CHAPTER 3

EXISTENCE OF ALMOST AUTOMORPHIC MILD SOLUTIONS TO NON-AUTONOMOUS NEUTRAL STOCHASTIC DIFFERENTIAL EQUATIONS

3.1 INTRODUCTION

The existence of almost automorphic, almost periodic, asymptotically almost periodic and pseudo almost periodic solutions are certainly one of the most attractive topics in qualitative theory of differential equations due to their significance and various applications. The concept of almost automorphy is a natural generalization of the notion of almost periodicity. Since then, such a topic has generated several developments and extensions. For the most recent developments, we refer the reader to (Boukli-Hacene & Ezzinbi 2011, Chang et al 2011e, Chang et al 2012, Ding et al 2010, Zhang et al 2012). More specifically, in (Liu & Song 2010), the authors introduced the concept of weighted pseudo almost automorphic functions, which generalizes both the classical concept of almost automorphy and that of weighted pseudo almost periodicity. Mophou (2011) studied the existence and uniqueness of weighted pseudo almost automorphic mild solution to the semilinear fractional equation by means of Banach fixed point theorem. The existence and uniqueness of weighted pseudo almost
automorphic mild solutions has been studied in (Blot et al 2009). Xia & Fan (2012) established a set of sufficient conditions for the existence and uniqueness of the weighted pseudo almost automorphic solution to a class of partial neutral functional differential equations.

However, the almost automorphic solutions for non-autonomous stochastic systems is still in its initial stage (Diagana 2011a, Diagana 2012, Fan et al 2012, Fu 2012). Cui & Yan (2012) studied the existence of square mean almost automorphic mild solutions for a class of non-autonomous stochastic evolution equations in real separable Hilbert spaces. The existence and uniqueness of square mean almost automorphic mild solutions for a class of non-autonomous stochastic differential equations in a real separable Hilbert space has been established in (Chang et al 2011b). Motivated by the above consideration, we study the existence and uniqueness of square mean and weighted pseudo almost automorphic mild solutions to a class of non-autonomous stochastic neutral differential equations with infinite delay in the following abstract form

$$d[x(t) + f(t, x_t)] = [A(t)x(t) + g(t, x_t)]dt + \sigma(t, x_t)dW(t), \quad t \in \mathbb{R}, \quad (3.1)$$

where $A(t) : D(A(t)) \subset L^2(\mathbb{P}, \mathbb{H}) \to L^2(\mathbb{P}, \mathbb{H})$ is a family of densely defined closed linear operators satisfying the so called “Acquistapace-Terrani” conditions; the coefficients $f, g, \sigma : \mathbb{R} \times \mathcal{B} \to L^2(\mathbb{P}, \mathbb{H})$ are appropriate functions, $\mathcal{B}$ is a abstract phase space. Also, the history $x_t : (-\infty, 0] \to L^2(\mathbb{P}, \mathbb{H})$, defined by $x_t(\theta) = x(t + \theta)$ for each $\theta \in (-\infty, 0]$. Further, $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, where $\mathcal{F}_t = \sigma\{W(u) - W(v); u, v \leq t\}$. 
In particular, $f \equiv 0$, then the equation (3.1) is called semilinear functional differential equations. However, Mishra & Bahuguna (2012) studied the existence of almost automorphic solutions to deterministic case of (3.1). Motivated by the works (Cao et al 2011, Chen & Lin 2011, Diagana 2008, Diagana 2011b, Mishra & Bahuguna 2012), in this chapter, we prove the existence and uniqueness of square mean almost automorphic and weighted pseudo almost automorphic mild solutions to (3.1). Our main result is established with the help of Banach contraction principle.

3.2 PRELIMINARIES

In this section, we provide some basic definitions, notations and lemma which will be used in this chapter. For more details, we refer the reader to (Chang et al 2011b, Chang et al 2011d).

Let $L^1_{\text{loc}}(\mathbb{R})$ denote the space of locally integrable functions on $\mathbb{R}$. $\mathbb{U}$ stands for the collection of functions (weights) $\rho : \mathbb{R} \rightarrow (0, \infty)$, which are in $L^1_{\text{loc}}(\mathbb{R})$ with $\rho > 0$ (a.e.). For a given $q > 0$ and each $\rho \in \mathbb{U}$, set

$$m(q, \rho) = \int_{-q}^{q} \rho(x)dx.$$ 

Let $\mathbb{U}_\infty$ denote the set of all $\rho \in \mathbb{U}$ with $\lim_{q \to \infty} m(q, \rho) = \infty$ and $\mathbb{U}_b$ denote the set of all $\rho \in \mathbb{U}_\infty$ such that $\rho$ is bounded with $\inf_{x \in \mathbb{R}} \rho(x) > 0$.

**Lemma 3.2.1.** (Fu & Liu 2010) If $x, x_1$ and $x_2$ are all square mean almost automorphic stochastic processes, then the following are true:

(i) $x_1 + x_2$ is square mean almost automorphic.

(ii) $\lambda x$ is square mean almost automorphic for every scalar $\lambda$. 
(iii) There exists a constant $M > 0$ such that $\sup_{t \in \mathbb{R}} ||x(t)||_2 \leq M$. That is, $x$ is bounded in $L^2(\mathbb{P}, \mathbb{H})$.

To study issues related to delay terms, we consider the new space of functions defined for each $p > 0$ by

$$BC_0(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}), p) = \left\{ x \in BC(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H})) : 
\lim_{q \to \infty} \frac{1}{2r} \int_{-q}^{q} \sup_{\theta \in [t-p,t]} (||x(\theta)||) \, dt = 0 \right\}.$$ 

In addition to the space $BC_0(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$ defined in chapter 2 and above mentioned space, the present setting requires the introduction of the following function spaces

$$BC_0(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H})) = \left\{ x \in BC(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H})) : 
\lim_{q \to \infty} \frac{1}{2r} \int_{-q}^{q} ||x(t, u)|| \, dt = 0 \right\},$$

$$BC_0(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H}), p) = \left\{ x \in BC(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H})) : 
\lim_{q \to \infty} \frac{1}{2r} \int_{-q}^{q} \sup_{\theta \in [t-p,t]} (||x(\theta, u)||) \, dt = 0 \right\},$$

where in both cases the limit (as $q \to \infty$) is uniform in compact subset of $L^2(\mathbb{P}, \mathbb{H})$.

In view of the previous definitions it is clear that $BC_0(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}), p)$ and $BC_0(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H}), p)$ are continuously embedded in $BC_0(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$ and $BC_0(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H}))$ respectively. Furthermore, it is not hard to see that $BC_0(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}), p)$ and $BC_0(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H}), p)$ are closed in $BC_0(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$ and $BC_0(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H}))$ respectively.

