CHAPTER 6

EXISTENCE RESULTS FOR SECOND ORDER
NON-AUTONOMOUS STOCHASTIC
DIFFERENTIAL EQUATIONS DRIVEN BY
FRACTIONAL BROWNIAN MOTION

6.1 INTRODUCTION

In recent years, the existence results on deterministic non-autonomous
differential equation have been extensively studied by many authors (Fu & Liu 2007,
Henríquez 2011 and references therein). More recently, there has been an increasing
interest in studying the non-autonomous stochastic differential equations (Fu 2009,
Chang et al 2011b). Fu (2009) established the results on existence and exponent
stability of solutions for a semilinear non-autonomous neutral stochastic evolution
equation with infinite delay. Moreover, when the delay is infinite, the selection of
the state (i.e. phase space) plays an important role in the study of both qualitative and
quantitative theory of stochastic differential equations. Therefore, there is a need to
discuss stochastic differential equations with infinite delay which heavily depends on a
choice of a phase space. In particular, various phase spaces have been considered and
each different phase space has needed a separate development of the theory. For more
detailed discussion on this topic, we refer the reader to the book (Hino & Murakami 1991). But the literature related to neutral stochastic differential equations in phase spaces is very limited and we refer the reader to (Ren & Sun 2010). Ren & Sakthivel (2012) studied the existence, uniqueness and stability of mild solutions to second order neutral stochastic evolution equations by means of successive approximation with Poisson jumps. More recently, Cui & Yan (2011) derived a set of sufficient conditions for the existence of mild solutions for a class of fractional neutral stochastic integrodifferential equations with infinite delay in Hilbert spaces under the non-Lipschitzian conditions.

Moreover, the fractional Brownian motion received much attention because of its huge range of potential applications in several fields like telecommunications networks, finance markets, biology and so on. The existence and uniqueness of mild solutions for a class of stochastic differential equations in a Hilbert space with a standard and cylindrical fractional Brownian motion with the Hurst parameter in the interval (1/2,1) has been studied in (Duncan et al 2002). Maslowski & Nualart (2003) studied the existence and uniqueness of a mild solution for nonlinear stochastic evolution equations in a Hilbert space driven by a cylindrical fractional Brownian motion under some regularity and boundedness conditions on the coefficients. Recently, Caraballo et al 2011 investigated the existence and uniqueness of mild solutions to stochastic delay equations driven by fractional Brownian motion with Hurst parameter $H \in (1/2,1)$. An existence and uniqueness result of mild solutions for a class of neutral stochastic differential equation with finite delay driven by a fractional Brownian motion in a Hilbert space has been investigated in (Boufoussi & Hajji 2012).
In many realistic cases, it is advantageous to treat the second order stochastic differential equations directly rather than to convert them to first order differential equations (Henríquez 2011). For instance, it is useful for engineers to model mechanical vibrations or charge on a capacitor or condenser subjected to white noise excitation through a second order stochastic differential equations (Prato & Zabczyk 2002). Moreover, non-autonomous models are important to deal with the changes in the vital rates through time as a result of environmental variation. Motivated by the above, we consider the second order non-autonomous neutral stochastic evolution equation with infinite delay driven by fractional Brownian motion in the following form

\[
\begin{cases}
  d[x'(t) - f(t, x_t)] = [A(t)x(t) + g(t, x_t)]dt + h(t, x_t)dW(t) \\
  +\sigma(t)dB^H_Q(t), \quad t \in I = [0, T], \\
  x_0 = \phi \in \mathcal{B}, \quad x'(0) = \xi,
\end{cases}
\]

(6.1)

where \( x(\cdot) \) takes value in a real separable Hilbert space \( \mathbb{H} \) with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \); \( A(t) : D(A(t)) \subset \mathbb{H} \to \mathbb{H} \) for \( t \in I \) denotes a closed linear operator which generates an evolution operator \( S(t, s) \); \( B^H_Q(t) \) is a cylindrical fractional Brownian motion with Hurst parameter \( H \in (1/2, 1) \) and \( \{W(t) : t \in I\} \) is a standard cylindrical Wiener process on \( \mathbb{K}_0; T \geq 0 \) is a fixed real number. In the sequel, let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a complete probability space and for \( t \geq 0, \mathcal{F}_t \) denote the \( \sigma \)-field generated by the random variable \( \{B^H_Q(s), W(s), s \in [0, t]\} \) and the \( \mathbb{P} \)-null sets. For equations with infinite delay, the segment \( x_t : (-\infty, 0] \to \mathbb{H} \) is defined by \( x_t(\theta) = x(t + \theta) \) for \( t \geq 0 \) belongs to the phase space \( \mathcal{B} \) which is defined axiomatically. Assume that \( f, g : I \times \mathcal{B} \to \mathbb{H}, \quad h : I \times \mathcal{B} \to \mathbb{L}^2_2 \) and \( \sigma : I \to \mathcal{L}^0_{Q}(\mathbb{K}, \mathbb{H}) \) are appropriate mappings specified later, here the spaces \( \mathbb{L}^2_2 \) and \( \mathcal{L}^0_{Q}(\mathbb{K}, \mathbb{H}) \) are defined in next section. The initial data \( \phi = \{\phi(t) : -\infty < t \leq 0\} \) is an \( \mathcal{F}_0 \) measurable, \( \mathcal{B} \)-valued stochastic
process independent of the Wiener process $W$. Further, we assume that $W$ and $B^H$ are independent.

It should be mentioned that the considered model (6.1) is new. However, to prove the existence result we borrow preliminaries from (Caraballo et al 2011) and follow the techniques similar to that of (Ren & Sakthivel 2012) with some necessary modifications so as to be compatible with fractional Brownian motion. More precisely, in this chapter we shall formulate and prove set of sufficient conditions for ensuring the existence, uniqueness and stability for second order non-autonomous nonlinear neutral stochastic evolution equations driven by fractional Brownian motion. First, the existence and uniqueness theorem of solutions to the stochastic equations (6.1) is proven by using the successive approximation procedure. Furthermore, the continuity of solutions to (6.1) with respect to the initial data is analyzed by using the Bihari inequality. Finally, an example is presented to illustrate the abstract theory.

\section{6.2 Preliminaries}

In this section, we provide some preliminaries needed to establish main results. For details of this section, we refer the reader to (Caraballo et al 2011, Hino & Murakami 1991, Ren & Sakthivel 2012 and references therein). Throughout this chapter, unless otherwise specified, we use the following notations.

