma = 0 \) and \( \kappa' = \sigma' = 0 \). It may be noted that in equation (21) the propagation vector is \( \mathbf{l}^a \) while in equation (22) it is \( \mathbf{n}^a \).

We shall now find the conditions that must be satisfied by the spin coefficients to admit type A (algebraically general) solutions of the Maxwell’s equations.

For the existence of a solution \( \phi \) of a non null electromagnetic field, the necessary and sufficient condition is that the commutators \([\mathcal{P}, \mathcal{D}], [\mathcal{P}, \mathcal{D}'], [\mathcal{P}', \mathcal{D}], [\mathcal{P}', \mathcal{D}'], [\mathcal{P}, \mathcal{P}'] \) and \([\mathcal{D}, \mathcal{D}']\) as computed from GHP commutators (cf. eqn.(24 - I)) agree with the commutators obtained from GHP field equations (23 - I). The agreement between the commutators exists if and only if the following equations are satisfied.

\[ \mathcal{P}' \kappa - \mathcal{D}' \sigma = (2 \tau' - \bar{\tau}) \sigma - \bar{\rho}' \kappa - 2 \Psi_1 \quad (23 \ a) \]
\[ \mathcal{P} \tau' - \mathcal{D}' \rho = \bar{\rho} \tau' + \sigma \tau - \bar{\tau}' \rho - \kappa \rho' \quad (23 \ b) \]
\[ \mathcal{P}' \tau - \mathcal{D} \rho' = \bar{\rho} \tau + \sigma \tau' - \bar{\tau}' \rho' - \kappa \rho \quad (23 \ c) \]
\[ \mathcal{P} \kappa' + \mathcal{D} \sigma' = \bar{\rho} (\tau' - \kappa') + \bar{\rho}' (\bar{\tau} - \bar{\tau}') + \sigma (2 \tau - \bar{\tau}) + \rho (\kappa' - \kappa) - \rho' \tau' - 2 \Psi_3 \quad (23 \ d) \]
\[ D\tau' - D\tau = \rho\rho' - \rho'\rho \quad (23\ e) \]
\[ \mathcal{P}\rho' - \mathcal{P}'\rho = \tau\bar{\tau}' - \tau'\bar{\tau}' \quad (23\ f) \]

The set of equations (23) has been obtained by using GHP commutators (24 - I) and GHP field equations (23 - I), e.g. equation (23 a) can be obtained as follows:

Consider the commutator \([\mathcal{P}, \mathcal{D}]\phi\) and use equation (20) to get

\[ [\mathcal{P}, \mathcal{D}]\phi = (\mathcal{PD} - \mathcal{DP})\phi = 2(\mathcal{P}\tau - \mathcal{D}\rho)\phi \quad (24) \]

But from equations (24 b - I), (17 a) and (20), we have

\[ [\mathcal{P}, \mathcal{D}]\phi = 2(\bar{\rho}'\tau + \sigma\tau' - \bar{\tau}'\rho - k\rho')\phi \quad (25) \]

Equations (24) and (25) thus lead to

\[ \mathcal{P}\tau - \mathcal{D}\rho = \bar{\rho}\tau + \sigma\tau' - \bar{\tau}'\rho - k\rho' \quad (26) \]

Now substituting the values of \(\mathcal{P}\tau\) and \(\mathcal{D}\rho\) from equations (23 c - I) and (23 d - I) in equation (26), we get after simplification

\[ \mathcal{P}'\kappa - \mathcal{D}'\sigma = 2\tau'\sigma - \bar{\tau}\sigma - \bar{\rho}'\kappa - 2\Psi_1 \]

which is equation (23 a).

