CHAPTER 5
EXISTENCE OF SOLUTIONS FOR NONLINEAR FRACTIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

5.1 INTRODUCTION

Differential equations with impulsive conditions constitute an important field of research due to their numerous applications in ecology, medicine, biology, electrical engineering and other areas of science. Many physical phenomena in evolution processes are modelled as impulsive fractional differential equations and existence results for such equations have been studied by several authors (Mophou, 2010, Shu et al 2011, Zhang et al 2010). Fečkan et al (2012) studied the existence of a solution for a class of impulsive differential equations with fractional derivative. The existence of mild solutions for a class of impulsive fractional semilinear differential equations has been discussed in (Shu et al 2011). The existence results for a class of impulsive fractional order semilinear evolution equations with infinite delay has been reported in (Dabas et al 2011). Moreover, it is known that the development of the theory of functional differential equations with infinite delay depends on a choice of a phase space. The existence of mild solution for fractional semilinear differential
equations with infinite delay has been discussed in (Mophou & N’Guérékata 2010b). Wang et al (2012) studied the solvability and optimal controls for a class of fractional integrodifferential evolution systems with infinite delay in Banach spaces. The stochastic differential equations with infinite delay have become important in recent years as mathematical models of phenomena in both physical and social sciences (Ren & Sun 2010, Wei & Wang 2007). Wei & Wang (2007) considered a class of stochastic functional differential equations with infinite delay in which the initial value belongs to the phase space. The existence, uniqueness and stability of mild solutions for time-dependent stochastic evolution equations with Poisson jumps and infinite delay has been investigated in (Ren et al 2011). However, only few works deal the existence result for stochastic differential equations of fractional order (Cui & Yan 2011, Pedjjeu & Ladde 2012, Shi & Wang 2012). Cui & Yan (2011) studied the existence of mild solutions for a class of fractional neutral stochastic integro-differential equations with infinite delay in Hilbert spaces by means of Sadovskii’s fixed point theorem. The existence, uniqueness and controllability results for a class of fractional stochastic delay and evolution differential equations has been established in (El-Borai et al 2010, El-Borai et al 2004, Ahmed 2009).

Since impulsive effects also widely exist in fractional stochastic differential equations, it is important and necessary to discuss the qualitative properties for stochastic fractional equations with impulsive perturbations and infinite delay. Motivated by the above considerations, first we shall discuss the existence of solutions for a class of impulsive fractional stochastic differential equations with infinite delay by using some appropriate fixed point theorems and evolution system theory. Moreover, the nonlocal conditions give a better description in applications than standard ones, the topic of nonlocal problems has been studied extensively for the
existence of fractional differential equations (Yan 2010, Yan 2011). As indicated in
(Wang et al 2011) and references there in, the Cauchy problem with nonlocal
initial condition can be applied in physics with better effect than the classical Cauchy
problem with traditional initial conditions. Taking this into account, in this chapter
we also study the existence of mild solutions for semilinear fractional stochastic
differential equations with nonlocal conditions.

5.2 PROBLEM FORMULATION AND PRELIMINARIES

Through out this chapter, we will use the same notation $\| \cdot \|$ to denote the
norms in $\mathbb{H}$, $\mathbb{K}$ and $\mathcal{L}(\mathbb{K}, \mathbb{H})$ and use $(\cdot, \cdot)$ to denote inner product of $\mathbb{H}$ and $\mathbb{K}$ without
any confusion. Let $W = (W_t)_{t \geq 0}$ be a $Q$-Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$
with the covariance operator $Q$ such that $\text{Tr}Q < \infty$. We assume that there exists
a complete orthonormal system $\{e_k\}_{k \geq 1}$ in $\mathbb{K}$, a bounded sequence of nonnegative
real numbers $\lambda_k$ such that $Qe_k = \lambda_ke_k$, $k = 1, 2, \ldots$, and a sequence of independent
Brownian motions $\{\beta_k\}_{k \geq 1}$ such that

$$(W(t), e)_{\mathbb{K}} = \sum_{k=1}^{\infty} \sqrt{\lambda_k} (e_k, e)_{\mathbb{K}, \mathbb{K}}(t), \quad e \in \mathbb{K}, \ t \geq 0.$$ 

Let $\mathcal{L}_Q^0(\mathbb{K}, \mathbb{H}) = \mathcal{L}_2(Q^\frac{1}{2}\mathbb{K}, \mathbb{H})$ be the space of all Hilbert-Schmidt operators from $Q^\frac{1}{2}\mathbb{K}$
to $\mathbb{H}$ with the inner product $(\varphi, \psi)_{\mathcal{L}_Q^0} = \text{Tr}[\varphi Q^{1/2}\psi]$.

We consider the following impulsive fractional stochastic differential
equations with infinite delay in the form
\[ D^\alpha_t x(t) = Ax(t) + f(t, x_t, B_1 x(t)) + \sigma(t, x_t, B_2 x(t)) \frac{dW(t)}{dt}, \quad t \in J = [0, T], \quad t \neq t_k, \]
\[ \Delta x(t_k) = I_k(x(t_k^-)), \quad k = 1, 2, \ldots, m, \] (5.1)

\[ x(t) = \phi(t), \quad \phi(t) \in B_h, \]

where \( T > 0; \ D^\alpha_t \) is the Caputo fractional derivative of order \( \alpha \), \( 0 < \alpha < 1; \)
\( x(\cdot) \) takes the value in the separable Hilbert space \( \mathbb{H} \); \( A : D(A) \subset \mathbb{H} \rightarrow \mathbb{H} \) is the
infinitesimal generator of an \( \alpha \)-resolvent family \( S_\alpha(t)_{t \geq 0} \). The history \( x_t : (-\infty, 0] \rightarrow \mathbb{H}, \ x_t(\theta) = x(t + \theta), \ \theta \leq 0, \) belongs to an abstract phase space \( B_h \); \( f : J \times B_h \times \mathbb{H} \rightarrow \mathbb{H} \) and \( \sigma : J \times B_h \times \mathcal{L}_Q^0 \rightarrow \mathbb{H} \) are appropriate functions to be specified later;
\( I_k : B_h \rightarrow H, \) (\( k = 1, 2, \ldots, m, \)) are appropriate functions. The terms \( B_1 x(t) \) and \( B_2 x(t) \) are given by \( B_1 x(t) = \int_0^t K(t, s)x(s)ds \) and \( B_2 x(t) = \int_0^t P(t, s)x(s)ds \) respectively, where \( K, P \in C(D, \mathbb{R}^+) \) are the set of all positive continuous functions on \( D = \{(t, s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\} \). Here \( 0 = t_0 \leq t_1 \leq \ldots \leq t_m \leq t_{m+1} = T, \) \( \Delta x(t_k) = x(t_k^+) - x(t_k^-), \ x(t_k^+) = \lim_{h \rightarrow 0} x(t_k + h) \) and \( x(t_k^-) = \lim_{h \rightarrow 0} x(t_k - h) \) represent the right and left limits of \( x(t) \) at \( t = t_k \), respectively. The initial data \( \phi = \{\phi(t), t \in (-\infty, 0]\} \) is an \( \mathcal{F}_0 \)-measurable, \( B_h \)-valued random variable independent of \( W \) with finite second moments.

Now, we present the abstract space phase \( B_h \). Assume that \( h : (-\infty, 0] \rightarrow (0, \infty) \) with \( l = \int_{-\infty}^0 h(t)dt < \infty \) a continuous function. We define the abstract phase space \( B_h \) by

\[ B_h = \{ \phi : (-\infty, 0] \rightarrow \mathbb{H}, \quad \text{for any } a > 0, \ (E|\phi(t)|^2)^{\frac{a}{2}} \text{ is bounded and measurable function on } [-a, 0] \text{ with } \phi(0) = 0 \text{ and } \int_{-\infty}^0 h(s) \sup_{s \leq t \leq 0} (E|\phi(t)|^2)^{\frac{a}{2}} ds < \infty \}. \]
If $\mathcal{B}_h$ is endowed with the norm
\[
\|\phi\|_{\mathcal{B}_h} = \int_{-\infty}^{0} h(s) \sup_{s \leq \theta \leq 0} (E|\phi(\theta)|^2)^{\frac{1}{2}} ds, \quad \phi \in \mathcal{B}_h,
\]
then $(\mathcal{B}_h, \| \cdot \|_{\mathcal{B}_h})$ is a Banach space (Ren & Sun 2010, Ren et al 2011).