Let $\mathbb{K}_0$ be an arbitrary separable Hilbert space and $L^2_0(\mathbb{K}_0, \mathbb{H})$ be a separable Hilbert space with respect to the Hilbert-Schmidt norm $\|\cdot\|_{L^2_0}$. Let $L^0_Q(\mathbb{K}, \mathbb{H})$ be the space of all $\psi \in L(\mathbb{K}, \mathbb{H})$ such that $\psi Q^1$ is a Hilbert-Schmidt operator. The norm is given by $\|\psi\|^2_{L^0_Q} = \left\| \psi Q^1 \right\|^2 = tr(\psi Q \psi^T)$. Then $\psi$ is called a $Q$-Hilbert-Schmidt operator from $\mathbb{K}$ to $\mathbb{H}$. In the sequel, $L^0_Q(\Omega, \mathbb{H})$ denotes the space of $\mathcal{F}_0$-measurable, $\mathbb{H}$-valued and square integrable stochastic processes.
Consider a time interval $[0, T]$ with arbitrary fixed horizon $T$ and let
\[ \{ \beta^H(t), t \in [0, T] \} \] be the one-dimensional fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. By definition, it means that $\beta^H$ is a centered Gaussian process with the covariance function:
\[ R_H(r, s) = \frac{1}{2} \left( s^{2H} + r^{2H} - |s - r|^{2H} \right). \]
Further, $\beta^H$ has the following Wiener integral representation:
\[ \beta^H(t) = \int_0^t K_H(t, s)d\beta(s), \]
where $\beta = \{ \beta(t) : t \in [0, T] \}$ is a Wiener process and $K_H(t, s)$ is the kernel given by
\[ K_H(t, s) = c_H s^{\frac{1}{2} - H} \int_s^t (x - s)^{H - 3/2} x^{H - 1/2} du \quad \text{for } t > s, \]
here $c_H = \sqrt{\frac{H(2H-1)}{B(2-2H, H-\frac{1}{2})}}$ with $B(\cdot)$ represents the Beta function. We take $K_H(t, s) = 0$ if $t \leq s$.

Let $\mathcal{H}$ be reproducing kernel Hilbert space of the fractional Brownian motion. Also, $\mathcal{H}$ is the closure of set of indicator functions $\{ \chi_{[0, t]}, t \in [0, T] \}$ with respect to the scalar product $\langle \chi_{[0, t]}, \chi_{[0, s]} \rangle_{\mathcal{H}} = R_H(t, s)$. The mapping $\chi_{[0, t]} \mapsto \beta^H(t)$ can be extended to an isometry between $\mathcal{H}$ and first Wiener chaos. Denote by $\beta^H(\varphi)$ the image of $\varphi$ by the previous isometry. Let us define the operator $K_H^t$ from $\mathcal{H}$ to $L^2([0, T])$ by
\[ (K_H^t \varphi)(s) = \int_s^T \varphi(u) \frac{\partial K_H}{\partial t}(u, s)du. \]
Then $K_H^t$ is an isometry between $\mathcal{H}$ and $L^2([0, T])$ (see Nualart 2006). Further, for any $\varphi \in \mathcal{H}$, we have $\beta^H(\varphi) = \int_0^T K_H(\varphi)d\beta(t)$. Let $\{ \beta^H_n(t) \}_{n \in \mathbb{N}}$ be a sequence of two-sided one-dimensional standard fractional Brownian motion mutually independent
on $\langle \Omega, \mathcal{F}, \mathbb{P} \rangle$. Consider the following series

$$\sum_{n=1}^{\infty} \beta^H_n(t)e_n, \ t \geq 0,$$

where $\{e_n\}_{n \in \mathbb{N}}$ is a complete orthonormal basis in $\mathbb{K}$, the series does not necessarily converge in the space $\mathbb{K}$. Therefore, we consider a $\mathbb{K}$-valued stochastic process $B^H_Q(t)$ given by the following series:

$$B^H_Q(t) = \sum_{n=1}^{\infty} \beta^H_n(t)Q^\frac{1}{2}e_n, \ t \geq 0.$$

Moreover, if $Q$ is a non-negative self-adjoint trace class operator, then this series converges in the space $K$, that is, it holds that $B^H_Q(t) \in L^2(\mathbb{P}, \mathbb{K})$. Then, we say that the above $B^H_Q(t)$ is a $\mathbb{K}$-valued $Q$-cylindrical fractional Brownian motion with covariance operator $Q$. For example, if $\{\sigma_n\}_{n \in \mathbb{N}}$ is a bounded sequence of non-negative real numbers such that $Qe_n = \sigma_n e_n$, assuming that $Q$ is a nuclear operator in $\mathbb{K}$, then the stochastic process

$$B^H_Q(t) = \sum_{n=1}^{\infty} \beta^H_n(t)Q^\frac{1}{2}e_n = \sum_{n=1}^{\infty} \sqrt{\sigma_n} \beta^H_n(t)e_n, \ t \geq 0,$$

is well-defined as a $\mathbb{K}$-valued $Q$-cylindrical fractional Brownian motion.

**Definition 6.2.1.** (Caraballo et al 2011) Let $\varphi : [0, T] \to L^0_Q(\mathbb{K}, \mathbb{H})$ such that

$$\sum_{n=1}^{\infty} ||K^+_H(\varphi Q^\frac{1}{2}e_n)||_{L^2(\Omega, \mathbb{P}, \mathbb{H})} < \infty.$$

Then, its stochastic integral with respect to the fractional Brownian motion $B^H_Q(t)$ is defined, for $t \geq 0$, as follows

$$\int_0^t \varphi(s)dB^H_Q(s) := \sum_{n=1}^{\infty} \int_0^t \varphi(s)Q^\frac{1}{2}e_n \beta^H_n ds = \sum_{n=1}^{\infty} \int_0^t (K^+_H(\varphi Q^\frac{1}{2}e_n))(s)ds.$$
Lemma 6.2.2. (Caraballo et al 2011) For any $\varphi : [0,T] \to \mathcal{C}_0^1(K, \mathbb{H})$ such that
\[
\sum_{n=1}^\infty \|\varphi Q^2 e_n\|_{L^1((0,T), \mathbb{H})} < \infty \text{ holds, and for any } \alpha, \beta \in [0, T] \text{ with } \alpha > \beta, \text{ we have }
\]
\[
E \left\| \int_\alpha^\beta \varphi(s) dB_H^Q(s) \right\|^2 \leq cH(2H - 1)(\alpha - \beta)^{2H-1} \sum_{n=1}^\infty \int_\alpha^\beta \|\varphi Q^2 e_n\|^2 ds,
\]
where $c = c(H)$. In addition, if $\sum_{n=1}^\infty \|\varphi(t)Q^2 e_n\|$ is uniformly convergent for $t \in [0, T]$, then
\[
E \left\| \int_\alpha^\beta \varphi(s) dB_H^Q(s) \right\|^2 \leq cH(2H - 1)(\alpha - \beta)^{2H-1} \int_\alpha^\beta \|\varphi\|^2_{L^2 Q} ds.
\]

In this work, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced in (Hino & Murakami 1991).