Now for $\rho \in \mathbb{U}_\infty$ define

$$BC_0(\mathbb{R}, \rho) = \left\{ x \in BC(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H})) : \lim_{q \to \infty} \frac{1}{m(q, \rho)} \int_{-q}^{q} ||x(t)|| \, dt = 0 \right\}.$$
Further, to investigate issues related to delay term we need the space of functions defined for each $p > 0$ by

$$BC_0(\mathbb{R}, \rho, p) = \left\{ x \in BC(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H})) : \lim_{q \to \infty} \frac{1}{m(q, \rho)} \int_{-q}^{q} \sup_{\theta \in [-p, \theta]} ||x(\theta)|| dt = 0 \right\}.$$

In addition to the above mentioned spaces, the present setting requires the introduction of the following function spaces

$$BC_0(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}), \rho, p) = \left\{ x \in BC(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H})) : \lim_{q \to \infty} \frac{1}{m(q, r)} \int_{-q}^{q} ||x(u)|| \rho(u) dt = 0 \right\},$$

$$BC_0(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}), \rho, p) = \left\{ x \in BC(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H})) : \lim_{q \to \infty} \frac{1}{m(q, r)} \int_{-q}^{q} \sup_{\theta \in [-p, \theta]} ||\rho(\theta)|| \rho(u) dt = 0 \right\},$$

where in both cases the limit (as $q \to \infty$) is uniform in compact subset of $L^2(\mathbb{P}, \mathbb{H})$.

**Definition 3.2.2.** Let $f(t) : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H}) \in BC(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$ is said to be a square mean weighted pseudo almost automorphic if it can be decomposed as $f = g + \phi$, where $g \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$ and $\phi \in BC_0(\mathbb{R}, \rho)$. The collection of all such functions will be denoted by $W P A A(\mathbb{R}, \rho)$.

**Lemma 3.2.3.** If $\rho \in \mathbb{U}_b$, $(W P A A(\mathbb{R}, \rho), || \cdot ||_{\infty})$ is a Banach space with the supremum norm given by

$$||f||_{\infty} = \left( \sup_{t \in \mathbb{R}} E||f(t)||^2 \right)^{\frac{1}{2}}.$$

**Lemma 3.2.4.** (Chang et al 2011d) Let $\Sigma_{\theta} = \{ \lambda \in \mathbb{C} : |\arg \lambda| \leq \theta \} \cup \{ 0 \} \subset \rho(A(t)), \theta \in (\frac{\pi}{2}, \pi)$ if there exists a constant $K_0$ and a set of real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \ldots, \beta_k$ with $0 \leq \beta_k < \alpha_i \leq 2, i = 1, 2, \ldots, k$ such that

$$||A(t)(\lambda - A(t))^{-1}(A(t)^{-1} - A(s)^{-1})|| \leq K_0 \sum_{i=1}^{k} (t - s)^{\alpha_i} |\lambda|^{|\beta_i| - 1},$$
for \( t, s \in T, \lambda \in \Sigma_\mathcal{A} \setminus \{0\} \) and there exists a constant \( K \geq 0 \) such that

\[
\left\| (\lambda - A(t))^{-1} \right\| \leq \frac{K}{1 + |\lambda|}, \quad \lambda \in \Sigma_\mathcal{A}.
\]

Then there exists a unique evolution family \( \{U(t, s), \ t \geq s > -\infty\} \).

Now, we recall the definition of fading memory space (phase space) \( \mathcal{B} \) axiomatically presented in (Hino & Murakami 1991, Mishra & Baluguna 2012).

Let, \( \mathcal{B} \) denote the vector space of functions \( x_t : (-\infty, 0] \to L^2(\mathbb{P}, \mathbb{H}) \), defined as \( x_t(s) = x(t + s) \) for \( s \in \mathbb{R}^- \), endowed with a seminorm denoted by \( \| \cdot \|_{\mathcal{B}} \). A Banach space \( (\mathcal{B}, \| \cdot \|_{\mathcal{B}}) \) which consists of such functions \( \psi : (-\infty, 0] \to L^2(\mathbb{P}, \mathbb{H}) \), is called a fading memory space, if it satisfies the following axioms.

(a) If \( x : (-\infty, r + a) \to L^2(\mathbb{P}, \mathbb{H}) \) with \( a > 0, r \in \mathbb{R} \), is continuous on \([r, r + a]\) and \( x_r \in \mathcal{B} \), then for each \( t \in [r, r + a] \) the following conditions hold:

(i) \( x_t \in \mathcal{B} \),

(ii) \( \|x(t)\| \leq L\|x_t\|_{\mathcal{B}} \),

(iii) \( \|x_t\|_\mathcal{B} \leq \sup\{G(t - r)\|x(s)\| : r \leq s \leq t\} + N(t - r)\|x_r\|_{\mathcal{B}} \),

where \( L > 0 \) is a constant, and \( G, N : [0, \infty) \to [1, \infty) \) are functions such that \( G(\cdot) \) and \( N(\cdot) \) are respectively continuous and locally bounded and \( L, G, N \) are independent of \( x(\cdot) \).

(aii) If \( x(\cdot) \) is a function as in (a), then \( x_t \) is a \( \mathcal{B} \) valued continuous function on \([r, r + a]\).

(aiii) The space \( \mathcal{B} \) is complete.

(aiv) If \( (\zeta^n)_{n \in \mathbb{N}} \) is a sequence of continuous functions with compact support defined
from \((-\infty, 0]\) into \(L^2(\mathbb{P}, \mathbb{H})\), which converges to \(\varsigma\) uniformly on compact subsets of \((-\infty, 0]\) and if \(\{\varsigma^n\}\) is a cauchy sequence in \(\mathcal{B}\), then \(\varsigma \in \mathcal{B}\) and \(\varsigma^n \to \varsigma\) in \(\mathcal{B}\).

**Definition 3.2.5.** (Mishra & Bahuguna 2012) Let \(S(t) : \mathcal{B} \to \mathcal{B}\) be a \(C_0\) semigroup defined by \(S(t)\varsigma(\theta) = \varsigma(0)\) on \([-t, 0]\) and \(S(t)\varsigma(\theta) = \varsigma(t + \theta)\) on \((-\infty, -t]\). The phase space \(\mathcal{B}\) is called a fading memory space if \(\|S(t)\varsigma\|_\mathcal{B} \to 0\) as \(t \to \infty\) for each \(\varsigma \in \mathcal{B}\) with \(\varsigma(0) = 0\).

Also, by axiom (aiv), there exists a constant \(\mathcal{K} > 0\) such that \(\|\varsigma\|_\mathcal{B} \leq \mathcal{K} \sup_{\theta \leq 0} \|\varsigma(\theta)\|\) for every \(\varsigma \in \mathcal{B}\) bounded continuous. Moreover, if \(\mathcal{B}\) is a fading memory, we assume that \(\max\{G(t), N(t)\} \leq \mathcal{K}\) for \(t \geq 0\). Further, it should be mentioned that the phase \(\mathcal{B}\) is a uniform fading memory space if and only if axiom (aiv) holds, the function \(G\) is bounded and \(\lim_{t \to \infty} N(t) = 0\). For more details (Mishra & Bahuguna 2012).

**Lemma 3.2.6.** (Lin et al 2010) Let \(x : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})\) be an \(\mathcal{F}_t\)-adapted measurable process such that for \(t \geq a\) the \(\mathcal{F}_t\)-adapted process \(x_a = \phi \in L^2(\Omega, \mathcal{B})\), then

\[
E\|x_a\|_\mathcal{B} \leq \mathcal{D} E\|\phi\|_\mathcal{B} + \mathcal{L} \sup_{s \in \mathbb{R}} E\|x(s)\|,
\]

where \(\mathcal{D} = \sup_{t \in \mathbb{R}} \{N(t)\}\) and \(\mathcal{L} = \sup_{t \in \mathbb{R}} \{G(t)\}\).