The remaining equation of the set of equations (23) can be obtained by similar arguments.
Although the set of equations (23) appears to be complicated but important conclusions can be made under some special choices of the spin coefficients and we have

**Theorem 1.** Let a non-null electromagnetic field satisfy the source-free Maxwell's equations. Suppose it is possible to propagate the complex null tetrad along the null geodesic congruence \( C(l) \) and \( C(n) \) then the set of equations (23) reduces to

\[
\begin{align*}
\mathcal{D}' \sigma &= 2 \Psi_1, \\
\mathcal{D}' \sigma' &= -2 \Psi_3 \\
\mathcal{D}' \rho &= 0 = \mathcal{D} \rho' \\
\rho \rho' &= \rho' \rho \\
\mathcal{P} \rho' &= \mathcal{P}' \rho
\end{align*}
\]

**Proof.** The hypothesis of the theorem, equation (23) and the use of Theorems 3 and 4 of Chapter II immediately lead to the proof of the theorem.

**Remark:** It may be noted that equation (27) describes the propagation of the shear of the congruence \( C(l) \) and \( C(n) \). The propagation of expansion and twist is given by equation (28), while equations (29) and (30) describe the coupling of the expansion and twist.

The above coupling of expansion and twist do exist even under weaker conditions as described by the following theorem.

**Theorem 2.** Let a non-null electromagnetic field satisfy the source-free Maxwell's equations and suppose that the tetrad \( Z^a \) can be chosen such that \( \tau \) and \( \tau' \) are constant then the set of equations (23) reduces to

\[
\mathcal{D}' \sigma = -\tau' \sigma + (\tau' - \tau) \rho + \rho' \kappa + \Psi_1 - \Phi_{01}
\]
\[ D'\rho = -\bar{\rho}' - \sigma \tau + \tau' \rho + \kappa \rho' \]  
\[ D\rho' = -\bar{\rho} - \sigma \tau' - \tau' \rho' + \kappa \rho \]  
\[ D\sigma' = \bar{\rho}(\tau' - \kappa') + \tau'(\rho - \bar{\rho}') + \rho(\kappa' - \kappa) - \rho'\tau' + \sigma \tau - \Psi_3 + \Phi_2 \]  
\[ \rho \bar{\rho}' - \rho'\bar{\rho} = 0 \]  
\[ \mathcal{P}\rho' - \mathcal{P}'\rho = \tau \bar{\tau} - \tau' \bar{\tau}' \]  

**Proof.** Since \( \tau \) and \( \tau' \) are constant, equation (23 c - I) leads to

\[ \mathcal{P}'\kappa = (\tau - \tau')\rho - \sigma(\bar{\tau} - \tau') - \Psi_1 - \Phi_0 \]

Substitute it in equation (23 a) to get equation (31). As \( \tau \) and \( \tau' \) are constant, equations (23 b), (23 c) and (23 e) immediately lead to equations (32), (33) and (35) respectively. Equation (34) can be obtained by substituting equation (23 c' - I) in equation (23 d); while equation (36) is identically satisfied in view of the equations (23 f - I) and (23 f' - I). This completes the proof of the theorem.

**3. Modified Lie Derivative Operator**

In Chapters I and II it is seen that in GHP formalism, the operator \( \Theta_a \) plays a crucial role and the use of \( \Theta_a \) in place of \( \nabla_a \) enable us to eliminate the spin coefficients \( \alpha, \beta, \epsilon, \gamma \) which behave badly under boost-rotations. For the same reason, the modified Lie differentiation operator \( L_\xi \) is defined in which \( \nabla_a \) is replaced by \( \Theta_a \) and thus the modified Lie derivative of a vector \( u^a \) is given by
\[ L_\xi u^a = \xi^b \Theta_b u^a - u^b \Theta_b \xi^a \]  \hspace{1cm} (37)

Since \( \Theta_a \) may be written as
\[ \Theta_a = \nabla_a - p \zeta_a - q \tilde{\zeta}_a \]  \hspace{1cm} (38)

where
\[ \zeta_a = \gamma \xi_a + \epsilon \kappa_a - \alpha m_a - \beta \bar{m}_a \]  \hspace{1cm} (39)

The modified Lie derivative \( L_\xi \) and the Lie derivative \( \mathcal{L}_\xi \) are related by
\[ L_\xi = \mathcal{L}_\xi - \xi^a (p \zeta_a + q \tilde{\zeta}_a) \]  \hspace{1cm} (40)

The action of this operator on tetrad vectors may be found in [40].