**Definition 6.2.3.** $\mathcal{B}$ is a linear space of family of $\mathcal{F}_0$-measurable functions from $(-\infty, 0]$ into $\mathbb{H}$ endowed with a norm $\| \cdot \|$ which satisfies the following axioms:

(a) If $x : (-\infty, T) \to \mathbb{H}$, $T > 0$, is such that $x_0 \in \mathcal{B}$, then, for every $t \in [0, T]$, the following conditions hold:

(a) $x_t \in \mathcal{B}$;

(b) $\|x(t)\| \leq m \|x_t\|$;

(c) $\|x_t\|_B \leq R(t) \sup_{0 \leq s \leq t} \|x(s)\| + N(t) \|x_0\|_B$,

where $m > 0$ is a constant, $R, N : [0, +\infty) \to [1, +\infty)$, $R$ is continuous, $N$ is locally bounded, $R, N$ are independent of $x(\cdot)$.

(aii) The space $\mathcal{B}$ is complete.
The next result is a consequence of the definition of phase space.

**Lemma 6.2.4.** (Ren & Sun 2010) Let \( x : (-\infty, T] \to \mathbb{H} \) be an \( \mathcal{F}_t \)-adapted measurable process such that the \( \mathcal{F}_0 \)-adapted process \( x_0 = \varphi \in L_2^0(\Omega, \mathcal{B}) \), then

\[
E\|x_0\|_B \leq NE\|\varphi\|_B + \mathcal{R} \sup_{0 \leq s \leq T} E\|x(s)\|,
\]

where \( N = \sup_{t \in I}\{N(t)\} \) and \( \mathcal{R} = \sup_{t \in I}\{\mathcal{R}(t)\} \).

**Definition 6.2.5.** (Ren & Sun 2010) Denote by \( \mathcal{M}^2(((-\infty, 0, \mathbb{H}) \) be the space of all \( \mathbb{H} \)-valued continuous \( \mathcal{F}_t \)-adapted process \( x = \{x(t)\}_{-\infty < t \leq T} \) such that

(i) \( x_0 = \varphi \in \mathcal{B} \) and \( x(t) \) is continuous on \([0, T]\);

(ii) define the norm \( \| \cdot \|_{\mathcal{M}} \) in \( \mathcal{M}^2((-\infty, T], \mathbb{H}) \) by

\[
\|x\|_{\mathcal{M}}^2 = E\|\varphi\|_B^2 + E \int_0^T \|x(s)\|^2 \, dt < \infty. \tag{6.2}
\]

Then, \( \mathcal{M}^2((-\infty, 0), \mathbb{H}) \) with the norm (6.2) is a Banach space.

**Definition 6.2.6.** (Henriques 2011) A map \( S : I \times I \to \mathcal{L}(\mathbb{H}) \) is said to be an evolution operator for Eq.(6.1) if the following conditions are fulfilled:

(D1) For each \( x \in \mathbb{H} \) the map \( (t, s) \mapsto S(t, s)x \) is continuously differentiable and

(a) For each \( t \in I \), \( S(t, t) = 0 \).

(b) For all \( t, s \in I \) and each \( x \in \mathbb{H} \),

\[
\frac{\partial}{\partial t} S(t, s)x \bigg|_{t=s} = x \text{ and } \frac{\partial}{\partial s} S(t, s)x \bigg|_{t=s} = -x.
\]

(D2) For all \( s, t \in I \), if \( x \in D \), then \( S(t, s)x \in D \), the map \( (t, s) \mapsto S(t, s)x \) is of class \( C^2 \) and
\[(a) \ \frac{\partial}{\partial t} S(t, s)x = A(t)S(t, s)x,\]
\[(b) \ \frac{\partial}{\partial s} S(t, s)x = S(t, s)A(s)x,\]
\[(c) \ \frac{\partial}{\partial \tau} S(t, s)x \bigg|_{t=s} = 0.\]

\[\text{(D3) For all } s, t \in I, \text{ if } x \in D, \text{ then } \frac{\partial}{\partial t} S(t, s)x \in D, \text{ there exist } \frac{\partial}{\partial \tau} S(t, s)x, \]
\[\frac{\partial}{\partial \tau} S(t, s)x \text{ and}\]
\[(a) \ \frac{\partial}{\partial \tau} S(t, s)x = A(t)\frac{\partial}{\partial \tau} S(t, s)x,\]
\[\text{Moreover, the map } (t, s) \mapsto A(t)\frac{\partial}{\partial \tau} S(t, s)x \text{ is continuous}\]
\[(b) \ \frac{\partial}{\partial \tau} S(t, s)x = \frac{\partial}{\partial \tau} S(t, s)A(s)x.\]

Moreover, we assume that there exists an evolution operator \(S(t, s)\) associated to the operators \(A(t)\). Also, we introduce the operator \(C(t, s) = -\frac{\partial}{\partial s} S(t, s)\).

Furthermore, we set a positive constant \(M\) such that \(\sup_{0 \leq s, t \leq T} \|S(t, s)\| \leq M\) and \(\sup_{0 \leq s, t \leq T} \|C(t, s)\| \leq M\).

### 6.3 Existence, Uniqueness and Stability Results

In this section, we will study the existence, uniqueness and stability of mild solutions for non-autonomous neutral stochastic equation (6.1) driven by fractional Brownian motion under assumptions that the coefficients satisfy non-Lipschitzian conditions. In order to prove the result, we impose the conditions on \(f, g, h\) and \(\sigma\) as follows:

\[(H1) \text{ The functions } f, g : I \times \mathcal{B} \to \mathbb{H} \text{ and } h : I \times \mathcal{B} \to L^0_2 \text{ satisfy for all } t \in I, \]
\[\varphi, \psi \in \mathcal{B}, \]
\[\|f(t, \varphi) - f(t, \psi)\|^2 \vee \|h(t, \varphi) - h(t, \psi)\|^2 \leq \kappa(\|\varphi - \psi\|_{\mathcal{B}}^2)\]
\[\text{and} \quad \|g(t, \varphi) - g(t, \psi)\|^2 \vee \|h(t, \varphi) - h(t, \psi)\|^2 \leq \kappa(\|\varphi - \psi\|_{\mathcal{B}}^2),\]
where $\kappa(\cdot)$ is a concave, nondecreasing, continuous function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\kappa(0) = 0$, $\kappa(x) > 0$ for $y > 0$ and $\int_0^y \frac{du}{\kappa(u)} = \infty$.