Now we consider the space

\[
\mathcal{B}_b = \{ x : (-\infty, T] \to \mathbb{H} \text{ such that } x|_{J_k} \in C(J_k, \mathbb{H}) \text{ and there exist } x(t_k^+) \text{ and } x(t_k^-) \text{ with } x(t_k) = x(t_k^-), \ x_0 = \phi \in \mathcal{B}_h, \ k = 1, 2, \ldots, m \},
\]

where $x|_{J_k}$ is the restriction of $x$ to $J_k = (t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, m$. The function $\| \cdot \|_{\mathcal{B}_b}$ to be a seminorm in $\mathcal{B}_b$, it is defined by

\[
\|x\|_{\mathcal{B}_b} = \|\phi\|_{\mathcal{B}_h} + \sup_{s \in [0, T]} (E|x(s)|^2)^{\frac{1}{2}}, \ x \in \mathcal{B}_b.
\]

We recall the following lemma (Ren et al 2011):

**Lemma 5.2.1.** Assume that $x \in \mathcal{B}_b$, then for $t \in J$, $x_t \in \mathcal{B}_b$. Moreover,

\[
l(E\|x(t)|^2)^{\frac{1}{2}} \leq l \sup_{s \in [0, t]} (E|x(s)|^2)^{\frac{1}{2}} + \|x_0\|_{\mathcal{B}_b},
\]

where $l = \int_{-\infty}^{0} h(s) ds < \infty$.

A two parameter function of the Mittag-Leffler type is defined by the series expansion

\[
E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_{C} \frac{e^{-\mu} e^{\mu z}}{\mu^\alpha} \mu^{\beta} d\mu, \quad \alpha, \beta \in C, \ \text{Re}(\alpha) > 0, \quad (5.2)
\]

where $C$ is a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \leq |z|^\frac{1}{\beta}$ counter clockwise. For short, $E_{\alpha}(z) = E_{\alpha, 1}(z)$. It is an entire function which provides
a simple generalization of the exponent function: \( E_1(z) = e^z \) and the cosine function: \( E_2(z^2) = \cosh(z), E_2(-z^2) = \cos(z) \), and plays an vital role in the theory of fractional differential equations. The most interesting properties of the Mittag-Leffler functions are associated with their Laplace integral

\[
\int_0^{\infty} e^{-\lambda t} t^{\alpha-1} E_{\alpha,\beta}(\omega t^\alpha) dt = \frac{\lambda^{\alpha-\beta}}{\lambda^{\alpha} - \omega}, \quad \text{Re} \lambda > \frac{\omega}{\omega^{\frac{1}{\alpha}}}, \quad \omega > 0,
\]

and for more details see (Kilbas et al 2006).

**Definition 5.2.2.** (Huase 2006) A closed and linear operator \( A \) is said to be sectorial if there are constants \( \omega \in \mathbb{R} \), \( \theta \in [\frac{\pi}{2}, \pi] \), \( M > 0 \), such that the following two conditions are satisfied:

1. \( \rho(A) \subset \sum_{(\theta, \omega)} = \{ \lambda \in \mathbb{C} : \lambda \neq \omega, \ |\arg(\lambda - \omega)| < \theta \} \),

2. \( \|\mathbb{R}(\lambda, A)\| \leq \frac{M}{|\lambda - \omega|}, \lambda \in \sum_{(\theta, \omega)} \).

**Definition 5.2.3.** (Araya & Lizama 2008) Let \( A \) be a closed and linear operator with the domain \( D(A) \) defined in a Banach space \( X \). Let \( \rho(A) \) be the resolvent set of \( A \). We say that \( A \) is the generator of an \( \alpha \)-resolvent family if there exist \( \omega \geq 0 \) and a strongly continuous function \( S_\alpha : \mathbb{R}_+ \to \mathcal{L}(X) \), where \( L(X) \) is Banach space of all bounded linear operators from \( X \) into \( X \) and the corresponding norm is denoted by \( \| \cdot \| \), such that \( \{ \lambda^\alpha : \text{Re} \lambda > \omega \} \subset \rho(A) \) and

\[
(\lambda^\alpha I - A)^{-1} x = \int_0^{\infty} e^{\lambda t} S_\alpha(t)x dt, \quad \text{Re} \lambda > \omega, x \in X,
\]

where \( S_\alpha(t) \) is called the \( \alpha \)-resolvent family generated by \( A \).

**Definition 5.2.4.** (Dabas et al 2011) Let \( A \) be a closed linear operator with the domain \( D(A) \) defined in a Banach space \( X \) and \( \alpha_1 > 0 \). We say that \( A \) is the generator of
a solution operator if there exist \( \omega \geq 0 \) and a strongly continuous function 
\( S_\alpha : \mathbb{R}_+ \to \mathcal{L}(X) \) such that \( \{ \lambda \alpha : \text{Re} \lambda > \omega \} \subset \rho(A) \) and 
\[
\chi^{\alpha-1}(\chi^\alpha I - A)^{-1}x = \int_0^\infty e^{\lambda t}S_\alpha(t)x\,dt, \quad \text{Re} \lambda > \omega, \ x \in X, \tag{5.5}
\]
where \( S_\alpha(t) \) is called the solution operator generated by \( A \).

The concept of the solution operator is closely related to the concept of 
a resolvent family (Dabas et al 2011). For more details on \( \alpha \)-resolvent family and 
solution operators, we refer to (Chauhan & Dabas 2011, Dabas et al 2011) and 
references therein.

**Definition 5.2.5.** (Podlubny 1999) The fractional integral of order \( q \) with the lower 
limit 0 for a function \( f \) is defined as 
\[
\mathcal{I}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0, \quad q > 0
\]
provided the righthand side is pointwise defined on \([0, \infty)\), where \( \Gamma(\cdot) \) is the gamma 
function.

**Definition 5.2.6.** (Podlubny 1999) The Caputo’s derivative of order \( \alpha \) for a function 
\( f : [0, \infty) \to \mathbb{R} \), which is atleast \( n \)-times differentiable can be defined as 
\[
D_0^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds = I^{n-\alpha} f^{(n)}(t), \tag{5.6}
\]
for \( n-1 \leq \alpha < n, \ n \in \mathbb{N} \). If \( 0 < \alpha \leq 1 \), then 
\[
D_0^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f^{(1)}(s) ds. \tag{5.7}
\]
Obviously, Caputo’s derivative of a constant is equal to zero. The Laplace transform
of the Caputo derivative of order $\alpha > 0$ is given as

$$L\{D_t^\alpha f(t); \lambda\} = \lambda^\alpha \hat{f}(-\lambda) - \sum_{k=0}^{n-1} \lambda^{\alpha-k} f^{(k)}(0); \ n - 1 \leq \alpha < n.$$  

If $f$ is an abstract function with values in $\mathbb{H}$, then integrals which appears in the above definitions are taken in Bochner’s sense.

**Lemma 5.2.7.** (Dabas et al 2011) If $f$ satisfies the uniform Holder condition with the exponent $\beta \in (0, 1]$ and $A$ is a sectorial operator, then the unique solution of the Cauchy problem

$$D_t^\alpha x(t) = Ax(t) + f(t, x(t), Bx(t)); \ t > t_0, \ t_0 \geq 0, 0 < \alpha < 1, \ (5.8)$$

$$x(t) = \phi(t), \ t \leq t_0, \ (5.9)$$

is given by

$$x(t) = T_\alpha(t - t_0)(x(t_0^+)) + \int_{t_0}^{t} S_\alpha(t - s)f(s, x(s), Bx(s))ds, \ (5.10)$$

where

$$T_\alpha(t) = E_{\alpha, \lambda}(At^\alpha) = \frac{1}{2\pi i} \int_{\tilde{B}_r} e^{\lambda t} \frac{\Gamma(\alpha - 1)}{\lambda^\alpha - A} d\lambda, \ (5.11)$$

$$S_\alpha(t) = t^{\alpha - 1} E_{\alpha, \lambda}(At^\alpha) = \frac{1}{2\pi i} \int_{\tilde{B}_r} e^{\lambda t} \frac{1}{\lambda^\alpha - A} d\lambda. \ (5.12)$$

Here $\tilde{B}_r$ denotes the Bromwich path; $S_\alpha(t)$ is called the $\alpha$-resolvent family and $T_\alpha(t)$ is the solution operator generated by $A$.

Next, we mention statement of Krasnoselskii’s fixed point theorem (Dabas et al 2011).

**Theorem 5.2.8.** Let $B$ be a nonempty closed convex of a Banach space $(X, \| \cdot \|)$. Suppose that $P$ and $Q$ map $B$ into $X$ such that

1. $Px + Qy \in B$ whenever $x, y \in B$;
(ii) $P$ is compact and continuous;

(iii) $Q$ is a contraction mapping.

Then there exists $z \in B$ such that $z = Pz + Qz$.

5.3 STOCHASTIC FRACTIONAL EQUATIONS WITH INFINITE DELAY AND IMPULSES

In this section, we first establish existence of mild solutions to nonlinear fractional stochastic equation (5.1). More precisely, we will formulate and prove sufficient conditions for the existence of solutions to (5.1) with infinite delay and impulses. In order to establish the results, we impose the following conditions:

(H1) If $\alpha \in (0, 1)$ and $A \in \mathcal{B}^{\alpha}(\theta_0, \omega_0)$, then for any $x \in \mathbb{H}$ and $t > 0$ we have

\[ ||T_\alpha(t)|| \leq Me^{\omega t} \quad \text{and} \quad ||S_\alpha(t)|| \leq Ce^{\omega t}(1 + t^{\alpha - 1}), \quad \omega > \omega_0. \]

Thus, we have

\[ ||T_\alpha(t)|| \leq \tilde{M}_T \quad \text{and} \quad ||S_\alpha(t)|| \leq t^{\alpha - 1}\tilde{M}_S, \]

where $\tilde{M}_T = \sup_{0 \leq t \leq T} ||T_\alpha(t)||$ and $\tilde{M}_S = \sup_{0 \leq t \leq T} C e^{\omega t}(1 + t^{1-\alpha})$ (Shu et al 2011).