**Example 3.2.7.** (Diagana 2008) (The phase space \(C_r \times L^p(\rho, L^2(\mathbb{P}, \mathbb{H}))\)). Let \(r \geq 0\), \(1 \leq p < \infty\) and let \(\rho : (-\infty, -r] \mapsto \mathbb{R}\) be a nonnegative measurable function which satisfies the conditions (Hino & Murakami 1991, (g-5)-(g-6)). Basically, this means that \(\rho\) is locally integrable and there exists a nonnegative locally bounded function \(\gamma\) on \((-\infty, 0]\) such that \(\rho(\xi + \theta) \leq \gamma(\xi)\rho(\theta)\) for all \(\xi \leq 0\) with \(\theta \in (-\infty, -r)\setminus N_\xi\), where \(N_\xi \subseteq (-\infty, -r)\) is a subset whose Lebesgue measure is zero. The space \(\mathcal{B} = C_r \times L^p(\rho, L^2(\mathbb{P}, \mathbb{H}))\) consists of all classes of functions
\[ \phi : (-\infty, 0] \mapsto L^2(\mathbb{P}, \mathbb{H}) \] such that \( \phi \) is continuous on \([-r, 0]\), Lebesgue-measurable, and \( \rho^p |\phi|^p \) is Lebesgue integrable on \((-\infty, -r)\). The seminorm in \( C_r \times L^p(\rho, L^2(\mathbb{P}, \mathbb{H})) \) is defined as follows:

\[
\|\phi\|_B := \sup\{ |\phi(\theta)| : -r \leq \theta \leq 0 \} + \left( \int_{-\infty}^{-r} \rho^p(\theta) |\phi(\theta)|^p d\theta \right)^{1/p}.
\]

The space \( B = C_r \times L^p(\rho, L^2(\mathbb{P}, \mathbb{H})) \) satisfies axioms (ai)-(aiv). Moreover, when \( r = 0 \) and \( p = 2 \), one can then take \( L = 1 \), \( G(t) = \gamma(-t)^{1/2} \) and \( N(t) = 1 + \left( \int_{-t}^{0} \rho(\theta)^{1/2} d\theta \right)^{1/2} \) for \( t \geq 0 \) (Hino & Murakami 1991, Theorem 1.3.8).

It is worth mentioning that if the conditions (Hino & Murakami 1991, (g-5)-(g-7)) hold, then \( B \) is a uniform fading memory.

**Definition 3.2.8.** (Mishra & Bahluguna 2012) A continuous stochastic function \( x : \mathbb{R} \times \Omega \to L^2(\mathbb{P}, \mathbb{H}), \ a > 0, \) is called a mild solution of (3.1), provided that \( \sup_{t \in \mathbb{R}} E\|x(t)\|^2 < \infty, \) the function \( s \to A(s)U(s,t) f(s, x_s) \) is integrable on \( \mathbb{R} \) and the following conditions hold:

(i) \( x_s \in B \) for every \( s \in \mathbb{R}. \)

(ii) for \( t \geq a, \ a \in \mathbb{R}, \) \( x(t) \) satisfies the following integral equation

\[
x(t) = U(t, a)[\phi(a) + f(a, \phi)] - f(t, x_t) - \int_{a}^{t} A(s)U(t, s)f(s, x_s)ds \\
+ \int_{a}^{t} U(t, s)g(s, x_s)ds + \int_{a}^{t} U(t, s)\sigma(s, x_s)dW(s).
\]

### 3.3 Existence of Almost Automorphic Mild Solutions

In this section, we will establish the existence of square mean almost automorphic and square mean weighted pseudo almost automorphic mild solutions
for non-autonomous neutral functional differential equations with infinite delay. In particular, using definitions and lemmas given in Section 3.2, we formulate and prove conditions for square mean almost automorphic and square mean weighted pseudo almost automorphic mild solutions for the non-autonomous neutral functional stochastic differential equations. Further, it should be mentioned that the evolution family \( \{U(t, s), \ t \geq s\} \) generated by \( A(t) \) is exponentially stable that is, there exist \( K \geq 1 \) and \( \delta > 0 \) such that \( \|U(t, s)\| \leq Ke^{-\delta(t-s)} \), for \( t \geq s \).

In order to establish our result, we impose the following conditions:

(H1) Assume that the operators \( A(t) \) and \( U(r, s) \) commute; that is
\[
A(t)U(r, s) = U(r, s)A(t),
\]

(H2) The function \( s \to A(s)U(t, s) \) defined from \( [-\infty, t] \) into \( \mathcal{L}(L^2(\mathbb{P}, \mathbb{H})) \) is strongly measurable and there exists a non increasing function \( \mathcal{H} : [0, \infty) \to [0, \infty) \) and \( \nu > 0 \) with \( s \to e^{-\nu s}\mathcal{H}(s) \in L^1[0, \infty) \) such that
\[
\|A(s)U(t, s)\| \leq e^{-\nu(t-s)}\mathcal{H}(t-s), \ t > s.
\]

(H3) \( U(t, s), \ t \geq s \) satisfies the condition that, for every sequence of real numbers \( (s'_n)_{n \in \mathbb{N}} \), there exists a subsequence \( (s_n)_{n \in \mathbb{N}} \) such that for any \( \epsilon > 0 \), there exists an \( N \in \mathbb{N} \) such that, for all \( n > N \) and \( t \geq s \), it follows that
\[
\|U(t + s_n, s + s_n) - U(t, s)\| \leq \epsilon e^{-\frac{\nu}{2}(t-s)} , \text{ for all } t \geq s \in \mathbb{R},
\]
and
\[
\|U(t - s_n, s - s_n) - U(t, s)\| \leq \epsilon e^{-\frac{\nu}{2}(t-s)} , \text{ for all } t \geq s \in \mathbb{R}.
\]

(H4) For any sequence of real numbers \( (s'_n)_{n \in \mathbb{N}} \), there exists a subsequence \( (s_n)_{n \in \mathbb{N}} \) such that, for each \( \epsilon > 0 \), one can find \( N > \mathbb{N} \), such that
\[
\|A(s + s_n)U(t + s_n, s + s_n) - A(s)U(s)\| \leq \epsilon \mathcal{H}(t - s)
\]
whenever \(n > N, \ t, s \in \mathbb{R}, \ t > s\). Moreover

\[ ||A(s - s_n)U(t - s_n, s - s_n) - A(s)U(s)|| \leq c \mathcal{H}(t - s) \]

whenever \(n > N, \ t, s \in \mathbb{R}, \ t > s\).

(H5) The functions \(f, g\) and \(\sigma\) satisfy Lipschitz conditions in the second variable and uniformly in the first variable, that is there exists a positive constant \(L_f\) and continuous functions \(L_g, L_{\sigma} : \mathbb{R} \to [0, \infty)\) such that

\[
E||f(t, \phi_1) - f(t, \phi_2)||^2 \leq L_f E||\phi_1 - \phi_2||^2,
\]

\[
E||g(t, \phi_1) - g(t, \phi_2)||^2 \leq L_g(t)E||\phi_1 - \phi_2||^2,
\]

\[
E||\sigma(t, \phi_1) - \sigma(t, \phi_2)||^2 \leq L_{\sigma}(t)E||\phi_1 - \phi_2||^2,
\]

for all \(t \in \mathbb{R}, \ \phi_i \in \mathcal{B}, \ i = 1, 2\).