(H2) The function $\sigma : I \rightarrow L_Q^{\mu}$ satisfies the following

(i) there exists a positive constant $L$ such that $||\sigma(t)||^2_{L_Q^{\mu}} \leq L$ uniformly in $I$.

(ii) for the complete orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ in $\mathbb{K}$, we have

$$\sum_{n=1}^{\infty} ||\sigma \gamma^{1/2} e_n|| < \infty,$$

$$\sum_{n=1}^{\infty} ||\sigma \gamma^{1/2} e_n||$$

is uniformly convergent for $t \in I$, which imply that $\int_0^t ||\sigma(s)||^2_{L_Q^{\mu}} ds < \infty$ for every $t \in I$.

(H3) For all $t \in I$, there exists a positive constant $\Lambda$. $||f(t,0)||^2 \vee ||g(t,0)||^2 \vee ||h(t,0)||^2 \leq \Lambda$.

**Definition 6.3.1.** A continuous stochastic process $x : (-\infty, T] \rightarrow \mathbb{H}$ is said to be a mild solution of (6.1) if

(i) $x(t)$ is $\mathcal{F}_t$-adapted and $\{x_t : t \in [0,T]\}$ is $\mathcal{B}$-valued;

(ii) $\int_0^T ||x(t)||^2 ds < \infty$, $\mathbb{P}$-a.s.;

(iii) for each $t \in I$, $x(t)$ satisfies the following integral equation:

$$x(t) = C(t,0)\xi(0) + S(t,0)[\xi - f(0,\phi)]$$

$$+ \int_0^t C(t,s)f(s,x_s)ds + \int_0^t S(t,s)g(s,x_s)ds$$

$$+ \int_0^t S(t,s)h(s,x_s)dW(s) + \int_0^t S(t,s)\sigma(s)d\mathbb{U}_Q(s);$$

(iv) $x_0 = \phi \in \mathcal{B}$.
Note 6.3.2. Let us give some concrete functions \( \kappa(\cdot) \). Let \( C > 0 \) and \( \delta \in (0, 1) \) be sufficiently small. Define

\[
\kappa_1(x) = Cx, x \geq 0.
\]

\[
\kappa_2(x) = \begin{cases} 
    y \log(x^{-1}), & 0 \leq x \leq \delta, \\
    \delta \log(\delta^{-1}) + \kappa_2(\delta-)(x-\delta), & x > \delta.
\end{cases}
\]

\[
\kappa_3(x) = \begin{cases} 
    x \log(x^{-1}) \log(\delta^{-1}), & 0 \leq x \leq \delta, \\
    \delta \log(\delta^{-1}) \log(\delta^{-1}) + \kappa_3(\delta-)(x-\delta), & x > \delta.
\end{cases}
\]

All the above functions are concave nondecreasing and satisfy \( \int_{0^+}^\infty \frac{ds}{\kappa_i(s)} = +\infty \) \((i = 1, 2, 3)\). Also, it can be seen that the Lipschitz condition is a special case of the proposed conditions.

In order to obtain the uniqueness of solutions, we present the Bihari inequality which is appeared in (Ren & Sakthivel 2012)

**Lemma 6.3.3.** Let \( T > 0 \) and \( u_0 \geq 0 \), \( u(t) \), \( v(t) \) be continuous functions on \([0, T]\). Let \( \kappa : \mathbb{R}_+ \to \mathbb{R}_+ \) be a concave continuous and nondecreasing function such that \( \kappa(r) > 0 \) for \( r > 0 \). If \( u(t) \leq u_0 + \int_0^t v(s)\kappa(u(s))ds \) for all \( 0 \leq t \leq T \), then

\[
u(t) \leq G^{-1} \left( G(u_0) + \int_0^t v(s)ds \right),
\]

and for all \( t \in [0, T] \), it holds that \( G(u_0) + \int_0^t v(s)ds \in \text{Dom}(G^{-1}) \), where \( G(r) = \int_1^r \frac{ds}{\kappa(s)} \), \( r \geq 0 \) and \( G^{-1} \) is the inverse function of \( G \). In particular, if, moreover, \( u_0 = 0 \) and \( \int_{0^+} \frac{ds}{\kappa(s)} = \infty \), then \( u(t) = 0 \) for all \( 0 \leq t \leq T \).

Further, in order to prove the existence and uniqueness result, we construct the sequence of successive approximations as follows:

\[
x^0(t) = C(t, 0)\phi(0) + S(t, 0)[\zeta - f(0, \phi)], \quad t \in I,
\]
\[ x^n(t) = C(t, 0)\phi(0) + S(t, 0)[\xi - f(0, \phi)] + \int_0^t C(t, s)f(s, x_s^{n-1})ds + \int_0^t S(t, s)g(s, x_s^{n-1})ds + \int_0^t S(t, s)h(s, x_s^{n-1})dW(s) + \int_0^t S(t, s)\sigma(s)d\mathcal{B}_Q^H(s), \quad t \in I, \quad n \geq 1, \quad (6.4) \]

\[ x^n(t) = \phi(t), \quad -\infty < t \leq 0, \quad n \geq 1. \]

The following theorem establishes the existence and uniqueness of mild solution to (6.1).

**Theorem 6.3.4.** Assume that the conditions (H1)–(H3) hold. Then, there exists a unique mild solution of (6.1) on \( \mathcal{M}^2((-\infty, T]; \mathbb{H}) \).

**Proof.** The proof of this theorem is long and technical, therefore it is convenient to divide it into three steps:

**Step 1.** For all \( t \in (-\infty, T] \) and \( n \geq 0 \), it holds that \( x^n(t) \in \mathcal{M}^2((-\infty, T]; \mathbb{H}) \), i.e., there exists a positive constant \( C_1 \) which is independent of \( n \) such that \( ||x^n(t)||^2 \leq C_1 \).