(H2) There exist $\mu_1, \mu_2 > 0$ such that

\[ E||f(t, \gamma, x) - f(t, \gamma, y)||_{\mathbb{H}}^2 \leq \mu_1 ||\gamma - \psi||_{\mathbb{B}_0}^2 + \mu_2 E||x - y||_{\mathbb{H}}^2. \]

(H3) There exist $\nu_1, \nu_2 > 0$ such that

\[ E||\sigma(t, \gamma, x) - \sigma(t, \gamma, y)||_{\mathbb{C}_2}^2 \leq \nu_1 ||\gamma - \psi||_{\mathbb{B}_0}^2 + \nu_2 E||x - y||_{\mathbb{H}}^2. \]

(H4) For each $k = 1, 2, \ldots, m$, there exists $n_k > 0$ such that

\[ E||I_k(x) - I_k(y)||_{\mathbb{H}}^2 \leq n_k E||x - y||_{\mathbb{H}}^2, \text{ for all } x, y \in \mathbb{H}. \]
Now, we present the definition of mild solutions for the system (5.1) based on (Dabas et al 2011).

**Definition 5.3.1.** An $\mathcal{F}_t$-adapted stochastic process $x : (-\infty, T] \rightarrow \mathbb{H}$ is called a mild solution of the system (5.1) if $x_0 = \phi \in \mathcal{B}$ satisfying $x_0 \in L^0_0(\Omega, \mathbb{H})$ and the following conditions hold:

(i) $\{x_t : t \in J\}$ is $x(t)$ is $\mathcal{B}_\mathbb{H}$-valued and the restriction of $x(\cdot)$ to the interval $(t_k, t_{k+1}]$, $k = 1, 2, \cdots, m$ is continuous.

(ii) for each $t \in J$, $x(t)$ satisfies the following integral equation

$$x(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\
\int_0^t S_\alpha(t-s)f(s, x(s), B_1 x(s))ds \\
+ \int_0^t S_\alpha(t-s)\sigma(s, x(s), B_2 x(s))dW(s), & t \in [0, t_1], \\
T_\alpha(t-t_1)(x(t_1^-) + I_1(x(t_1^-))) \\
+ \int_{t_1}^t S_\alpha(t-s)f(s, x(s), B_1 x(s))ds \\
+ \int_{t_1}^t S_\alpha(t-s)\sigma(s, x(s), B_2 x(s))dW(s), & t \in (t_1, t_2], \\
\vdots, \\
T_\alpha(t-t_m)(x(t_m^-) + I_m(x(t_m^-))) \\
+ \int_{t_m}^t S_\alpha(t-s)f(s, x(s), B_1 x(s))ds \\
+ \int_{t_m}^t S_\alpha(t-s)\sigma(s, x(s), B_2 x(s))dW(s), & t \in (t_m, T]. 
\end{cases}$$

(iii) $\Delta x|_{t=t_k} = I_k(x(t_k^-)), k = 1, \ldots, m$ the restriction of $x(\cdot)$ to the interval $[0, T) \setminus \{t_1, \ldots, t_m\}$ is continuous.

Now, we are in a position to state the existence theorem. Our first theorem is based on the Banach contraction principle.
Theorem 5.3.2. Assume that the conditions (H1)-(H4) hold. If \( A \in \mathcal{A}^\alpha(\theta_0, \omega_0) \), then the system (5.1) has a unique mild solution provided that

\[
\max_{1 \leq i \leq m} \left( 4\hat{m}_i^2(1 + n_i) + 4\hat{m}_i^2 T^{2\alpha} \left[ \frac{1}{\alpha^2}(\mu_1 l + \mu_2 B_i^2) + \frac{1}{T(2\alpha - 1)}(\mu_1 l + \mu_2 B_i^2) \right] \right) < 1,
\]

(5.14)

where \( B_i^1 = \sup_{t \in [0,T]} \int_0^t K(t, s)ds < \infty \) and \( B_i^2 = \sup_{t \in [0,T]} \int_0^t P(t, s)ds < \infty \).

Proof. Define the operator \( \Pi : B_b \to B_b \) by

\[
(\Pi x)(t) = \begin{cases} 
\phi(t), & t \in (-\infty, 0], \\
\int_0^t S_\alpha(t-s)f(s, x_s, B_1 x(s))ds \\
+ \int_0^t S_\alpha(t-s)\sigma(s, x_s, B_2 x(s))dW(s), & t \in [0, t_1], \\
T_\alpha(t-t_1)(x(t_1^-) + I_1(x(t_1^-))) \\
+ \int_{t_1}^t S_\alpha(t-s)f(s, x_s, B_1 x(s))ds \\
+ \int_{t_1}^t S_\alpha(t-s)\sigma(s, x_s, B_2 x(s))dW(s), & t \in (t_1, t_2], \\
\vdots \\
T_\alpha(t-t_m)(x(t_m^-) + I_m(x(t_m^-))) \\
+ \int_{t_m}^t S_\alpha(t-s)f(s, x_s, B_1 x(s))ds \\
+ \int_{t_m}^t S_\alpha(t-s)\sigma(s, x_s, B_2 x(s))dW(s), & t \in (t_m, T].
\end{cases}
\]

(5.15)

For \( \phi \in B_b \), define

\[
g(t) = \begin{cases} 
\phi(t), & t \in (-\infty, 0], \\
0, & t \in J,
\end{cases}
\]

then \( g_0 = \phi \). Next we define the function

\[
\varpi(t) = \begin{cases} 
0, & t \in (-\infty, 0], \\
z(t), & t \in J,
\end{cases}
\]
for each \( z \in C(J, \mathbb{R}) \) with \( z(0) = 0 \). If \( x(\cdot) \) satisfies (5.13) then \( x(t) = g(t) + \bar{z}(t) \) for \( t \in J \), which implies \( x_t = g_t + \bar{z}_t \) for \( t \in J \) and the function \( z(\cdot) \) satisfies

\[
z(t) = \begin{cases} 
\frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, g_s + \bar{z}_s, B_1(g(s) + \bar{z}(s))) \, ds \\
+ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s, g_s + \bar{z}_s, B_2(g(s) + \bar{z}(s))) \, dW(s), \enspace t \in [0, t_1], \\
T_\alpha(t-t_1)(g(t_1^-) + \bar{z}(t_1^-)) + I_1(g(t_1^-) + \bar{z}(t_1^-)) \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, g_s + \bar{z}_s, B_1(g(s) + \bar{z}(s))) \, ds \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s, g_s + \bar{z}_s, B_2(g(s) + \bar{z}(s))) \, dW(s), \enspace t \in (t_1, t_2), \\
\vdots \\
T_\alpha(t-t_m)(g(t_m^-) + \bar{z}(t_m^-)) + I_m(g(t_m^-) + \bar{z}(t_m^-)) \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_m}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, g_s + \bar{z}_s, B_1(g(s) + \bar{z}(s))) \, ds \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_m}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s, g_s + \bar{z}_s, B_2(g(s) + \bar{z}(s))) \, dW(s), \enspace t \in (t_m, T]. 
\end{cases}
\]

Set \( B_0^0 = \{ z \in B_0, \text{ such that } z_0 = 0 \} \) and for any \( z \in B_0^0 \), we have

\[
\|z\|_{B_0^0} = \|z_0\|_{B_0} + \sup_{t \in J} (E||z(t)||^2)^{\frac{1}{2}} = \sup_{t \in J} (E||z(t)||^2)^{\frac{1}{2}},
\]

thus \( (B_0^0, \| \cdot \|_{B_0^0}) \) is a Banach space.