In order to prove our main result, we need the following lemma:

**Lemma 3.3.1.** Let \(x \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))\). If \(\Gamma\) is the function defined by

\[
\Gamma x(t) := \int_{-\infty}^{t} U(t, s)x(s)ds, \ \forall t \in \mathbb{R}, \text{ then } \Gamma x(t) \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H})).
\]

**Proof.** First we observe that \(\Gamma\) is well defined. Since \(x \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))\), then \(x\) is bounded, we assume that there exists \(C > 0\) such that \(E||x(t)||^2 < C\). Hence

\[
E||\Gamma x(t)||^2 = E \left| \left| \int_{-\infty}^{t} U(t, s)x(s)ds \right| \right|^2 \\
\leq K^2 \left( \int_{-\infty}^{t} e^{-\delta(t-s)}ds \right) \int_{-\infty}^{t} e^{-\delta(t-s)}E||x(t)||^2ds \\
\leq \frac{K^2C}{\delta^2} < \infty.
\]
This implies \( \Gamma x(t) \) is bounded. Now we show that \( \Gamma x(t) \) is almost automorphic with respect to \( \mathbb{R} \). Since \( x \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H})) \), for any sequence \((s_n)_{n \in \mathbb{N}}\) of real numbers, there exists a subsequence \((s_n)_{n \in \mathbb{N}}\) and \( \tilde{x} \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H})) \) such that

\[
\lim_{n \to \infty} E\|x(t + s_n) - \tilde{x}(t)\|^2 = 0 \quad \text{and} \quad \lim_{n \to \infty} E\|x(t) - \tilde{x}(t)\|^2 = 0.
\]

Moreover let \( \hat{\Gamma} x(t) := \int_{-\infty}^{t} U(t, s)\tilde{x}(s)ds \), we have

\[
E\|\hat{\Gamma} x(t) - \hat{\Gamma} x(t)\|^2 \\
\leq E\left\| \int_{-\infty}^{t} U(t + s_n, s + s_n)x(s + s_n)ds - \int_{-\infty}^{t} U(t, s)\tilde{x}(s)ds \right\|^2 \\
\leq 2E\left\| \int_{-\infty}^{t} U(t + s_n, s + s_n)[x(s + s_n) - \tilde{x}(s)]ds \right\|^2 \\
+ 2E\left\| \int_{-\infty}^{t} [U(t + s_n, s + s_n) - U(t, s)]\tilde{x}(s)ds \right\|^2 \\
\leq 2\left( \int_{-\infty}^{t} Ke^{-\delta(t-s)}ds \right)^2 \sup_{t \in \mathbb{R}} E\|x(t + s_n) - \tilde{x}(t)\|^2 \\
+ 2\left( \int_{-\infty}^{t} ce^{-\frac{\delta}{2}(t-s)}ds \right)^2 \sup_{t \in \mathbb{R}} E\|\tilde{x}(t)\|^2 \\
\leq \frac{2K^2}{\delta^2} \sup_{t \in \mathbb{R}} E\|x(t + s_n) - \tilde{x}(t)\|^2 + \frac{8c^2}{\delta^2} \sup_{t \in \mathbb{R}} E\|\tilde{x}(t)\|^2.
\]

For arbitrary \( c \), right hand side of the above inequality tends to zero as \( n \to \infty \).

Similarly, we can show that \( E\|\hat{\Gamma} x(t) - \Gamma x(t)\|^2 \to 0 \) as \( n \to \infty \). Thus we conclude that \( \Gamma x(t) \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H})) \). This completes the proof of the lemma.

**Lemma 3.3.2.** For \( x \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H})) \), the function \( \Gamma u \), defined by \( \Gamma x(t) := \int_{-\infty}^{t} A(s)U(t, s)x(s)ds \), \( \forall t \in \mathbb{R} \), is also square mean almost automorphic.

**Proof.** First, we note that \( \Gamma \) is well defined. Since \( x \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H})) \), then \( x \) is bounded, we assume that there exists \( C > 0 \) such that \( E\|x(t)\|^2 < C \). Hence
\[ E[|\Gamma x(t)|^2] = E\left[\left|\int_{-\infty}^{t} A(s)U(t, s)x(s)ds\right|^2\right] \]
\begin{align*}
\leq & \left(\int_{-\infty}^{t} e^{-\nu(t-s)}\mathcal{H}(t-s)ds\right)\int_{-\infty}^{t} e^{-\nu(t-s)}\mathcal{H}(t-s)E\|x(t)\|^2ds \\
\leq & C\left(\int_{-\infty}^{t} e^{-\nu(t-s)}\mathcal{H}(t-s)ds\right)^2 \\
\leq & (\zeta_n^H)^2 C < \infty
\end{align*}

with \((\zeta_n^H)^2 = \left(\sup_{t \in \mathbb{R}} \int_{-\infty}^{t} e^{-\nu(t-s)}\mathcal{H}(t-s)ds\right)^2 < \infty\). Hence \(\Gamma x(t)\) is bounded. Now, we show that \(\Gamma x(t)\) is square mean almost automorphic with respect to \(\mathbb{R}\).

Since \(x \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))\), for any sequence \((s'_n)_{n \in \mathbb{N}}\) of real numbers, there exists a subsequence \((s_n)_{n \in \mathbb{N}}\) and \(\tilde{x} \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))\) such that \(\lim_{n \to \infty} E\|x(t + s_n) - \tilde{x}(t)\|^2 = 0\) and \(\lim_{n \to \infty} E\|\tilde{x}(t-s_n) - x(t)\|^2 = 0\). Moreover let \(\tilde{\Gamma} x(t) := \int_{-\infty}^{t} A(s)U(t, s)\tilde{x}(s)ds\), we have

\[ E[\|\tilde{\Gamma} x(t + s_n) - \tilde{\Gamma} x(t)\|^2] \]
\begin{align*}
\leq & 2E\left[\left|\int_{-\infty}^{t} A(s + s_n)U(t + s_n, s + s_n)[x(s + s_n) - \tilde{x}(s)]ds\right|^2\right] \\
& + 2E\left[\left|\int_{-\infty}^{t} [A(s + s_n)U(t + s_n, s + s_n) - A(s)U(t, s)]\tilde{x}(s)ds\right|^2\right] \\
\leq & 2\left(\int_{-\infty}^{t} e^{-\nu(t-s)}\mathcal{H}(t-s)ds\right)\int_{-\infty}^{t} e^{-\nu(t-s)}\mathcal{H}(t-s)E\|x(s + s_n) - \tilde{x}(s)\|^2ds \\
& + 2\left(\int_{-\infty}^{t} e^{-\nu(t-s)}\mathcal{H}(t-s)ds\right)\int_{-\infty}^{t} e^{-\nu(t-s)}\mathcal{H}(t-s)E\|\tilde{x}(s)\|^2ds \\
\leq & 2(\zeta_n^H)^2 \sup_{t \in \mathbb{R}} E\|x(t + s_n) - \tilde{x}(t)\|^2 + 2\nu^2 \left(\int_{-\infty}^{t} e^{-\nu(t-s)}\mathcal{H}(t-s)ds\right)^2 \sup_{t \in \mathbb{R}} E\|\tilde{x}(t)\|^2.
\end{align*}

For arbitrary \(\nu\) and \(\left(\int_{-\infty}^{t} e^{-\nu(t-s)}\mathcal{H}(t-s)ds\right)^2 < \infty\), right hand side of the above inequality tends to zero as \(n \to \infty\). Arguing in a similar way, we can get \(E[|\tilde{\Gamma} x(t - s_n) - \tilde{\Gamma} x(t)|^2] \to 0\) as \(n \to \infty\). This implies that \(\Gamma x(t) \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))\).
The proof is complete.