It is obvious that \( x^n(t) \in \mathcal{M}^2((-\infty, T]; \mathbb{H}) \). By induction, we prove that \( x^n(t) \in \mathcal{M}^2((-\infty, T]; \mathbb{H}) \). It follows from (6.4), Lemma 6.2.2, Hölder inequality and Doob martingale inequality that

\[
E\left( \sup_{0 \leq s \leq t} ||x^n(s)||^2 \right) \\
\leq 6M^2E||\phi(0)||^2 + 12M^2E||\xi||^2 + 12M^2E||f(0, \phi)||^2 \\
+ 6M^2TE\int_0^t ||f(s, x_s^{n-1})||^2ds + 6TM^2E\int_0^t ||g(s, x_s^{n-1})||^2ds \\
+ 6M^2E\int_0^t ||h(s, x_s^{n-1})||^2ds + 6M^2cH(2H - 1)T^{2H-1}\int_0^t ||\sigma(s)||^2_{\mathcal{B}_Q}ds
\]
\[ \leq M^2 E||\phi||_B^6 + 12M^2 E||\xi||^2 + 24M^2 E\kappa(||\phi||_B^2) + 24M^2 \Lambda \\
+ 12M^2 T\kappa(\int_0^t \kappa(||x_s^{n-1}||^2_B)ds) + 12M^2 T^2 \Lambda \\
+ 12M^2 T E \int_0^t \kappa(||x_s^{n-1}||^2_B)ds + 12M^2 T^2 \Lambda \\
+ 12M^2 \int_0^t \kappa(||x_s^{n-1}||^2_B)ds + 12M^2 T \Lambda + 6M^2 cH(2H-1)T^{2H} L \\
\leq K_1 + 12M^2(2T + 1)E \int_0^t \kappa(||x_s^{n-1}||^2_B)ds, \]

where \( K_1 = 6M^2 E||\phi||_B^6 + 12M^2 E||\xi||^2 + 24M^2 E\kappa(||\phi||_B^2) + 24M^2 \Lambda + 12M^2 T(2T + 1) \Lambda + 6M^2 cH(2H - 1)T^{2H} L \). Given that \( \kappa(\cdot) \) is concave and \( \kappa(0) = 0 \), we can find positive constants \( a \) and \( b \) such that \( \kappa(x) \leq a + bx \), for all \( y \geq 0 \). Also, by using the Lemma 6.2.4 in the above inequality, we obtain

\[ E \left( \sup_{0 \leq r \leq s} ||x^n(s)||^2 \right) \leq K_1 + 24M^2 a T^2 + 24M^2 T b E \int_0^t ||x^n_s||^2_B ds \\
+ 12M^2 a T + 12M^2 b E \int_0^t ||x^n_s||^2_B ds \\
\leq K_1 + 12M^2 a T(2T + 1) \\
+ 12b M^2 (2T + 1) E \int_0^t \left( \sup_{0 \leq r \leq s} \left( ||x^{n-1}(r)|| + N ||x^n_0||_B \right)^2 \right) ds \\
\leq K_3 + 24b M^2 (2T + 1) \int_0^t E \left( \sup_{0 \leq r \leq s} ||x^{n-1}(r)|| \right)^2 ds, \quad (6.5) \]

where \( K_3 = K_1 + 12M^2 a T(2T + 1) + 24M^2 b T(2T + 1) N^2 E||\phi||_B^2 \). From any \( k \geq 1 \),

it follows from (6.5) that

\[ \max_{1 \leq n \leq k} E \left( \sup_{0 \leq s \leq t} ||x^n(s)||^2 \right) \]

\[ \leq K_3 + 24M^2 b(2T + 1) \int_0^t \left[ E||x^0(s)||^2 + E \max_{1 \leq n \leq k} \left( \sup_{0 \leq r \leq s} ||x^n(r)|| \right)^2 \right] ds \]
\[ \leq K_3 + 48M^2b(2T + 1) \int_0^t \left[ M^2E\|\phi\|_B^2 + 2M^2E\|\zeta\|^2 + 4M^2I_n(\|\phi\|_B^2) + 4M^2A \right] ds \\
+ 24M^2b(2T + 1) \int_0^t E \max_{1 \leq n \leq k} \left( \sup_{0 \leq r \leq s} \|x^n(r)\|^2 \right) ds \\
\leq K_4 + 24M^2b(2T + 1) \int_0^t E \max_{1 \leq n \leq k} \left( \sup_{0 \leq r \leq s} \|x^n(r)\|^2 \right) ds, \]

where \( K_4 = K_3 + 48M^4b(2T + 1)T[E\|\phi\|_B^2 + 2M^2E\|\zeta\|^2 + 4a + 4bE\|\phi\|_B^2 + 4A]. \)

Using the Gronwall inequality in the above inequality, we get

\[
\max_{1 \leq n \leq k} E \left( \sup_{0 \leq s \leq t} \|x^n(s)\|^2 \right) \leq K_4e^{24M^2b(2T + 1)T}.
\]

Since \( k \) is arbitrary, we have

\[
E \left( \sup_{0 \leq s \leq t} \|x^n(s)\|^2 \right) \leq K_4e^{24M^2b(2T + 1)T}, \text{ for all } 0 \leq t \leq T, \; n \geq 1.
\]

Hence by the above result, we obtain

\[
\|x^n\|^2 = E\|u^n_0\|_B^2 + E \int_0^T \|x^n(s)\|^2 ds \leq C_1 < \infty,
\]

where \( C_1 = E\|\phi\|_B^2 + TK_4e^{24M^2b(2T + 1)T} \). The proof of Step 1 is complete.

**Step 2.** Next we show that there exists a positive constant \( C_2 \) which is independent of \( n \) such that

\[
E \left( \sup_{0 \leq s \leq t} \|x^{n+m}(s) - x^n(s)\|^2 \right) \leq C_2 \int_0^t \kappa \left( E \left( \sup_{0 \leq r \leq s} \|x^{n+m-1}(r) - x^{n-1}(r)\|^2 \right) \right) ds,
\]

for all \( 0 \leq t \leq T, \; n, m \geq 1. \)

For \( n, m \geq 1 \), From (6.4), we obtain

\[
E \left( \sup_{0 \leq s \leq t} \|x^{n+m}(s) - x^n(s)\|^2 \right) \leq 3M^2(2T + 1)E \int_0^t \kappa \left( \|x_s^{n+m-1} - x_s^{n-1}\|^2 \right) ds
\]
\[ \leq 3M^2(2T + 1) E \int_0^t \kappa \left( \sup_{0 \leq r \leq s} \| x^{n+m-1}(r) - x^{n-1}(r) \|^2 \right) ds. \]

Further, it follows from the above and Jensen’s inequality that

\[ E \left( \sup_{0 \leq s \leq t} \| x^{n+m}(s) - x^n(s) \|^2 \right) \leq 2C_2 \int_0^t \kappa \left( \sup_{0 \leq r \leq s} \| x^{n+m-1}(r) - x^{n-1}(r) \|^2 \right) ds, \]

where \( C_2 = 3M^2(2T + 1) \). The proof of Step 2 is complete.