In this section, we shall investigate the behaviour of the modified Lie derivative operator on the electromagnetic field tensor \( F_{ij} \), the Ricci tensor \( R_{ij} \) and the metric tensor \( g_{ij} \) for the non null electromagnetic fields.

From equations (18 - II), (38) and (40), the modified Lie derivative of \( F_{ij} \) with respect to the principal null direction \( l^a \) is
\[ L_l F_{ij} = \Theta_a F_{ij} l^a + F_{ia} \Theta_j l^a + F_{aj} \Theta_i l^a \]  \hspace{1cm} (41)

The definition of \( \Theta_a \) (cf. eqn.(18 - II)) now enable us to write equation (41) as

\[ \text{42} \]
Using the properties of the null tetrad, equations (14 - II), (13) and (20), after lengthy calculations, equation (42) now takes the form

\[ L_i F_{ij} = (l_a \mathcal{P}' + n_a \mathcal{P} - m_a \mathcal{D}' - \bar{m}_a \mathcal{D}) F_{ij \ell a} + F_{ia} (l_i \mathcal{P}' + n_i \mathcal{P} - m_i \mathcal{D}' - \bar{m}_i \mathcal{D}) l^a + F_{aj} (l_i \mathcal{P}' + n_i \mathcal{P} - m_i \mathcal{D}' - \bar{m}_i \mathcal{D}) l^a \]  \hspace{1cm} (42)

which is clearly non zero for non null electromagnetic fields. However, a considerable amount of simplification results in equation (43) under the hypothesis of Theorem 1 if we consider the congruence \( C(l^a) \) to be expansion-free, and we have

**Theorem 3.** Let the null geodesic congruence \( C(l^a) \) and \( C(n^a) \) satisfy the Maxwell's equations for a non null electromagnetic filed and the tetrad is parallely propagated along the congruences. If \( C(l^a) \) is expansion-free, then

\[ L_i F_{ij} = -2i \Im \phi \{ \bar{\sigma} m_i [m_j] + \sigma \bar{m}_i [m_j] \} \] \hspace{1cm} (43)

In the spin coefficient formalism [53], the field equations

\[ R_{ij} = -\frac{1}{4\pi} (F_{ik} F^k_j - \frac{1}{4} g_{ij} F^{rs} F_{rs}) \]

for a purely electromagnetic distribution takes the following form [4] for different types
Type A : \( R_{ij} = \chi \phi_1 \bar{\phi}_1 \{ l_i n_j + m_i \bar{m}_j \} \) \hspace{1cm} (45a)

Type B : \( R_{ij} = \frac{1}{2} \chi \phi_2 \bar{\phi}_2 \ l_i l_j \) \hspace{1cm} (45b)

Type C : \( R_{ij} = \frac{1}{2} \chi \phi_0 \bar{\phi}_0 \ n_i n_j \) \hspace{1cm} (45c)

It may be noted that equations (45b) and (45c) are the well known Lichnerowicz conditions [3] for total gravitational radiation having \( l_i \) and \( n_i \), respectively, as the propagation vectors.

From equations (14 - II), (45a) and (20) it is not hard to obtain (although the calculations are very lengthy)

\[
L_l \ R_{ij} = \chi \left\{ 2(\rho + \bar{\rho})l_i (n_j) + (\bar{\tau} - 2\tau')l_i (m_j) + (\tau - 2\bar{\tau}')l_i (\bar{m}_j) - \kappa n_i (m_j) \\
- \kappa n_i (\bar{m}_j) - \bar{\sigma} m_i (m_j) - \sigma \bar{m}_i (\bar{m}_j) + (\rho + \bar{\rho}) m_i (\bar{m}_j) \right\} \phi \bar{\phi} \quad (46)
\]

for non null electromagnetic fields.