**Lemma 3.3.3.** For \( x \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H})) \), the function \( \Gamma x \), defined by \( \Gamma x(t) := \int_{-\infty}^{t} U(t, s)x(s)dW(s) \), \( \forall t \in \mathbb{R} \), is also square mean almost automorphic.

**Proof.** The proof of this lemma is similar to that of Lemma 3.3.1 and hence it is omitted.

**Lemma 3.3.4.** (Diagana 2012) Suppose (H2) hold and \( x(t) \in BC_0(\mathbb{R}, \rho) \), then the function \( \Gamma x(t) := \int_{-\infty}^{t} A(s)U(t, s)x(s)ds \) is in \( BC_0(\mathbb{R}, \rho) \).

**Proof.** Let \( \mathcal{H}_p = \sup_{s \geq p} \mathcal{H}(s) \). For a positive number \( p \), we obtain

\[
\frac{1}{m(q, \rho)} \int_{-q}^{q} \sup_{\theta \in [t-p, t]} E \left\| \int_{-\infty}^{\theta} A(s)U(\theta, s)x(s)ds \right\|^2 \rho(t)dt \\
\leq \frac{1}{m(q, \rho)} \int_{-q}^{q} \left( \sup_{\theta \in [t-p, t]} \int_{-\infty}^{\theta} e^{-\gamma(\theta-s)} \mathcal{H}((\theta-s)ds) \right) \\
\times \left( \sup_{\theta \in [t-p, t]} \int_{-\infty}^{\theta} e^{-\gamma(\theta-s)} \|x(s)\|^2 ds \right) \rho(t)dt \\
\leq \frac{1}{m(q, \rho)} \int_{-q}^{q} \left( \mathcal{H}_p \sup_{\theta \in [t-p, t]} \int_{-\infty}^{\theta-p} e^{-\gamma(\theta-s)}ds + \sup_{\theta \in [t-p, t]} \int_{-\theta-p}^{\theta} \mathcal{H}((\theta-s)ds) \right) \\
\times \left( \mathcal{H}_p \sup_{\theta \in [t-p, t]} \int_{-\infty}^{\theta-p} e^{-\gamma(\theta-s)} \|x(s)\|^2 ds + \sup_{\theta \in [t-p, t]} \int_{-\theta-p}^{\theta} \mathcal{H}((\theta-s)ds) \right) \rho(t)dt \\
\leq \left( \mathcal{H}_p \frac{e^{\gamma p}}{p} + \int_{0}^{p} \mathcal{H}(s)ds \right) \left( \mathcal{H}_p \frac{e^{\gamma p}}{m(q, \rho)} \int_{-\infty}^{q} \int_{-\infty}^{q} e^{-\gamma(\theta-s)}E\|x(s)\|^2 \rho(s)dsdt \right) \\
+ \mathcal{H}_p \frac{e^{\gamma p}}{m(q, \rho)} \int_{-q}^{q} \int_{s}^{q} e^{-\gamma(\theta-s)}E\|x(s)\|^2 \rho(s)dsdt \\\n+ \int_{0}^{p} \mathcal{H}(s)ds \frac{1}{m(q, \rho)} \int_{-q}^{q} \sup_{\theta \in [t-p, t]} E\|x(\theta)\|^2 \rho(t)dt \right) \\
\leq \left( \frac{e^{\gamma p}}{p} + \int_{0}^{p} \mathcal{H}(s)ds \right) \left( \frac{e^{\gamma p}}{m(q, \rho)} \sup_{t \in \mathbb{R}} E\|x(t)\|^2 \|\rho\|_{L^\infty(\mathbb{R})} \int_{-\infty}^{q} e^{\gamma(q+s)}ds \right) \\
+ \frac{e^{\gamma p}}{m(q, \rho)} \int_{-q}^{q} E\|x(s)\|^2 \rho(s)ds + \int_{0}^{p} \mathcal{H}(s)ds \frac{1}{m(q, \rho)} \int_{-q}^{q} \sup_{\theta \in [t-p, t]} E\|x(\theta)\|^2 \rho(t)dt \right)
\[
\leq \left( \frac{e^{\nu p}}{\nu} + \int_0^p \mathcal{H}(s) ds \right) \left( \frac{e^{\nu p}}{m(q, \rho) \nu} \sup_{t \in \mathbb{R}} E\|x(t)\|^2 \rho(t) \right)_{L_{\text{loc}}(\mathbb{R})} \\
+ \frac{e^{\nu p}}{m(q, \rho) \nu} \int_{-q}^q E\|x(s)\|^2 \rho(s) ds + \int_0^p \mathcal{H}(s) ds \frac{1}{m(q, \rho)} \int_{-q}^q \sup_{\theta \in [-p, t]} E\|x(\theta)\|^2 \rho(t) dt \right).
\]

Since \( p \) is arbitrary, the right hand side of above inequality tend to zero as \( q \to \infty \).

The proof is complete.

The following Lemma can be seen as immediate consequence of the above Lemma 3.3.4.

**Lemma 3.3.5.** Suppose (H1) hold. If \( x(t) \in BC_0(\mathbb{R}, \rho) \), then the function \( \Gamma x(t) := \int_{-\infty}^t U(t, s) x(s) ds \) in \( BC_0(\mathbb{R}, \rho) \).

**Lemma 3.3.6.** (Diagana 2011a) Let \( x(t) \in BC_0(\mathbb{R}, \rho) \), then the function \( \Gamma x(t) \) is defined by \( \Gamma x(t) := \int_{-\infty}^t U(t, s) x(s) dW(s) \) which also belongs to \( BC_0(\mathbb{R}, \rho) \).

**Proof.** For positive numbers \( p \) and \( q \), we find that

\[
\frac{1}{m(q, \rho)} \int_{-q}^q \sup_{\theta \in [-p, t]} \|U(\theta, t) x(t) dW(s)\|^2 \rho(t) dt \\
\leq \frac{K^2}{m(q, \rho)} \int_{-q}^q \sup_{\theta \in [-p, t]} \int_{-\infty}^\theta e^{-2\delta(\theta - s)} E\|x(s)\|^2 \rho(s) ds dt \\
\leq \frac{K^2}{m(q, \rho)} e^{-2\delta p} \int_{-q}^q \int_{-\infty}^\theta e^{-2\delta(t - s)} E\|x(s)\|^2 \rho(s) ds dt \\
\leq \frac{K^2}{m(q, \rho)} e^{-2\delta p} \sup_{\theta \in \mathbb{R}} \int_{-\infty}^\theta E\|x(s)\|^2 \rho(s) ds \\
+ \frac{K^2}{m(q, \rho)} e^{-2\delta p} \frac{2\delta}{2\delta} \int_{-q}^q E\|x(s)\|^2 \rho(s) ds \\
\leq \frac{K^2}{m(q, \rho)} e^{-2\delta p} \frac{2\delta}{4\delta^2} \sup_{\theta \in \mathbb{R}} \int_{-\infty}^\theta E\|x(s)\|^2 \rho(s) ds \\
+ \frac{K^2}{m(q, \rho)} e^{-2\delta p} \frac{2\delta}{2\delta} \int_{-q}^q E\|x(s)\|^2 \rho(s) ds.
\]
Since \( p \) is arbitrary, the right-hand side of the above inequality tends to zero as \( q \to \infty \). The proof is complete.