**Step 3.** By using a similar procedure as in proof of (Lemma 15, Ren & Sakthivel 2012), we can prove that there exists a positive constant \( C_3 \) such that

\[ E \left( \sup_{0 \leq s \leq t} \| x^{n+m}(s) - x^n(s) \|^2 \right) \leq C_3 t, \]

for all \( 0 \leq t \leq T, \ n, m \geq 1 \). Next, we define

\[
\begin{align*}
\varphi_1(t) &= C_3 t, \\
\varphi_{n+1}(t) &= C_2 \int_0^t \kappa(\varphi_n(s)) ds, \ n \geq 1, \\
\varphi_{n,m}(t) &= E \left( \sup_{0 \leq r \leq t} \| x^{n+m}(r) - x^n(r) \|^2 \right), \ n, m \geq 1.
\end{align*}
\]

Choose \( T_1 \in [0, T) \) such that \( C_2 \kappa(C_3 t) \leq C_3 \), for all \( 0 \leq t < T_1 \). Next, by following the similar procedure as in (Lemma 16, Ren & Sakthivel 2012), by induction one can show that there exists a positive \( 0 \leq T_1 < T \) such that for all \( n, m \geq 1, \ 0 \leq \varphi_{n,m}(t) \leq \varphi_n(t) \leq \varphi_{n-1}(t) \leq \ldots \leq \varphi_1(t) \) for all \( 0 \leq t \leq T_1 \).

Next we prove the existence and uniqueness results:

**Uniqueness:** Let \( x(t) \) and \( z(t) \) be any two solutions of (6.1). By employing the similar procedure as in Step 2, on the interval \((-\infty, 0]\) and for \( 0 \leq t \leq T \), we have

\[ E \left( \sup_{0 \leq s \leq t} \| x(s) - z(s) \|^2 \right) \leq 3M^2(2T + 1) \int_0^t \kappa \left( E \left( \sup_{0 \leq r \leq s} \| x(r) - z(r) \|^2 \right) \right) ds. \]
Further, it follows from the Bihari inequality that

\[
E \left( \sup_{0 \leq s \leq t} \| x(s) - z(s) \|^2 \right) = 0, \quad t \in [0, T].
\] (6.6)

This implies that \( x(t) = z(t) \) for all \( 0 \leq t \leq T \). Therefore, for all \( -\infty \leq t \leq T \), \( x(t) = z(t) \) a.s. This gives the uniqueness.

**Existence:** In order to prove the existence result, we claim that

\[
E \left( \sup_{0 \leq s \leq t} \| x^{n+m}(s) - x^n(s) \|^2 \right) \to 0,
\] (6.7)

for all \( -\infty \leq t \leq T_1 \), as \( n, m \to \infty \). Notice that \( \varphi_n \) is continuous on \([0, T_1]\).

Also, we note that for each \( n \geq 1 \), \( \varphi_n(\cdot) \) is decreasing on \([0, T_1]\), and for each \( t \), \( \varphi_n(t) \) is a decreasing sequence. Now, we can define the function \( \varphi(t) \) as

\[
\varphi(t) = \lim_{n \to \infty} \varphi_n(t) = \lim_{n \to \infty} C_2 \int_0^t \kappa(\varphi_{n-1}(s))ds = C_2 \int_0^t \kappa(\varphi(s))ds \text{ for all } 0 \leq t \leq T_1.
\]

It follows from Lemma 6.3.3 that \( \varphi(t) = 0 \) for all \( 0 \leq t \leq T_1 \). Also, from Step 3, we have \( \varphi_n(t) \leq \sup_{0 \leq t \leq T_1} \varphi_n(t) \leq \varphi_n(T_1) \to 0 \) as \( n \to \infty \). That is \( x^n(t) \) is a Cauchy sequence in \( L^2 \) on \(( -\infty, T_1)\). Also, from Step 1, we can obtain that \( \| x(t) \|^2 \leq C \), where \( C \) is a positive constant. Moreover, for all \( 0 \leq t \leq T_1 \), by using the property of the function \( \kappa(\cdot) \), we can obtain

\[
E \left\| \int_0^t C(t, s)[f(s, x^n_s) - f(s, x_s)]ds \right\|^2 \to 0, \text{ as } n \to \infty,
\]

\[
E \left\| \int_0^t S(t, s)[g(s, x^n_s) - g(s, x_s)]ds \right\|^2 \to 0, \text{ as } n \to \infty
\]

and

\[
E \left\| \int_0^t S(t, s)[h(s, x^n_s) - h(s, x_s)]dW(s) \right\|^2 \to 0, \text{ as } n \to \infty.
\]
For all $0 \leq t \leq T_1$, by taking limit on both sides of (6.4), we obtain

$$ x(t) = C(t, 0)e(0) + S(t, 0)[\zeta - f(0, \psi)] + \int_0^t C(t, s)f(s, x_s)ds + \int_0^t S(t, s)g(s, x_s)ds + \int_0^t S(t, s)h(s, x_s)dW(s) + \int_0^t S(t, s)\sigma(s)dB^H_Q(s). $$

The above expression reveals that $x(t)$ is one solution of (6.1) on $[0, T_1]$. By iteration technique, the existence of solutions of (6.1) on $[0, T]$ can be obtained. Thus, the proof of this theorem is completed.

Next, we investigate the stability of mild solutions for the stochastic equations (6.1). In particular, we will provide the continuous dependence of solutions on the initial value by using the Bihari inequality. In order to establish the stability result, we impose the following condition on $f$:

(H4) For all $t \in I$, $\varphi, \psi \in \mathcal{B}$, the function $f$ satisfies $\|f(t, \varphi) - f(t, \psi)\|^2 \leq \mathcal{L}\|\varphi - \psi\|^2_{\mathcal{B}}$

where $\mathcal{L} > 0$ is a constant.

**Definition 6.3.5.** A mild solution $x^{\xi, \eta}(t)$ of (6.1) with initial value $(\xi, \eta)$ is said to be stable in mean square if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$ E\left(\sup_{0 \leq s \leq T} ||x^{\xi, \eta}(s) - z^{\eta, \zeta}(s)||^2\right) \leq \epsilon, \text{ when } E||\xi - \eta||^2_{\mathcal{B}} + E||x - z||^2 \leq \delta. \quad (6.8)$$

where $z^{\eta, \zeta}(t)$ is another solution of (6.1) with initial value $(\eta, \zeta)$.

To obtain the stability of solutions, we need the following lemmas.

**Lemma 6.3.6.** (Ren & Sakthivel 2012) Let the assumptions of Lemma 6.3.3 hold.

If $u(t) \leq u_0 + \int_t^T v(s)K(u(s))ds$ for all $0 \leq t \leq T$, then $u(t) \leq G^{-1}(G(u_0))$
+ \int_t^T v(s) \, ds$, and for all $t \in [0, T]$, it holds that

$$G(u_0) + \int_t^T v(s) \, ds \in \text{Dom}(G^{-1}),$$

where $G(r) = \int_1^r \frac{ds}{k(s)}$, $r \geq 0$ and $G^{-1}$ is the inverse function of $G$.