Define the operator \( \Psi : B_0^0 \rightarrow B_0^0 \) by

\[
(\Psi z)(t) = \begin{cases} 
\frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, g_s + \bar{z}_s, B_1(g(s) + \bar{z}(s))) \, ds \\
+ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s, g_s + \bar{z}_s, B_2(g(s) + \bar{z}(s))) \, dW(s), \enspace t \in [0, t_1], \\
T_\alpha(t-t_1)(z(t_1^-)) + I_1(z(t_1^-)) \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, g_s + \bar{z}_s, B_1(g(s) + \bar{z}(s))) \, ds \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s, g_s + \bar{z}_s, B_2(g(s) + \bar{z}(s))) \, dW(s), \enspace t \in (t_1, t_2), \\
\vdots \\
T_\alpha(t-t_m)(z(t_m^-)) + I_m(z(t_m^-)) \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_m}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, g_s + \bar{z}_s, B_1(g(s) + \bar{z}(s))) \, ds \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_m}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s, g_s + \bar{z}_s, B_2(g(s) + \bar{z}(s))) \, dW(s), \enspace t \in (t_m, T]. 
\end{cases}
\]
In order to prove the existence result, it is enough to show that $\Psi$ has a unique fixed point. Let $z, z^t \in \mathcal{B}_0^\alpha$, then for all $t \in [0, t_1]$, we have

$$E\| (\Psi z)(t) - (\Psi z^t)(t) \|_{\mathcal{H}}^2 \leq 2E \left\| \int_0^t S_{\alpha}(t - s) [f(s, g_s + \overline{z}_s, B_1(g(s) + \overline{z}(s))) - f(s, g_s + \overline{z}^t_s, B_1(g(s) + \overline{z}^t(s))] ds \right\|_{\mathcal{H}}^2 + 2E \left\| \int_0^t S_{\alpha}(t - s) [\sigma(s, g_s + \overline{z}_s, B_2(g(s) + \overline{z}(s))) - \sigma(s, g_s + \overline{z}^t_s, B_2(g(s) + \overline{z}^t(s))] d\mathcal{W}(s) \right\|_{\mathcal{H}}^2$$

$$\leq 2 \int_0^t \| S_{\alpha}(t - s) \| ds \int_0^t \| S_{\alpha}(t - s) \| E \| f(s, g_s + \overline{z}_s, B_1(g(s) + \overline{z}(s))) - f(s, g_s + \overline{z}^t_s, B_1(g(s) + \overline{z}^t(s))] \| ds$$

$$+ 2 \int_0^t \| S_{\alpha}(t - s) \|^2 E \| \sigma(s, g_s + \overline{z}_s, B_2(g(s) + \overline{z}(s))) - \sigma(s, g_s + \overline{z}^t_s, B_2(g(s) + \overline{z}^t(s))] \| ds$$

$$\leq 2 \widetilde{M}_2 \int_0^t (t - s)^{\alpha - 1} ds \int_0^t (t - s)^{\alpha - 1} \mu_1 \| \overline{z}_s - \overline{z}^t_s \|_{\mathcal{H}}^2$$

$$+ \mu_2 E \| B_1(g(s) + \overline{z}(s)) - B_1(g(s) + \overline{z}^t(s)) \|_{\mathcal{H}}^2$$

$$+ 2 \widetilde{M}_2 \int_0^t (t - s)^{2(\alpha - 1)/\alpha} \| \nu_1 \| \overline{z}_s - \overline{z}^t_s \|_{\mathcal{H}}^2$$

$$+ \nu_2 E \| B_2(g(s) + \overline{z}(s)) - B_2(g(s) + \overline{z}^t(s)) \|_{\mathcal{H}}^2.$$
For $t \in (t_1, t_2]$, we have

\[
E \|((\Psi z)(t) - (\Psi z^+)(t))\|_{\mathbb{H}}^2 \leq 4\|T_\alpha(t - t_1)\|^2 E\|z(t_1^-) - z^+(t_1^-)\|_{\mathbb{H}}^2
+ 4\|T_\alpha(t - t_1)\|^2 E\|I_1(z(t_1^-)) - I_1(z^+(t_1^-))\|_{\mathbb{H}}^2
+ 4E\left\| \int_{t_1}^t S_\alpha(t - s)[f(s, g_s + \bar{z}_s, B_1(g(s) + \bar{z}(s)))] - f(s, g_s + \bar{z}_s, B_1(g(s) + \bar{z}_s))ds \right\|_{\mathbb{H}}^2
+ 4E\left\| \int_{t_1}^t S_\alpha(t - s)[\sigma(s, g_s + \bar{z}_s, B_2(g(s) + \bar{z}(s)))] - \sigma(s, g_s + \bar{z}_s, B_2(g(s) + \bar{z}_s))]dW(s) \right\|_{\mathbb{H}}^2
\]
Similarly, when \( t \in (t_i, t_{i+1}] \), \( i = 2, \ldots, m \), we get

\[
E \| (\Psi z)(t) - (\Psi z^*)(t) \|_H^2 \\
\leq \left( 4\tilde{M}_1^2 (1 + n_i) + 4\tilde{M}_3^2 T^{2\alpha} \left[ \frac{1}{\alpha^2} (\mu_1 l + \mu_2 B_1^t) \\
+ \frac{1}{T(2\alpha - 1)} (\nu_1 l + \nu_2 B_2^t) \right] \right) \| z - z^* \|_{g_0^*}^2.
\]

Thus for all \( t \in [0, T] \), we have

\[
E \| (\Psi z)(t) - (\Psi z^*)(t) \|_H^2 \\
\leq \max_{1 \leq i \leq m} \left( 4\tilde{M}_1^2 (1 + n_i) + 4\tilde{M}_3^2 T^{2\alpha} \left[ \frac{1}{\alpha^2} (\mu_1 l + \mu_2 B_1^t) \\
+ \frac{1}{T(2\alpha - 1)} (\nu_1 l + \nu_2 B_2^t) \right] \right) \| z - z^* \|_{g_0^*}^2.
\]

Hence, \( \Psi \) is a contraction map by the condition (5.14) and therefore it has a unique fixed point \( z \in B_0^* \), which is a mild solution of (5.1) on \((-\infty, T]\). The proof is complete.

The second result is established using the Krasnoselkii’s fixed point theorem. Now, we make the following assumptions:

(15) \( f : J \times \mathcal{B}_h \times \mathbb{H} \rightarrow \mathbb{H} \) is continuous and there exist two continuous functions \( \mu_1, \mu_2 : J \rightarrow (0, \infty) \) such that

\[
E\|f(t, \psi, x)\|_H^2 \leq \mu_1(t)\|\psi\|_{\mathcal{B}_h}^2 + \mu_2(t)E\|x\|_H^2, \quad (t, \psi, x) \in J \times \mathcal{B}_h \times \mathbb{H},
\]

where \( \mu_1^t = \sup_{s \in [0, t]} \mu_1(s) \) and \( \mu_2^t = \sup_{s \in [0, t]} \mu_2(s) \).

(16) \( \sigma : J \times \mathcal{B}_h \times L_2^0 \rightarrow \mathbb{H} \) is continuous and there exist two continuous functions
\[ \nu_1, \nu_2 : J \rightarrow (0, \infty) \] such that

\[
E\left[\|\sigma(t, \psi, x)\|_{\mathcal{C}^2}\right] \leq \nu_1(t) \|\psi\|^2_{B_h} + \nu_2(t) E\left[\|x\|^2_{\mathcal{H}}\right], \quad (t, \psi, x) \in J \times B_h \times \mathcal{L}_2^0,
\]

where \( \nu^+_1 = \sup_{s \in [0,t]} \nu_1(s) \) and \( \nu^+_2 = \sup_{s \in [0,t]} \nu_2(s) \).

(H7) The functions \( I_k : \mathbb{H} \rightarrow \mathbb{H}, k = 1, 2, \ldots, m, \) are continuous and there exists a \( \Lambda > 0 \) such that

\[
\Lambda = \max_{1 \leq k \leq m, x \in B_q} \{ E[\|I_k(x)\|^2_{\mathcal{H}}] \},
\]

where \( B_q = \{ y \in \mathcal{B}_h, \|y\|^2_{\mathcal{B}_h} \leq q, \ q > 0 \} \).

The set \( B_q \) is clearly a bounded closed convex set in \( \mathcal{B}_h \) for each \( q \) and for each \( y \in B_q \).

From Lemma 5.2.1, we have

\[
\|y_t + \xi_t\|^2_{\mathcal{B}_h} \leq 20\left(\|y(t)\|^2_{\mathcal{B}_h} + \|\xi(t)\|^2_{\mathcal{B}_h}\right) \leq 4 \left( I^2 \sup_{t \in [0,T]} E[\|y(t)\|^2_{\mathcal{H}}] + \|y_0\|^2_{\mathcal{B}_h}\right) + 4 \left( I^2 \sup_{t \in [0,T]} E[\|\xi(t)\|^2_{\mathcal{H}}]\right) \leq 4 \left( \|\phi\|^2_{\mathcal{B}_h} + I^2 q \right). \tag{5.18}
\]

**Theorem 5.3.3.** Suppose that the assumptions (H1)-(H3) and (H5)-(H7) are satisfied with

\[
q \geq 4\tilde{M}_7^2 (q + \Lambda) + 4\tilde{M}_8^2 T^{2\alpha} \left[ \frac{\lambda_1}{\alpha^2} + \frac{\lambda_2}{T(2\alpha - 1)} \right] \tag{5.19}
\]

and

\[
2\tilde{M}_9^2 T^{2\alpha} \left[ \frac{1}{\alpha^2} (\mu_1 l + \mu_2 B_1^1) + \frac{1}{T(2\alpha - 1)} (\nu_1 l + \nu_2 B_2^1) \right] < 1. \tag{5.20}
\]

Then the impulsive stochastic fractional differential equation (5.1) has at least one mild solution on \((-\infty, T]\).
Proof. Let $\Theta_1 : B_q \to B_q$ and $\Theta_2 : B_q \to B_q$ be defined as

$$
(\Theta_1 z)(t) = \begin{cases}
0, & t \in [0, t_1], \\
T_{\alpha}(t - t_1)(Z(t_1^+) + I_1(Z(t_1^+))), & t \in (t_1, t_2], \\
\vdots \\
T_{\alpha}(t - t_m)(Z(t_m^+) + I_m(Z(t_m^+))), & t \in (t_m, T].
\end{cases}
$$