**Lemma 3.3.7.** (Diagana 2012) Let \( x \in WPAA(\mathbb{R}, \rho) \) and assume that \( B \) is a uniform fading memory space. Then the function \( t \to x_t \) belongs to \( WPAA(\mathbb{R}, \rho) \).

**Proof.** Assume that \( x = h + g \) where \( h \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H})) \) and \( g \in BC_0(\mathbb{R}, \rho) \).

Clearly, \( x_t = h_t + g_t \) and from (Lemma 3.6, Mishra & Bahuguna 2012) we infer that \( t \to h_t \) is square mean almost automorphic.

To complete the proof, we need to prove that \( t \to g_t \in BC_0(\mathbb{R}, \rho) \). Let \( p > 0 \) and \( \epsilon > 0 \). Since \( B \) is a uniform fading memory space, there exists \( \tau_\epsilon > p \) such that \( N(\tau) < \epsilon \) for every \( \tau > \tau_\epsilon \) and consequently \( \mathfrak{D} = \sup N(\tau) < \epsilon \). Under these conditions, for \( q > 0 \) and \( \tau > \tau_\epsilon \) we find that

\[
\frac{1}{m(q, \rho)} \int_{-q}^{q} \sup_{t \in [t-p, t]} \|g_t\|_{\mathcal{B}_\rho}^2 dt \\
\leq \frac{1}{m(q, \rho)} \int_{-q}^{q} \left( \sup_{t \in [-p, t]} \mathfrak{D} \|g_{t-p}\|_{\mathcal{B}}^2 + \mathcal{L} \sup_{s \in [t-p, t]} E\|g(s)\|^2 \right) \rho(t) dt \\
\leq \mathfrak{R} \sup_{t \in \mathbb{R}} E\|g(t)\|^2 + \frac{\mathcal{L}}{m(q, \rho)} \int_{-q}^{q} \sup_{t \in [t-p, t]} E\|g(t)\|^2 \rho(t) dt
\]

which enables us to complete the proof, since \( \epsilon \) is arbitrary and \( g \in BC_0(\mathbb{R}, \rho) \).

**Theorem 3.3.8.** Assume that (H1)-(H5) hold. Further, if \( f, g, \sigma \) are square mean almost automorphic then there exists a unique square mean almost automorphic mild solutions to (3.1) provided that

\[
\mathfrak{R} \left( 4L_f + 4L_f(\mathfrak{C}^2) + \frac{4K^2}{\delta} \int_{-\infty}^{t} e^{-\delta(t-s)} L_g(s) + 4K^2 \int_{-\infty}^{t} e^{-2\delta(t-s)} L_g(s) \right) < 1.
\]

(3.2)
Proof. For any $x(t) \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$, define the operator $T$ by

$$
T x(t) = -f(t, x_t) - \int_{-\infty}^{t} A(s)U(s, t)f(s, x_s)ds + \int_{-\infty}^{t} U(s, t)g(s, x_s)ds
+ \int_{-\infty}^{t} U(t, s)\sigma(s, x_s)dW(s), \ t \in \mathbb{R}.
$$

It is easy to see that $T x$ is well defined and continuous. Moreover, from Lemmas 3.3.1–3.3.4 and (Lemma 3.6, Mishra & Bahuguna 2012), it follows that $T x \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$, that is, $T : AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H})) \to AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$.

Now, the remaining task is to prove that $T$ is a contraction mapping on $AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$. Indeed, for each $x, y \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$, we get

$$
E\|T x(t) - T y(t)\|^2 \leq 4E\|f(t, x_t) - f(t, y_t)\|^2
+ 4E\left|\int_{-\infty}^{t} A(s)U(t, s)[f(s, x_s) - f(s, y_s)]ds\right|^2
+ 4E\left|\int_{-\infty}^{t} U(t, s)[g(s, x_s) - g(s, y_s)]ds\right|^2
+ 4E\left|\int_{-\infty}^{t} U(t, s)[\sigma(s, x_s) - \sigma(s, y_s)]dW(s)\right|^2
$$

$$
\leq 4L_f E\|x_t - y_t\|_{\mathbb{B}}^2 + 4L_f \left(\int_{-\infty}^{t} e^{-\lambda(t-s)} \mathcal{H}(t-s)ds\right)\times \int_{-\infty}^{t} e^{-\lambda(t-s)} \mathcal{H}(t-s)E\|x_s - y_s\|_{\mathbb{B}}^2ds
+ \frac{4K^2}{\delta} \int_{-\infty}^{t} e^{-\delta(t-s)} L_{g}(s)E\|x_s - y_s\|_{\mathbb{B}}^2ds
+ 4K^2 \int_{-\infty}^{t} e^{-2\delta(t-s)} L_{\sigma}(s)E\|x_s - y_s\|_{\mathbb{B}}^2ds
$$

$$
\leq \lambda \left(4L_f + 4L_f \left(\frac{\lambda}{\delta}\right)^2 + \frac{4K^2}{\delta} \int_{-\infty}^{t} e^{-\delta(t-s)} L_{g}(s)ds\right)\sup_{t \in \mathbb{R}} E\|x - y\|^2
$$

which implies that $T$ is a contraction mapping by (3.2). Hence, by Banach fixed point
theorem, we can conclude that $T$ has a unique fixed point $x \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$ such that $Tx = x$, that is,

$$x(t) = -f(t, x_t) - \int_{-\infty}^{t} A(s)U(s, t)f(s, x_s)ds + \int_{-\infty}^{t} U(s, t)g(s, x_s)ds + \int_{-\infty}^{t} U(t, s)\sigma(s, x_s)dW(s).$$

If we let

$$x(a) = -f(a, x_a) - \int_{-\infty}^{a} A(s)U(s, a)f(s, x_s)ds + \int_{-\infty}^{a} U(s, a)g(s, x_s)ds + \int_{-\infty}^{a} U(a, s)\sigma(s, x_s)dW(s),$$

then

$$U(t, a)x(a) = -U(t, a)f(a, x_a) - \int_{-\infty}^{a} A(s)U(t, s)f(s, x_s)ds + \int_{-\infty}^{a} U(t, s)g(s, x_s)ds + \int_{-\infty}^{a} U(t, s)\sigma(s, x_s)dW(s).$$

However, for $a \leq t$, we have

$$\int_{a}^{t} U(t, s)\sigma(s, x_s)dW(s) = \int_{-\infty}^{t} U(t, s)\sigma(s, x_s)dW(s) - \int_{-\infty}^{a} U(t, s)\sigma(s, x_s)dW(s) = x(t) + f(t, x_t) + \int_{-\infty}^{t} A(s)U(t, s)f(s, x_s)ds$$

$$- \int_{-\infty}^{t} U(t, s)g(s, x_s)ds - U(t, a)[x(a) - f(a, x_a)]$$

$$- \int_{-\infty}^{a} A(s)U(t, s)f(s, x_s)ds + \int_{-\infty}^{a} U(t, s)g(s, x_s)ds + \int_{-\infty}^{a} U(t, s)\sigma(s, x_s)dW(s)$$

$$= x(t) - U(t, a)[x(a) - f(a, x_a)] + f(t, x_t) + \int_{a}^{t} A(s)U(t, s)f(s, x_s)ds - \int_{a}^{t} U(t, s)g(s, x_s)ds.$$ 

In conclusion $x(t)$ is mild solution of (3.1) and $x(\cdot) \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$. This completes the proof.
Next, we investigate the existence of weighted square mean pseudo almost automorphic solution for equation (3.1). Moreover, we need the following lemma.