**Lemma 6.3.7.** Let the assumptions of Lemma 6.3.3 hold and $v(t) \geq 0$ for $t \in [0, T]$. If for all $\epsilon > 0$, there exists $t_1 \geq 0$, for all $0 \leq u_0 < \epsilon$, $\int_{t_1}^T v(s) \, ds \leq \int_{u_0}^\epsilon \frac{ds}{r(s)}$ holds. Then for every $t \in [t_1, T]$, the estimate $u(t) \leq \epsilon$ holds.

**Theorem 6.3.8.** Assume that the conditions of Theorem 6.3.4 are satisfied and $f$ satisfies (H4) instead of (H1), then the solution of (6.1) is stable in mean square.

**Proof.** Let $x^{\xi, c}(t)$ and $z^{\eta, c}(t)$ be solutions of (6.1) with initial value $(\xi, c)$ and $(\eta, c)$, respectively. This implies that $x(t)$ and $z(t)$ be two solutions of (6.1) with initial value $(\xi, c)$ and $(\eta, c)$. Then, we can prove the theorem by following the same lines as in the proof of Theorem 18 in (Ren & Sakthivel 2012) with $\kappa_1(u) = 5M(T + 1)\kappa_1(u) + 5MT\mathcal{L}u$ and $u_0 = 5M(1 + 2\mathcal{L})E||\xi - \eta||_B^2 + 10ME||x - z||^2$.

Next, we consider the autonomous case of equation (6.1). When $A(t) = A$, the second order non-autonomous stochastic equation (6.1) becomes

$$\begin{cases}
d[x'(t) - f(t, x_t)] = [Ax(t) + g(t, x_t)] \, dt + h(t, x_t) \, dW(t) \\
+ \sigma(t) dB^H_Q(t), \quad t \in I = [0, T], \\
x_0 = \phi \in \mathcal{B}, \quad x'(0) = \xi, \quad (6.9)
\end{cases}$$

where $A$ is the infinitesimal generator of a strongly continuous cosine family $C(t)$ on $\mathbb{R}$, $f, g, h, \sigma$ and $B^H_Q(t)$ are defined as in equation (6.1).
Now, we will present some facts about cosine families of operators.

**Definition 6.3.9.** *(Ren & Sakthivel 2012)* The one parameter family \( \{C(t) : t \in \mathbb{R}\} \)
\[ C \in \mathcal{L}(\mathbb{H};\mathbb{H}) \] satisfying that

(i) \( C(0) = I \),

(ii) \( C(t)x \) is continuous in \( t \) on \( \mathbb{R} \), for all \( x \in \mathbb{H} \),

(iii) \( C(t + s) + C(t - s) = 2C(t)C(s) \) for all \( t, s \in \mathbb{R} \),

is called a strongly continuous cosine family.

Then, the corresponding strongly continuous sine family \( \{S(t) : t \in \mathbb{R}\} \)
\[ S \in \mathcal{L}(\mathbb{H};\mathbb{H}) \] is defined by \( S(t)x = \int_0^t C(s)xds \), \( t \in \mathbb{R}, x \in \mathbb{H} \). Also, the generator
\( A : \mathbb{H} \to \mathbb{H} \) of \( \{C(t) : t \in \mathbb{R}\} \) is given by \( Ax = \frac{d}{dt}C(t)x|_{t=0} \) for all \( x \in D(A) \)
\[ = \{ x \in \mathbb{H} : C(\cdot)x \in C^2(\mathbb{R};\mathbb{H}) \} \). For more details about the cosine family of operators one can see *(Ren & Sakthivel 2012)*.

Now, we present the mild solution of the problem (6.9).

**Definition 6.3.10.** A continuous stochastic process \( x : (-\infty, T] \to \mathbb{H} \) is called the mild solution of (6.9) if

(i) \( x(t) \) is \( \mathcal{F}_t \)-adapted and \( \{x_t : t \in [0, T]\} \) is \( \mathcal{B} \)-valued;

(ii) \( \int_0^T ||x(t)||^2 ds < \infty, \mathbb{P} \)-a.s.:

(iii) for each \( t \in I \), \( x(t) \) satisfies the following integral equation:
\[
x(t) = C(t)x(0) + S(t)[\xi - f(0, \phi)] + \int_0^t C(t-s)f(s, x_s)ds \\
+ \int_0^t S(t-s)g(s, x_s)ds + \int_0^t S(t-s)h(s, x_s)dW(s) \\
+ \int_0^t S(t-s)\sigma(s)dB^H_Q(s); \tag{6.10}
\]
(iv) \( x_0 = \phi \in \mathcal{B} \).

The following corollary provide the existence and uniqueness of the mild solution to the autonomous stochastic evolution equations (6.9):

**Corollary 6.3.11.** Assume that the cosine family of operators \( \{ C(t) : t \in [0, T] \} \) on \( \mathbb{H} \) and the corresponding sine family \( \{ S(t) : t \in [0, T] \} \) satisfy the conditions \( ||C(t)||^2 \leq M, \ ||S(t)||^2 \leq M, \ t \geq 0 \) for a positive constant \( M \). Further, if the conditions (H1)–(H3) hold. Then, there exists a unique mild solution of (6.9) in \( \mathcal{M}^2([-\infty, T], \mathbb{H}) \).

**Proof.** First, we consider the sequence of successive approximations as follows:

\[
\begin{align*}
    x^0(t) &= C(t)\phi(0) + S(t)[\xi - f(0, \phi)], \ t \in I, \\
    x^n(t) &= C(t)\phi(0) + S(t)[\xi - f(0, \phi)] + \int_0^t C(t-s)f(s, x^{n-1}_s)ds \\
    &\quad + \int_0^t S(t-s)g(s, x^{n-1}_s)ds + \int_0^t S(t-s)h(s, x^{n-1}_s)dW(s) \\
    &\quad + \int_0^t S(t-s)\sigma(s)dB^H_Q(s), \ t \in I, n \geq 1, \\
    x^n(t) &= \phi(t), \ -\infty < t \leq 0, \ n \leq 1.
\end{align*}
\]

The proof of this corollary with some modifications is an immediate consequence of Theorem 6.3.4 and hence it is omitted.

The following corollary can be seen as immediate consequence of the Theorem 6.3.8.

**Corollary 6.3.12.** Let \( x^{\xi, \nu}(t) \) and \( z^{\eta, \varsigma}(t) \) be solutions of (6.9) with initial value \((\xi, x)\) and \((\eta, z)\) respectively. Assume the assumptions (H1)–(H3) hold and \( f \) satisfies (H4) instead of (H1), then the solution of (6.9) is stable in mean square.