(5.21)

and

$$
(\Theta_2 z)(t) = \begin{cases}
\int_0^t S_\alpha(t - s)f(s, g_s + \varpi_s, B_1(g(s) + \varpi(s)))ds \\
+ \int_0^t S_\alpha(t - s)\sigma(s, g_s + \varpi_s, B_2(g(s) + \varpi(s)))dW(s), & t \in [0, t_1], \\
\int_0^t S_\alpha(t - s)f(s, g_s + \varpi_s, B_1(g(s) + \varpi(s)))ds \\
+ \int_0^t S_\alpha(t - s)\sigma(s, g_s + \varpi_s, B_2(g(s) + \varpi(s)))dW(s), & t \in (t_1, t_2], \\
\vdots \\
\int_0^t S_\alpha(t - s)f(s, g_s + \varpi_s, B_1(g(s) + \varpi(s)))ds \\
+ \int_0^t S_\alpha(t - s)\sigma(s, g_s + \varpi_s, B_2(g(s) + \varpi(s)))dW(s), & t \in (t_m, T].
\end{cases}
$$

(5.22)

In order to use Theorem (5.2.8) we will verify that $\Theta_1$ is compact and continuous while $\Theta_2$ is a contraction operator. For the sake of convenience, we divide the proof into several steps.

Step 1. We show that $\Theta_1 z + \Theta_2 z^+ \in B_q$ for $z, z^+ \in B_q$. For $t \in [0, t_1]$, we have

$$
E \| (\Theta_1 z)(t) + (\Theta_2 z^+)(t) \|^2_{L^2} \\
\leq 2E \left\| \int_0^t S_\alpha(t - s)f(s, g_s + \varpi_s, B_1(g(s) + \varpi(s)))ds \right\|^2_{L^2} \\
+ 2E \left\| \int_0^t S_\alpha(t - s)\sigma(s, g_s + \varpi_s, B_2(g(s) + \varpi(s)))dW(s) \right\|^2_{L^2} \\
\leq 2 \int_0^t |S_\alpha(t - s)| ds \int_0^t |S_\alpha(t - s)| E \| f(s, g_s + \varpi_s, B_1(g(s) + \varpi(s))) \|^2_{L^2} ds \\
+ 2 \int_0^t |S_\alpha(t - s)|^2 E \| \sigma(s, g_s + \varpi_s, B_2(g(s) + \varpi(s))) \|^2_{L^2} ds
$$
\[
\leq 2\tilde{M}_2^2 \int_0^t (t - s)^{\alpha - 1} ds \int_0^t (t - s)^{\alpha - 1} \left[ \mu_1(s) ||g_s + \varpi_s||_{B_h}^2 \\
+ \mu_2(s) E ||B_1(g(s) + \varpi(s))||_{B_h}^2 \right] ds \\
+ 2\tilde{M}_2^2 \int_0^t (t - s)^{2(\alpha - 1)} [4\mu_1^2(||\varphi||_{B_h}^2 + l^2 q) + \mu_2^2 B_1^q] \sup_{s \in [0, T]} E ||z^+(s)||_{\mathbb{H}}^2 ds \\
\leq 2\tilde{M}_2^2 \frac{T^{\alpha}}{\alpha} \int_0^t (t - s)^{\alpha - 1} [4\mu_1^2(||\varphi||_{B_h}^2 + l^2 q) + \mu_2^2 B_1^q] \\
+ 2\tilde{M}_2^2 \frac{T^{2\alpha - 1}}{2\alpha - 1} [4\mu_1^2(||\varphi||_{B_h}^2 + l^2 q) + \mu_2^2 B_1^q] \\
= 2\tilde{M}_2^2 T^{2\alpha} \left[ \frac{\lambda_1}{\alpha^2} + \frac{\lambda_2}{T(2\alpha - 1)} \right].
\]

Thus, by the condition (5.19), we obtain \(||\Theta_1 z + \Theta_2 z^+||_{\mathcal{B}_h} \leq q\).

Similarly, for \(t \in (t_i, t_{i+1}], i = 1, 2, \ldots, m\), we have the estimate

\[
E ||(\Theta_1 z)(t) + (\Theta_2 z^+)(t)||_{\mathbb{H}}^2 \\
\leq 4||T_\alpha(t - t_i)||^2 E \left[ ||z(t_i^-)||_{\mathbb{H}}^2 + 4||T_\alpha(t - t_i)||^2 E ||I_1(z(t_i^-))||_{\mathbb{H}}^2 \\
+ 4 E \left[ \int_{t_i}^t S_\alpha(t - s) f(s, g_s + \varpi_s, B_1(g(s) + \varpi(s))) ds \right] ||_{\mathbb{H}}^2 \\
+ 4 E \left[ \int_{t_i}^t S_\alpha(t - s) \sigma(s, g_s + \varpi_s, B_2(g(s) + \varpi(s))) dW(s) \right] ||_{\mathbb{H}}^2 \\
\leq 4\tilde{M}_T^2 \left[ ||z||_{B_h}^2 + E ||I_1(z(t_i^-))||_{\mathbb{H}}^2 \right] + 4\tilde{M}_2^2 T^{2\alpha} \left[ \frac{\lambda_1}{\alpha^2} + \frac{\lambda_2}{T(2\alpha - 1)} \right] \\
\leq 4\tilde{M}_T^2 (q + \lambda) + 4\tilde{M}_2^2 T^{2\alpha} \left[ \frac{\lambda_1}{\alpha^2} + \frac{\lambda_2}{T(2\alpha - 1)} \right] \\
\leq q.
\]

This implies that \(||\Theta_1 z + \Theta_2 z^+||_{\mathcal{B}_h} \leq q\) with \(\lambda_1 = 4\mu_1^2(||\varphi||_{B_h}^2 + l^2 q) + \mu_2^2 B_1^q\) and \(\lambda_2 = 4\mu_1^2(||\varphi||_{B_h}^2 + l^2 q) + \mu_2^2 B_1^q\). Hence, we get \(\Theta_1 z + \Theta_2 z^+ \in B_q\).
Step 2. The map $\Theta_1$ is continuous on $B_q$.

Let $\{z^n\}_{n=1}^\infty$ be a sequence in $B_q$ with $\lim z^n \to z \in B_q$. Then for $t \in (t_i, t_{i+1})$, $i = 0, 1, \ldots, m$, we have

$$E ||(\Theta_1 z^n)(t) - (\Theta_1 z)(t)||_H^2 \leq 2||T_\alpha(t - t_i)||^2 E ||z^n(t_i^-) - z(t_i^-)||_H^2 \leq E ||I_i(z^n(t_i^-)) - I_i(z(t_i^-))||_H^2.$$

Since the functions $I_i, i = 1, 2, \ldots, m$ are continuous, hence

$$\lim_{n \to \infty} E ||(\Theta_1 z^n - \Theta_1 z)||^2 = 0,$$

which implies that the map $\Theta_1$ is continuous on $B_q$.

Step 3. $\Theta_1$ maps bounded sets into bounded sets in $B_q$.

Let us prove that for $q > 0$ there exists a $\tilde{r} > 0$ such that for each $z \in B_q$, we have $E ||(\Theta_1 z)(t)||_H^2 \leq \tilde{r}$ for $t \in (t_i, t_{i+1})$, $i = 0, 1, \ldots, m$. Now, we have

$$E ||(\Theta_1 z)(t)||_H^2 \leq 2||T_\alpha(t - t_i)||^2 E ||z(t_i^-)||_H^2 + E ||I_i(z(t_i^-))||_H^2 \leq 2\tilde{M}_q^2 (q + \Lambda) = \tilde{r},$$

which proves the desired result.

Step 4. The map $\Theta_1$ is equicontinuous.

Let $u, v \in (t_i, t_{i+1})$, $t_i \leq u < v \leq t_{i+1}$, $i = 0, 1, \ldots, m$, $z \in B_q$, we obtain
\[ E \| \left( \Theta_1 z \right)(v) - \left( \Theta_1 z \right)(u) \|^2_H \leq 2 \| T_\alpha (v - t_i) - T_\alpha (u - t_i) \|^2 \times \left[ E \| z(t_i^-) \|^2_H + E \| J_i(z(t_i^-)) \|^2_H \right] \]
\[ \leq 2(q + \Lambda) \| T_\alpha (v - t_i) - T_\alpha (u - t_i) \|^2. \]

Since \( T_\alpha \) is strongly continuous and it allows us to conclude that \( \lim_{i \to \infty} \| T_\alpha (v - t_i) - T_\alpha (u - t_i) \|^2 = 0 \), which implies that \( \Theta_1(B_q) \) is equicontinuous. Finally, combining Step 1 to Step 4 together with the Ascoli’s theorem, we conclude that the operator \( \Theta_1 \) is compact.