**Lemma 3.3.9.** (Blot et al 2009) Let $f = g + \phi \in WPAA(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}), \rho)$, where $\rho \in U_\infty$, $g \in AA(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H}))$ and $\phi \in BC_0(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}), \rho)$. Assume both $f$ and $g$ are Lipschitzian in $x \in L^2(\mathbb{P}, \mathbb{H})$ uniformly in $t \in \mathbb{R}$. If $x(t) \in WPAA(\mathbb{R}, \rho)$ then the function $f(\cdot, x(\cdot)) \in WPAA(\mathbb{R}, \rho)$.

The above lemma is a stochastic generalized version of corollary 2.11 in (Blot et al 2009).

**Theorem 3.3.10.** Let the conditions (H1)-(H5) and (3.2) hold. Suppose that $f$, $g$ and $\sigma$ are square mean weighted pseudo almost automorphic, then the problem (3.1) has a unique square mean weighted pseudo almost automorphic mild solution.

**Proof.** Now, we define the operator $T$ as in Theorem 3.3.8. It can be easily seen that $Tx$ is well-defined and continuous. Moreover, from Lemmas 3.3.1-3.3.7 and 3.3.9, it follows that $Tx(t) \in WPAA(\mathbb{R}, \rho)$, that is, $T$ maps $WPAA(\mathbb{R}, \rho)$ into itself. The proof for the contraction property of $T$ on the space $WPAA(\mathbb{R}, \rho)$, one can easily follow along the same lines as in the proof of Theorem 3.3.8 and hence it is omitted. Thus, by Banach fixed point principle, there exists a unique fixed point which is a mild solution of (3.1). The proof is complete.

**Example 3.3.11.** Consider the following the stochastic neutral partial differential equation in the form
\[
d\left[ u(t, x) + \int_{-\infty}^{t} \int_{0}^{\bar{\sigma}} \bar{b}(t - s, \eta, x)u(s, \eta)\,d\eta\,ds \right]
= \left[ \frac{\partial^2 u(t, x)}{\partial x^2} + a_0(t, x)u(t, x) + \int_{-\infty}^{t} a_1(t - s)u(s, x)\,ds \right] \,dt
+ \int_{-\infty}^{t} a_2(t - s)u(s, x)\,dsdW(s), \, t \in \mathbb{R}, \, x \in I = [0, \pi],
\]
where \( a_1, a_2 : \mathbb{R} \rightarrow \mathbb{R} \) are continuous functions that satisfy appropriate conditions, \( W(t) \) is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\). Take \( \mathbb{H} = L^2[0, \pi] \) and \( \mathcal{B} = C_0 \times L^p(\rho, \mathbb{H}) \) (see Diagana et al 2007). Define the linear operator \( A \) by
\[
D(A) := \{ \varphi \in L^2[0, \pi] : \varphi'' \in L^2[0, \pi], \varphi(0) = \varphi(\pi) = 0 \}
\]
and
\[
A_{\varphi} = \varphi'', \quad \forall \varphi \in D(A).
\]

It is well known that \( A \) is the infinitesimal generator of an analytic semigroup \((T(t))_{t \geq 0}\) on \( \mathbb{H} \). Furthermore, \( A \) has a discrete spectrum with eigenvalues of the form \(-n^2, n \in \mathbb{N},\) and corresponding normalized eigenfunctions given by \( z_n(x) := \sqrt{\frac{1}{\pi}} \sin(nx).\)

Also, the following properties hold:

(a) \( \{z_n : n \in \mathbb{N}\} \) is an orthonormal basis for \( \mathbb{H};\)

(b) For \( \varphi \in \mathbb{H}, \quad T(t)\varphi = \sum_{n=1}^{\infty} e^{-n^2 t} \langle \varphi, z_n \rangle z_n \) and \( A_{\varphi} = -\sum_{n=1}^{\infty} n^2 \langle \varphi, z_n \rangle z_n, \) for all \( \varphi \in D(A).\)

Define the class of operators \( A(t) \) by:
\[
A(t)_{\varphi}(x) = A_{\varphi}(x) + a_0(t, x)\varphi \text{ for each } \varphi \in D(A(t)) = D(A).
\]
By assuming that \( x \mapsto a_0(t, x) \) is continuous for each \( t \in \mathbb{R} \) with \( a_0(t, x) \leq -\delta_0 \) \((\delta_0 > 0)\) for all \( t \in \mathbb{R}, x \in [0, \pi] \). Clearly, the system

\[
\begin{cases}
u'(t) = A(t)u(t), \ t \geq s, \\ u(s) = x \in \mathbb{H},
\end{cases}
\]

has an associated evolution family \((U(t, s))_{t \geq s}\) on \( \mathbb{H} \), which is given by

\[ U(t, s) \psi = T(t - s) e^{\int_s^t a_0(\tau, x)d\tau} \psi. \]

Moreover,

\[ \|U(t, s)\| \leq e^{-(1+\delta_0)(t-s)} \text{ for every } t \geq s. \]

Further, we define \( f : \mathbb{R} \times \mathcal{B} \rightarrow \mathbb{H}, \ g : \mathbb{R} \times \mathcal{B} \rightarrow \mathbb{H} \text{ and } \sigma : \mathbb{R} \times \mathcal{B} \rightarrow \mathbb{H} \) by

\[
\begin{align*}
f(t, \psi)(x) &= \int_{-\infty}^0 \int_0^\pi b(\theta, \eta, x)u(\theta, \eta)d\eta d\theta, \\
g(t, \psi)(x) &= \int_{-\infty}^0 a_1(s)\psi(s, \xi)ds, \\
\sigma(t, \psi)(x) &= \int_{-\infty}^0 a_2(s)\psi(s, \xi)ds.
\end{align*}
\]

In view of the above, the system (3.3) can be rewritten in the abstract form of (3.1). Hence, the conditions of Theorem 3.3.10 are satisfied. Therefore, by Theorem 3.3.10, the system (3.3) has a unique square mean weighted pseudo almost automorphic mild solution.

Next, we consider the neutral stochastic differential equation in the following form

\[
d[x(t) + f(t, h_1x(t))] = [A(t)x(t) + g(t, h_2x(t))]dt + \sigma(t, h_3x(t))dW(t), \ t \in \mathbb{R},
\]

(3.5)
under suitable conditions on the operator $A(t)$, the coefficient functions $f, g, \sigma : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}) \to L^2(\mathbb{P}, \mathbb{H})$ and the bounded linear operators $h_i, i = 1, 2, 3$.

Next we present the definition of mild solution to (3.5).

**Definition 3.3.12.** (Chang et al 2011c) Let $x_a$ be a $F_a$-measurable, square integrable random variable with values in $L^2(\mathbb{P}, \mathbb{H})$. A predictable process $x : \mathbb{R} \times \Omega \to L^2(\mathbb{P}, \mathbb{H})$ is called a mild solution to (3.5) if $\sup_{t \in \mathbb{R}} E \|x(t)\|^2 < \infty$ and

$$x(t) = U(t, a)[x(a) - B(t, h_1x(a))] + B(t, h_1x(t)) + \int_a^t A(s)U(t, s)B(s, h_1x(s))ds + \int_a^t U(t, s)F(s, h_2x(s))ds + \int_a^t U(t, s)G(s, h_3x(s))dW(s), \quad (3.6)$$

for all $t \geq a$ and for each $a \in \mathbb{R}$.