Under some special circumstances equation (46) do admit a simpler form and we have

**Theorem 4.** Let the null geodesic congruences \( C(l^a) \) and \( C(n^a) \) satisfy the Maxwell's equations for non null electromagnetic fields. If the tetrad is parallely propagated along \( C(l^a) \) and \( C(n^a) \), and \( C(l^a) \) is expansion-free, then

\[
L_l \ R_{ij} = -\chi \left\{ \bar{\sigma} m_i (m_j) + \sigma \bar{m}_i (\bar{m}_j) \right\} \phi \bar{\phi} \quad (47)
\]

which is non zero as \( \sigma \neq 0 \).

Finally, from the definition of modified Lie derivative, we have

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\[ L_t g_{ij} = \Theta_i l_j + \Theta_j l_i \]

which on using equation (19 - II) reduces to

\[
L_t g_{ij} = 2\left\{ -\tau l_i (m_{mj}) - \tau l_i (\tilde{m}_{mj}) - \kappa n_i (m_{mj}) - \kappa n_i (\tilde{m}_{mj}) + \sigma m_i (m_{mj}) + \sigma m_i (\tilde{m}_{mj}) + \rho m_i (\tilde{m}_{mj}) + \tilde{\rho} m_i m_{mj} \right\} \tag{48}
\]

so that we have

**Theorem 5.** Let the null geodesic congruences \( C(l^n) \) and \( C(n^n) \) satisfy the Maxwell’s equations for non null electromagnetic fields and the tetrad is parallely propagated along the congruences, then

\[
L_t g_{ij} = 2\left\{ \sigma m_i (m_{mj}) + \sigma m_i (\tilde{m}_{mj}) + \rho m_i (\tilde{m}_{mj}) + \tilde{\rho} m_i m_{mj} \right\} \tag{49}
\]

**Remark:** It is interesting to note that for the Reissner-Nördstrom black hole \([12]\), equations (43), (46) and (48) take the following forms respectively

\[
L_t F_{ij} = 2\left\{ \Re (\rho + \bar{\rho}) - 2\Re (\rho) l_i n_{mj} + 2i \Im (\rho) m_i (\tilde{m}_{mj}) \right\}\phi
-4i\Im (\rho + \bar{\rho}) m_i (\tilde{m}_{mj}) \tag{50}
\]

\[
L_t R_{ij} = \chi (\rho + \bar{\rho}) \left\{ 2l_i (n_{mj}) + m_i (\tilde{m}_{mj}) \right\}\phi \bar{\phi} \tag{51}
\]

\[
L_t g_{ij} = 2(\rho + \bar{\rho}) m_i (\tilde{m}_{mj}) \tag{52}
\]

These equations (50) - (52) suggest that for the Reissner-Nördstrom black hole, the modified Lie derivatives of the electromagnetic field tensor, the Ricci tensor and the metric tensor depend on the radial coordinate (as \( \rho = -1/r \)) and thus for large \( r \),
Therefore, for large $r$, the Reissner-Nördstrom black hole admits Maxwell's collineation, Ricci collineation and motion.

4. Conclusion

In this Chapter, the non null electromagnetic fields have been studied using the compacted spin coefficient formalism. The Maxwell's equations have been translated into the language of GHP formalism (cf. equations (15), (19) - (22)). The equations describing the propagation of shear (equations (27), (31) and (34)), expansion and twist (equations (28), (32) and (33)) of the null congruences $C(l^a)$ and $C(n^a)$ associated with the non null electromagnetic fields have been obtained and the conditions (equations (29), (30) and (35)) under which the expansion and twist of the congruence can be coupled together have also been given. Moreover, the propagation of shear (equation (27)) is seen to be related with the longitudinal wave component of the gravitational field in $n^a$ and $l^a$ directions. The role of the modified Lie derivative operator on the electromagnetic field tensor, Ricci tensor and metric tensor has been explored. For Reissner-Nördstrom black hole these derivatives are seen to depend only on one spin coefficient $\rho$ (cf. equations (50) - (52)).