Now, it only remains to show that the map \( \Theta_2 \) is a contraction mapping. Let \( z, z^* \in B_q \) and \( t \in (t_i, t_{i+1}] \), \( i = 0, 1, \ldots, m \), we have

\[ E \| \left( \Theta_2 z \right)(t) - \left( \Theta_2 z^* \right)(t) \|^2_H \]
\[ \leq 2E \left\| \int_{t_i}^{t} S_\alpha (t - s) \left[ f(s, g_s + \bar{x}_s, B_1(g(s) + \bar{x}(s))) - f(s, g_s + \bar{x}_s, B_1(g(s) + \bar{x}(s))) \right] ds \right\|^2_H \]
\[ + 2E \left\| \int_{t_i}^{t} S_\alpha (t - s) \left[ \sigma(s, g_s + \bar{x}_s, B_2(g(s) + \bar{x}(s))) - \sigma(s, g_s + \bar{x}_s, B_2(g(s) + \bar{x}(s))) \right] dW(s) \right\|^2_H \]
\[ \leq 2 \int_{t_i}^{t} \| S_\alpha (t - s) \| ds \left\| \int_{t_i}^{t} \| S_\alpha (t - s) \| E \| f(s, g_s + \bar{x}_s, B_1(g(s) + \bar{x}(s))) - f(s, g_s + \bar{x}_s, B_1(g(s) + \bar{x}(s))) \|_H^2 ds \right\| \]
\[ + 2 \int_{t_i}^{t} \| S_\alpha (t - s) \| E \| \sigma(s, g_s + \bar{x}_s, B_2(g(s) + \bar{x}(s))) - \sigma(s, g_s + \bar{x}_s, B_2(g(s) + \bar{x}(s))) \|_H^2 ds \]
\[ \leq 0. \]
\[
\leq 2\tilde{M}_S^2 \int_{t_i}^t (t-s)^{\alpha-1} \mu_1 \|z_s - \overline{z}_s\|^2_{B^h} ds
+ \mu_2 E\|B_1(g(s) + \overline{z}(s)) - B_1(g(s) + \overline{z}^i(s))\|^2_{B^h} ds
+ 2\tilde{M}_S^2 \int_{t_i}^t (t-s)^{2(\alpha-1)} \|\mu_1 \|z_s - \overline{z}_s\|^2_{B^h} ds
+ \nu_2 E\|B_2(g(s) + \overline{z}(s)) - B_2(g(s) + \overline{z}^i(s))\|^2_{B^h} ds
\leq 2\tilde{M}_S^2 T^{2\alpha} \left[ \frac{1}{\alpha_2} (\mu_1 l + \mu_2 B_1^1) + \frac{1}{T(2\alpha - 1)} (\nu_1 l + \nu_2 B_2^1) \right] \|z - \overline{z}^i\|^2_{B^h}.
\]

By the condition (5.20), we obtain that \(\Theta_2\) is a contraction mapping. Hence, by the Krasnoselkii’s fixed point theorem we can conclude that the problem (5.1) has at least one solution on \((-\infty, T]\). This completes the proof of the theorem.

**Example 5.3.4.** Here, we consider an example to illustrate our main theorem. We examine the existence of solutions for the following fractional stochastic partial differential equation of the form

\[
D_t^\alpha u(t, x) = \frac{\partial^2}{\partial x^2} u(x, t) + \int_{-\infty}^t H(t, x, s-t)Q(u(s, x))ds + \int_0^t k(s, t)e^{-u(s,x)}ds
+ \left[ \int_{-\infty}^t V(t, x, s-t)U(u(s, x))ds + \int_0^t p(s, t)e^{-u(s,x)}ds \right] \frac{d\beta(t)}{dt},
\]

\(x \in [0, \pi], t \in [0, b], t \neq t_k\)

\[u(t, 0) = u(t, \pi), \quad t \geq 0,\]

\[u(t, x) = \phi(t, x), \quad t \in (-\infty, 0], \quad x \in [0, \pi],\]

\[\Delta u(t_i)(x) = \int_{-\infty}^t q_i(t_i - s)u(s, x)ds, \quad x \in [0, \pi],\]

(5.23)
where $\beta(t)$ is a standard cylindrical Wiener process in $\mathbb{H}$ defined on a stochastic space $\langle \Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\} \rangle$; $D_t^q$ is Caputo’s fractional derivative of order $0 < q < 1$; $0 < t_1 < t_2 < \ldots < t_n < b$ are prefixed numbers; $H, Q, V$ and $U$ are continuous; $\phi \in \mathcal{B}_h$.

Let $\mathbb{H} = L^2([0, \pi])$ with the norm $\| \cdot \|$. Define $A : \mathbb{H} \rightarrow \mathbb{H}$ by $Az = z''$ with the domain $D(A) = \{ z \in \mathbb{H}, z, z'$ are absolutely continuous, $z'' \in \mathbb{H}$ and $z(0) = z(\pi) = 0 \}$. Then

$$Az = \sum_{n=1}^{\infty} n^2(z, z_n)z_n, \ z \in D(A),$$

where $z_n(x) = \sqrt{2\pi} \sin(nx)$, $n \in \mathbb{N}$ is the orthogonal set of eigenvectors of $A$. It is well known that $A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t \geq 0}$ in $\mathbb{H}$ and is given by

$$T(t)z = \sum_{n=1}^{\infty} e^{-n^2 t}(z, z_n)z_n, \ \text{for all} \ z \in \mathbb{H}, \ t > 0.$$

It follows from the above expressions that $(T(t))_{t \geq 0}$ is a uniformly bounded compact semigroup, so that, $R(\lambda, A) = (\lambda - A)^{-1}$ is a compact operator for all $\lambda \in \rho(A)$ i.e. $A \in \mathcal{A}(\theta_0, \omega_0)$. Let $h(s) = e^{2s}$, $s < 0$, then $l = \int_{-\infty}^{0} h(s)ds = \frac{1}{2}$ and define

$$\|\phi\|_{\mathcal{B}_h} = \int_{-\infty}^{0} h(s) \sup_{s \leq \theta \leq 0} (E|\phi(\theta)|^2)^{\frac{1}{2}} ds.$$

Hence for $(t, \phi) \in [0, b] \times \mathcal{B}_h$, where $\phi(\theta)(z) = \phi(\theta, z)$, $(\theta, z) \in (-\infty, 0] \times [0, \pi]$. Put $u(t) = u(t, \cdot)$, that is $u(t)(x) = u(t, x)$. Define $f : J \times \mathcal{B}_h \times L^2([0, \pi]) \rightarrow L^2([0, \pi])$ and $\sigma : J \times \mathcal{B}_h \times \mathcal{L}_Q^0 \rightarrow L^2([0, \pi])$ as follows

$$f(t, \phi, B_1 u(t))(x) = \int_{-\infty}^{0} H(t, x, \theta)Q(\phi(\theta))(x) d\theta + B_1 u(t)(x),$$

$$\sigma(t, \phi, B_2 u(t))(x) = \int_{-\infty}^{0} V(t, x, \theta)U(\phi(\theta))(x) d\theta + B_2 u(t)(x),$$
where \( B_1 u(t)(x) = \int_0^t k(s, t) e^{-u(s, x)} ds \) and \( B_2 u(t)(x) = \int_0^t p(s, t) e^{-u(s, x)} ds \). Then, with the above settings the considered equation (5.23) can be written in the abstract form of the equation (5.1). All conditions of Theorem 5.3.2 are now fulfilled, so we deduce that the system (5.23) has a mild solution on \((-\infty, T]\).

5.4 STOCHASTIC FRACTIONAL EQUATIONS WITH NONLOCAL CONDITIONS

The initial conditions usually represent the measurements at initial time. However, in various real world problems, it is possible to require more measurements at some instances in addition to standard initial data and therefore, the initial conditions changed to nonlocal conditions. Balasubramaniam et al (2009a) studied the existence of solutions for semilinear neutral stochastic functional differential equations with nonlocal conditions. In the last few years, there has been an increasing interest in study of fractional differential equations involving nonlocal conditions (Chauhan & Dabas 2011). In this section, we are concerned with the existence and uniqueness of mild solution for semilinear stochastic fractional differential equations with nonlocal conditions in the form

\[
D_t^\alpha x(t) + Ax(t) = f(t, x(t)) + \sigma(t, x(t)) \frac{dW(t)}{dt}, \quad t \in J = [0, T], \quad 0 < \alpha < 1,
\]

\[
x(0) + g(x) = x_0,
\]

(5.24)

where \( A; D(A) \subset H \rightarrow H \) and \( D_t^\alpha \) are defined as in equation (5.1). Further, the functions \( f, \sigma \) and \( g \) are given functions to be defined later. The collection of all strongly-measurable, square-integrable \( H \)-valued random variables, denoted by \( L^2(\Omega, H) \), is a Banach space equipped with norm \( \|x(\cdot)\|_{L^2} = (E\|x(\cdot; W)\|_H^2)^{1/2} \), where
$E$ is defined by $E(h) = \int_{\Omega} h(W) dP$, $W \in \Omega$. Let $C(J, L^2(\Omega, \mathbb{H}))$ be the Banach space of all continuous maps from $J$ into $L^2(\Omega, \mathbb{H})$ satisfying the condition $\sup_{t \in J} E||x(t)||^2 < \infty$.