Moreover, we introduce an assumption to prove the required result.

(H6) The operators $h_i : L^2(\mathbb{P}, \mathbb{H}) \to L^2(\mathbb{P}, \mathbb{H})$ for $i = 1, 2, 3$, are bounded linear operators and let $\varpi := \max_{i=1,2,3} \{\|h_i\|_{L^2(\mathbb{P}, \mathbb{H})}\}$.

Further, we recall the following lemma.

**Lemma 3.3.13.** (Chang et al 2011b) Let $\mathcal{L} \in L(H)$ and assume that $f \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$, then $\mathcal{L} f \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$.

**Corollary 3.3.14.** Assume the conditions (H1)-(H6) hold. Suppose that $f, g, \sigma$ are square mean almost automorphic, then the problem (3.5) has unique square mean almost automorphic mild solution on $\mathbb{R}$ provided that

$$\Theta := \varpi^2 \left(4L_f + 4L_f(\mathcal{L}^e)^2 + \frac{4K^2}{\delta} \int_{-\infty}^t e^{-\delta(t-s)}L_g(s)ds \right. + 4K^2 \left. \int_{-\infty}^t e^{-2\delta(t-s)}L_\sigma(s)ds \right) < 1. \quad (3.7)$$
Proof. Define the operator \( \Pi : AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H})) \rightarrow C(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H})) \) by

\[
\Pi x(t) := f(t, h_1 x(t)) + \int_{-\infty}^{t} A(s) U(t, s) f(s, h_1 x(s)) \, ds
+ \int_{-\infty}^{t} U(t, s) g(s, h_2 x(s)) \, ds + \int_{-\infty}^{t} U(t, s) \sigma(s, h_3 x(s)) \, dW(s).
\] (3.8)

It can be easily seen that \( \Pi x \) is well-defined and continuous. In view of Lemmas 3.3.1-3.3.3, 3.3.13 and 2.3.2, it follows that \( \Pi \) maps \( AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H})) \) into itself. It remains to prove that \( \Pi \) is a contraction on \( AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H})) \).

For \( x, y \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H})) \), we obtain

\[
E \| \Pi x(t) - \Pi y(t) \|^2 \leq 4L_f E \| h_1 x(t) - h_1 y(t) \|^2
+ 4L_f \left( \int_{-\infty}^{t} e^{-\|t-s\|} \mathcal{H}(t-s) \, ds \right)
\times \int_{-\infty}^{t} e^{-\|t-s\|} \mathcal{H}(t-s) E \| h_1 x(s) - h_1 y(s) \|^2 \, ds
+ \frac{4K^2}{\delta} \int_{-\infty}^{t} e^{-\|t-s\|} L_g(s) E \| h_2 x(s) - h_2 y(s) \|^2 \, ds
+ 4K^2 \int_{-\infty}^{t} e^{-2\|t-s\|} L_\sigma(s) E \| h_3 x(s) - h_3 y(s) \|^2 \, ds
\leq \xi \left( 4L_f + 4L_f e^{H} \right)^2 + \frac{4K^2}{\delta} \int_{-\infty}^{t} e^{-\|t-s\|} L_g(s) \, ds
+ 4K^2 \int_{-\infty}^{t} e^{-2\|t-s\|} L_\sigma(s) \, ds \sup_{t \in \mathbb{R}} E \| x - y \|^2.
\]

By the condition (3.7), \( \Pi \) is a contraction mapping. As a consequence of Banach fixed point theorem, we deduce that \( \Pi \) has a unique fixed point \( x \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H})) \) such that \( \Pi x = x \) which means that
\[ x(t) = f(t, h_1 x(t)) + \int_{-\infty}^{t} A(s)U(t, s)f(s, h_1 x(s))ds \]
\[ + \int_{-\infty}^{t} U(t, s)g(s, h_2 x(s))ds + \int_{-\infty}^{t} U(t, s)\sigma(s, h_3 x(s))dW(s). \]

Let

\[ x(a) = f(a, h_1 x(a)) + \int_{-\infty}^{a} A(s)U(a, s)f(s, h_1 x(s))ds \]
\[ + \int_{-\infty}^{a} U(a, s)g(s, h_2 x(s))ds + \int_{-\infty}^{a} U(a, s)\sigma(s, h_3 x(s))dW(s). \]

Further, \( U(t, s) = U(t, r)U(r, s) \), for \( s \leq r \leq t \) indicates that

\[ U(t, a)x(a) = -U(t, a)f(a, h_1 x(a)) - \int_{-\infty}^{a} A(s)U(t, s)f(s, h_1 x(s))ds \]
\[ + \int_{-\infty}^{a} U(t, s)g(s, h_2 x(s))ds + \int_{-\infty}^{a} U(t, s)\sigma(s, h_3 x(s))dW(s). \]

For \( a \leq t \), we have

\[ \int_{a}^{t} U(t, s)\sigma(s, h_3 x(s))dW(s) \]
\[ = \int_{-\infty}^{t} U(t, s)\sigma(s, h_3 x(s))dW(s) - \int_{-\infty}^{a} U(t, s)\sigma(s, h_3 x(s))dW(s) \]
\[ = x(t) + f(t, x_t) + \int_{-\infty}^{t} A(s)U(t, s)f(s, h_1 x(s))ds \]
\[ - \int_{-\infty}^{t} U(t, s)g(s, h_2 x(s))ds - U(t, a)[x(a) - f(a, x_a)] \]
\[ - \int_{-\infty}^{a} A(s)U(t, s)f(s, h_1 x(s))ds + \int_{-\infty}^{a} U(t, s)g(s, h_2 x(s))ds \]
\[ = x(t) - U(t, a)[x(a) - f(a, x_a)] + f(t, x_t) \]
\[ + \int_{a}^{t} A(s)U(t, s)f(s, h_1 x(s))ds - \int_{a}^{t} U(t, s)g(s, h_2 x(s))ds. \]
This implies that \( x(t) \) is mild solution of (3.5) and \( x \in AA(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H})) \). The proof is complete.

In order to investigate the existence of unique square mean weighted pseudo almost automorphic mild solution to the problem (3.5), we need the following lemma which can be seen as an immediate consequence of (Lemma 3.4, Mophou 2011).

**Lemma 3.3.15.** Let \( \rho \in U_\infty \), \( x \in WPAA(\mathbb{R}, \rho) \) and \( h \in L(\mathbb{K}, \mathbb{H}) \). If \( y(t) = hx(t) \) for each \( t \in \mathbb{R} \), then \( y \in WPAA(\mathbb{R}, \rho) \).

**Corollary 3.3.16.** Suppose that the conditions (H1)-(H6) hold. Further, if \( f \), \( g \) and \( \sigma \) are square mean weighted pseudo almost automorphic, then the problem (3.5) has unique square mean weighted pseudo almost automorphic mild solution on \( \mathbb{R} \) whenever \( \epsilon < 1 \).

**Proof.** Define the operator \( \Pi \) on \( WPAA(\mathbb{R}, \rho) \) as in equation (3.8). By using the Lemmas 3.3.1–3.3.7, 3.3.9 and 3.3.15, it can be easily obtain that \( \Pi \) is mapping from \( WPAA(\mathbb{R}, \rho) \) into itself. Then, by following the same lines as in the proof of Theorem 3.3.10 with the help of Banach fixed point principle, we can obtain that there exists a unique fixed point \( x(\cdot) \) which is a mild solution of (3.5). This completes the proof.