**Note 6.3.13.** In addition, neutral stochastic functional differential equations with Poisson jumps have become very popular in modelling the phenomena arising in the...
many fields such as economics, physics, biology, medicine, and so on. Especially, existence results for stochastic evolution equations with Poisson jumps have attracted the interest of many researchers (Sakthivel & Ren 2012). The autonomous case of the problem (6.1) without fractional Brownian motion by introducing Poisson jumps is studied in (Ren & Sakthivel 2012). The results in Theorem 6.3.4 can be extended to study the existence and stability of nonlinear non-autonomous stochastic evolution equations with fractional Brownian motion and Poisson jumps.

**Example 6.3.14.** Now, we apply the results established in the previous section to discuss the existence of solutions of the non-autonomous stochastic wave equation with infinite delay driven by fractional Brownian motion. Now, we consider only a simple type of stochastic wave equation driven by fractional Brownian motion in the following form

\[
\frac{\partial}{\partial \tau} \left[ \frac{\partial^2 z(t, \xi)}{\partial t^2} - G(t, z(t - \tau, \xi)) \right] = \left[ \frac{\partial^2 z(t, \xi)}{\partial \xi^2} + a(t) \frac{\partial z(t, \xi)}{\partial \xi} \right] \frac{\partial t}{\partial t} + F(t, z(t - \tau, \xi)) \frac{\partial t}{\partial t} + \Phi(t, z(t - \tau, \xi)) dW(t) + \Theta(t) dB^H_t(t),
\]

\[0 < \xi < 2\pi, \ \tau > 0, \ t \in [0, b],\]

\[z(\theta, \xi) = \phi(\theta, \xi), \ \theta \in (-\infty, 0], \ 0 \leq \xi \leq 2\pi, \]

\[z(t, 0) = z(t, 2\pi) = 0, \ t \in [0, b],\]

\[\frac{\partial z(0, \xi)}{\partial \xi} = \psi(\xi), 0 \leq \xi \leq 2\pi.\]

where \(a : [0, \infty) \to \mathbb{R}\) is a continuous function; \(b > 0; G, F, \Phi, \Theta, \phi \) and \(\psi\) are appropriate functions; set \(\beta = \sup_{0 \leq t \leq q} \|a(t)\|\), here \(q > 0\). Let \(H = K = L^2(T, \mathbb{C})\), where \(T\) is defined as the quotient \(\mathbb{R}/2\pi\mathbb{Z}\). Further \(H^2(T, \mathbb{C})\) denotes the Sobolev space of \(2\pi\)-periodic functions \(z : \mathbb{R} \to \mathbb{C}\) such that \(z'' \in H\). In order to define the operator \(Q : K \to K\), we choose a sequence \(\{\mu_n\}_{n \geq 1} \subset \mathbb{R}^+\) and set \(Qz_n = \mu_n z_n\).\]
and assume that \( tr(Q) = \sum_{n=1}^{\infty} \sqrt{\mu_n} < \infty \). Define the stochastic process \( B^H_Q(t) \) by
\[
B^H_Q(t) = \sum_{n=1}^{\infty} \sqrt{\mu_n} h_n^H(t) z_n, \ t \geq 0,
\]
where \( H \in (1/2, 1) \) and \( \{ h_n^H \}_{n \in \mathbb{N}} \) is a sequence of two-sided one-dimensional fractional Brownian motion mutually independent.

Consider the operator \( Az(\xi) = \frac{\partial^2}{\partial \xi^2} z(\xi) \) with domain \( D(A) = H^2(T, \mathbb{C}) \). It is known that \( A \) is the infinitesimal generator of a strongly continuous cosine function \( C_0(t) \) and is given by (Hemínez 2011)
\[
C_0(t)u = \sum_{n \in \mathbb{Z}} \cos(nt) \langle u, z_n \rangle z_n, \ t \in \mathbb{R},
\]
with associated sine function
\[
S_0(t)u = t \langle u, z_0 \rangle z_0 + \sum_{n \in \mathbb{Z}, n \neq 0} \frac{\sin(nt)}{n} \langle u, z_n \rangle z_n, \ t \in \mathbb{R}.
\]

Also, \( A \) has discrete spectrum and the spectrum of \( A \) consists of eigenvalues \(-n^2\) for \( n \in \mathbb{Z} \), with associated eigenvectors \( z_n(\xi) = \frac{1}{\sqrt{2\pi}} e^{in\xi} \), \( n \in \mathbb{Z} \). Furthermore, the set \( \{ z_n : n \in \mathbb{Z} \} \) is an orthonormal basis of \( \mathbb{H} \). In particular, \( Au = \sum_{n \in \mathbb{Z}} -n^2 \langle u, z_n \rangle z_n \) for \( u \in D(A) \). Also, it is clear that \( ||C_0(t)|| \leq 1 \) for all \( t \in \mathbb{R} \) and hence \( C_0(\cdot) \) is uniformly bounded on \( \mathbb{R} \). Take \( B(t)x(\xi) = a(t)x'(\xi) \). It is easy to see that \( A(t) = A + B(t) \) is a closed linear operator, and \( S(t, s) : \mathbb{H} \rightarrow \mathbb{H} \) is well defined and satisfies the conditions of Definition 6.2.6 (Hemínez 2011).

Let \( \phi(\theta)z = \phi(\theta, \xi), \ (\theta, \xi) \in (-\infty, 0] \times [0, 2\pi], \ x(t)(\xi) = z(t, \xi) \).

Consider the phase space \( \mathcal{B} \) with the norm
\[
||x||_{c_h} = \int_{-\infty}^{0} h(s) \sup_{s \leq \theta \leq 0} (E||\phi(\theta)||^2)^{\frac{1}{2}} ds,
\]
where \( h : (-\infty, 0) \rightarrow \mathbb{R} \) is a positive Lebesgue integrable function. Define \( f, g : [0, \infty) \times \mathcal{B} \rightarrow \mathbb{H}, h : [0, \infty) \times \mathcal{B} \rightarrow \mathcal{C}_Q^0(\mathbb{R}, \mathbb{H}) \) by \( f(t, z)(\cdot) = G(t, z(\cdot)) \),
\[ g(t, z)(\cdot) = F(t, z(\cdot)) \] and \[ h(t, z)(\cdot) = \Phi(t, z(\cdot)). \] Then, the system (6.11) can be rewritten as the abstract form of the system (6.1). Further, all the conditions of Theorem 6.3.4 have been fulfilled, so we can conclude that the system (6.11) has a unique mild solution.