Let $H_2$ be the closed subspace of all continuous process $x$ that belong the space $C(J, L^2(\Omega, \mathbb{H}))$ consisting of $\mathcal{F}_t$-adapted measurable processes such that the $\mathcal{F}_t$-adapted processes $x(0)$ with a seminorm $|| \cdot ||_{H_2}$ in $H_2$ be defined by

$$||x||_{H_2} = \left( \sup_{t \in J} ||x(t)||_{L^2}^2 \right)^{1/2}.$$

It is easy to verify that $H_2$ furnished with the norm topology as defined above, is a Banach space.

Now, we define the definition of mild solution of (5.24).

**Definition 5.4.1.** A continuous stochastic process $x : J \to \mathbb{H}$ is called a mild solution of (5.24) if the following conditions hold:

(i) $x(t)$ is measurable and $\mathcal{F}_t$-adapted.

(ii) $x(0) + g(x) = x_0$.

(iii) $x$ satisfies the following equation

$$x(t) = T_\alpha(t)(x_0 - g(x)) + \int_0^t S_\alpha(t-s)f(s, x(s))ds + \int_0^t S_\alpha(t-s)\sigma(s, x(s))dW(s),$$

where

$$T_\alpha(t) = E_{\alpha, 1}(At^\alpha) = \frac{1}{2\pi i} \int_{\mathcal{B}_r} e^{\lambda\alpha} \frac{\chi^{\alpha-1}}{\chi^\alpha - A} d\lambda,$$

$$S_\alpha(t) = t^{\alpha-1} E_{\alpha, 1}(At^\alpha) = \frac{1}{2\pi i} \int_{\mathcal{B}_r} e^{\lambda\alpha} \frac{1}{\chi^\alpha - A} d\lambda.$$
here $\bar{B}_e$ denotes the Bromwich path, $S_{\alpha}(t)$ is called the $\alpha$-resolvent family and $T_{\alpha}(t)$ is the solution operator generated by $-A$.

Further, we assume the following conditions:

(H8) There exists a constant $L_g > 0$ such that $E\|g(x) - g(y)\|_H^2 \leq L_g E\|x - y\|_H^2$.

(H9) The nonlinear map $f : J \times \mathbb{H} \rightarrow \mathbb{H}$ and $\sigma : J \times \mathbb{H} \rightarrow \mathcal{L}_Q^0$ are continuous and there exist constants $L_f$ and $L_\sigma$ such that

\[ E\|f(t, x) - f(t, y)\|_H^2 \leq L_f E\|x - y\|_H^2 \]

and

\[ E\|\sigma(t, x) - \sigma(t, y)\|_{\mathcal{L}_Q^0}^2 \leq L_\sigma E\|x - y\|_H^2, \]

for all $x, y \in \mathbb{H}$ and $t \in [0, T]$.

(H10) $f \in C(J \times \mathbb{H}, \mathbb{H})$, $g \in C(\mathbb{H}, \mathbb{H})$ and $\sigma \in C(J \times \mathbb{H}, \mathcal{L}_Q^0)$. Moreover, there exists a constant $C_1 > 0$ such that for $x \in \mathbb{H}$,

\[ E\|g(x)\|_H^2 \leq C_1 \]

and for $s \in J$, $x \in B_e$ there exist two continuous functions $\tilde{L}_f$, $\tilde{L}_\sigma : J \rightarrow (0, \infty)$ such that

\[ E\|f(t, x)\|_H^2 \leq \tilde{L}_f(t) \phi(E\|x\|_H^2), \]

\[ E\|\sigma(t, x)\|_{\mathcal{L}_Q^0}^2 \leq \tilde{L}_\sigma(t) \psi(E\|x\|_H^2). \]

(H11) Assume that the following relationship holds:

\[ \int_0^T \xi(s)ds \leq \int_c^\infty \frac{ds}{\phi(s) + \psi(s)}, \]

where

\[ \xi(t) = \max \left\{ \frac{4\tilde{M}_T^2 T_{\alpha}^{\alpha - 1}}{\alpha} \tilde{L}_f(t), \frac{4\tilde{M}_T^2 t^{2(\alpha - 1)}}{\alpha} \tilde{L}_\sigma(t) \right\}, \quad c = 4\tilde{M}_T^2 \|x_0\|_H^2 + C_1. \]
Theorem 5.4.2. Under the assumptions (H1), (H8) and (H9), the fractional stochastic equation (5.24) has a unique mild solution on $J$ provided that

$$3\tilde{M}_T^2 L_g + 3\tilde{M}_S^2 T^{2\alpha} \left(\frac{L_f}{\alpha^2} + \frac{L_\sigma}{T(2\alpha-1)}\right) \leq 1. \quad (5.25)$$

Proof. Let $\lambda : H_2 \to H_2$ be the operator defined by

$$(\lambda x)(t) = T_\alpha(t)(x_0 - g(x)) + \int_0^t S_\alpha(t-s)f(s, x(s))ds$$

$$+ \int_0^t S_\alpha(t-s)\sigma(s, x(s))dW(s).$$

It can be seen that $\lambda$ maps $H_2$ into itself. Let us show that $\lambda$ is a contraction on $H_2$. For $t \in J$, it follows from the assumptions (H1), (H8) and (H9) that

$$E\|\lambda x(t) - \lambda y(t)\|_H^2$$

$$\leq 3\|T_\alpha(t)\|^2 E\|g(x) - g(y)\|_H^2 + 3\int_0^t \|S_\alpha(t-s)\|^2 E\|f(s, x(s)) - f(s, y(s))\|_H^2 ds$$

$$+ 3\int_0^t \|S_\alpha(t-s)\|^2 E\|\sigma(s, x(s)) - \sigma(s, y(s))\|_{L_2}^2 ds$$

$$\leq 3\tilde{M}_T^2 L_g E\|x - y\|_H^2 + 3\tilde{M}_S^2 \frac{T^{2\alpha}}{\alpha} \int_0^t (t-s)^{\alpha-1} L_f E\|x - y\|_H^2 ds$$

$$+ 3\tilde{M}_S^2 \int_0^t (t-s)^{2(\alpha-1)} L_\sigma E\|x - y\|_H^2 ds$$

$$\leq 3\tilde{M}_T^2 L_g + 3\tilde{M}_S^2 L_f \frac{T^{2\alpha}}{\alpha^2} + 3\tilde{M}_S^2 L_\sigma \frac{T^{2\alpha-1}}{2\alpha - 1} E\|x - y\|_H^2$$

$$= \left[3\tilde{M}_T^2 L_g + 3\tilde{M}_S^2 T^{2\alpha} \left(\frac{L_f}{\alpha^2} + \frac{L_\sigma}{T(2\alpha-1)}\right)\right] E\|x - y\|_H^2.$$
Next our second result is based on the following Schaefer’s fixed point theorem.

**Theorem 5.4.3.** Let \( \tilde{K} \) be a closed convex subset of a Banach space \( X \) such that \( 0 \in \tilde{K} \). Let \( P : \tilde{K} \to \tilde{K} \) be a completely continuous map. Then the set \( \{ x \in \tilde{K} : x = \nu P x, 0 \leq \nu \leq 1 \} \) is unbounded or \( P \) has a fixed point.

**Theorem 5.4.4.** Assume that (I1), (I8), (I110) and (I111) hold. Then the fractional stochastic equation (5.24) has at least one mild solution on \([0, T]\).

**Proof.** Define the operator \( \lambda : H_2 \to H_2 \) as in Theorem 5.4.2. Now, we have to prove that \( \lambda \) is completely continuous operator. Note that \( \lambda \) is well defined in \( H_2 \). For the sake of convenience, we divide the proof into several steps.

**Step 1.** We prove that \( \lambda \) is continuous.

Let \( \{ x^n \}_{n=0}^{\infty} \) be a sequence in \( H_2 \) such that \( x^n \to x \) in \( H_2 \). Since the functions \( f, g \) and \( \sigma \) are continuous,

\[
\lim_{n \to \infty} E\| \lambda x^n(t) - \lambda x(t) \|_H^2 = 0
\]

in \( H_2 \) for every \( t \in J \). This implies that the mapping \( \lambda \) is continuous on \( H_2 \).

**Step 2.** Next we prove that \( \lambda \) maps bounded sets into bounded sets in \( H_2 \).

To prove that for any \( r > 0 \), there exists a \( \gamma > 0 \) such that for \( x \in B_r = \{ x \in H_2 : E\| x \|_H^2 \leq r \} \), we have \( E\| \lambda x \|_H^2 \leq \gamma \). For any \( x \in B_r, t \in J \), we have
\[ E\|\lambda x(t)\|_H^2 \leq 4\|T_\alpha(t)\|^2 E\|x_0\|_H^2 + 4\|T_\alpha(t)\|^2 E\|g(x)\|_H^2 \]
\[ + 4\int_0^t \|S_\alpha(t-s)\| ds \int_0^t \|S_\alpha(t-s)\| E\|f(s, x(s))\|_H^2 ds \]
\[ + 4\int_0^t \|S_\alpha(t-s)\|^2 E\|\sigma(s, x(s))\|_E^2 ds \]
\[ \leq 4\tilde{M}_r^2 r + 4\tilde{M}_r^2 C_1 + 4\tilde{M}_r^2 \frac{T_\alpha}{\tilde{\epsilon}} \psi(r) \int_0^t (t-s)^{\alpha-1} \tilde{L}_f(s) ds \]
\[ + 4\tilde{M}_r^2 \psi(r) \int_0^t (t-s)^{2(\alpha-1)} \tilde{L}_\sigma(s) ds \]
\[ = \gamma, \ t \in J. \]

**Step 3.** We show that \( \lambda \) maps bounded sets into equicontinuous sets of \( B_r \).

Let \( 0 < u < v \leq T \), for each \( x \in B_r \), we have

\[ E\|\lambda x(v) - \lambda x(u)\|_H^2 \]
\[ \leq 6\|T_\alpha(v) - T_\alpha(u)\|^2 E\|x_0\|_H^2 + 6\|T_\alpha(v) - T_\alpha(u)\|^2 E\|g(x)\|_H^2 \]
\[ + 6E \left\| \int_0^u [S_\alpha(v-s) - S_\alpha(u-s)] f(s, x(s)) ds \right\|_H^2 \]
\[ + 6E \left\| \int_u^v S_\alpha(v-s) f(s, x(s)) ds \right\|_H^2 \]
\[ + 6E \left\| \int_0^u [S_\alpha(v-s) - S_\alpha(u-s)] \sigma(s, x(s)) dW(s) \right\|_H^2 \]
\[ + 6E \left\| \int_u^v S_\alpha(v-s) \sigma(s, x(s)) dW(s) \right\|_H^2 . \]

Therefore we obtain

\[ E\|\lambda x(v) - \lambda x(u)\|_H^2 \leq 6(r + C_1)\|T_\alpha(v) - T_\alpha(u)\|^2 \]
\[ + 6 \int_0^u \|S_\alpha(v-s) - S_\alpha(u-s)\| ds \]
\[ \times \int_0^u \|S_\alpha(v-s) - S_\alpha(u-s)\| E\|f(s, x(s))\|_H^2 ds \]
\[ + 6 \int_u^v \| S_\alpha(v - s) \| ds \int_u^v \| S_\alpha(v - s) \| E \| f(s, x(s)) \|_{E}^2 ds \]
\[ + 6 \int_0^u \| S_\alpha(v - s) - S_\alpha(u - s) \|^2 E \| \sigma(s, x(s)) \|_{\mathcal{L}_Q}^2 ds \]
\[ + \int_u^v \| S_\alpha(v - s) \|^2 E \| \sigma(s, x(s)) \|_{\mathcal{L}_Q}^2 ds \]
\[ \leq 6(r + C_1) \| T_\alpha(v) - T_\alpha(u) \|^2 \]
\[ + 6c_\alpha(r) \int_0^u \| S_\alpha(v - s) - S_\alpha(u - s) \| ds \]
\[ \times \int_0^u \| S_\alpha(v - s) - S_\alpha(u - s) \| \tilde{L}_f(s) ds \]
\[ + 6\overline{M}_2 \alpha \beta(r) \int_0^u (v - s)^{\alpha - 1} \tilde{L}_f(s) ds \]
\[ + 6c_\alpha(r) \int_0^u \| S_\alpha(v - s) - S_\alpha(u - s) \|^2 \tilde{L}_\sigma(s) ds \]
\[ + 6\overline{M}_2 \alpha \beta(r) \int_0^u (v - s)^{2(\alpha - 1)} \tilde{L}_\sigma(s) ds. \]

Since \( T_\alpha(t) \) and \( S_\alpha(t) \) are strongly continuous, so \( \| T_\alpha(v) - T_\alpha(u) \| \to 0 \) and \( \| S_\alpha(v - s) - S_\alpha(u - s) \| \to 0 \) as \( u \to v \). Thus, from the above inequality we have \( \lim_{u \to v} E \| \lambda_x(v) - \lambda_x(u) \|_{E}^2 = 0 \). Thus, the set \( \{ \lambda_x, x \in B_r \} \) is equicontinuous. Finally, combining step 1 to step 3 with Ascoli’s theorem, we conclude that the operator \( \lambda \) is compact.

**Step 4.** Next, we show that the set

\[ N = \{ x \in H_2 \text{ such that } x = q \lambda_x(t) \text{ for some } 0 < q < 1 \} \]

is bounded. Let \( x \in N \), then \( x(t) = q \lambda_x(t) \), for some \( 0 < q < 1 \). Then for each \( t \in J \), we have

\[ x(t) = q \left( T_\alpha(t)(x_0 - g(x)) + \int_0^t S_\alpha(t - s)f(s, x(s))ds + \int_0^t S_\alpha(t - s)\sigma(s, x(s))dW(s) \right), \]
which implies that

\[ E\|x(t)\|^2 \leq 4\|T_\alpha(t)\|^2 E\|x_0\|^2 + 4\|T_\alpha(t)\|^2 E\|g(x)\|^2 \]

\[ + 4 \int_0^t \|S_\alpha(t - s)\| ds \int_0^t \|S_\alpha(t - s)\| E\|f(s, x(s))\|^2 ds \]

\[ + 4 \int_0^t \|S_\alpha(t - s)\|^2 E\|\sigma(s, x(s))\|^2 \varphi_q ds \]

\[ \leq 4\tilde{M}_r^2 E\|x_0\|^2 + 4\tilde{M}_r^2 C_1 \]

\[ + 4\tilde{M}_r^2 T^{\gamma_1}_\alpha \int_0^t (t - s)^{\gamma_1 - 1} \tilde{L}_f(s) \psi_\phi(E\|x(s)\|^2) ds \]

\[ + 4\tilde{M}_r^2 \int_0^t (t - s)^{2(\alpha - 1)} \tilde{L}_\sigma(s) \psi_\phi(E\|x(s)\|^2) ds. \]

Consider the function \( \mu(t) \) defined by

\[ \mu(t) = \sup \{ E\|x(s)\|^2, 0 \leq s \leq t \}, \quad 0 \leq t \leq T. \]

\[ \mu(t) \leq 4\tilde{M}_r^2 [E\|x_0\|^2 + C_1] + 4\tilde{M}_r^2 T^{\gamma_1}_\alpha \int_0^t (t - s)^{\gamma_1 - 1} \tilde{L}_f(s) \psi_\phi(\mu(s)) ds \]

\[ + 4\tilde{M}_r^2 \int_0^t (t - s)^{2(\alpha - 1)} \tilde{L}_\sigma(s) \psi_\phi(\mu(s)) ds. \]

Denoting by \( v(t) \) the right hand side of last inequality, we have \( v(0) = c = 4\tilde{M}_r^2[E\|x_0\|^2 + C_1], \mu(t) \leq v(t), t \in J. \) Moreover,

\[ v'(t) = 4\tilde{M}_r^2 T^{\gamma_1}_\alpha t^{\gamma_1 - 1} \tilde{L}_f(t) \psi_\phi(\mu(t)) + 4\tilde{M}_r^2 t^{2(\alpha - 1)} \tilde{L}_\sigma(t) \psi_\phi(\mu(t)) \]

\[ \leq 4\tilde{M}_r^2 T^{\gamma_1}_\alpha t^{\gamma_1 - 1} \tilde{L}_f(t) \psi_\phi(v(t)) + 4\tilde{M}_r^2 t^{2(\alpha - 1)} \tilde{L}_\sigma(t) \psi_\phi(v(t)) \]

or equivalently by (H11), we have

\[ \int_{v(0)}^{v(t)} \frac{ds}{\psi_\phi(s) + \psi'(s)} \leq \int_0^T \zeta(s) ds < \int_c^\infty \frac{ds}{\psi_\phi(s) + \psi'(s)}, \quad 0 \leq t \leq T. \]
This inequality implies that there is a constant \( k \) such that \( v(t) \leq k, \ t \in J, \) and hence, \( \mu(t) \leq k, \ t \in J. \) Furthermore, we get \( ||x(t)||^2 \leq \mu(t) \leq v(t) \leq k, \ t \in J. \) By the Schaefer’s fixed point theorem, we deduce that \( \lambda \) has a fixed point on \( J \) which is a solution to (5.24). This completes the proof of the theorem.

**Example 5.4.5.** Now, we present an example to illustrate the Theorem 5.4.4. Consider the fractional partial stochastic differential equation in the following form

\[
D_t^\alpha [z(t, x)] = \frac{\partial^2}{\partial t^2} [z(t, x)] + h(t, z(t, x)) + \psi(t, z(t, x))d\beta(t) \frac{dt}{dt},
\]

\[
0 \leq t \leq b, \ 0 \leq x \leq \pi
\]

\[
z(t, 0) = z(t, \pi) = 0,
\]

\[
z(0, x) + \sum_{\ell=0}^p \int_0^{\pi} K(x, y)z(t, y)dy = z_0(x), \ 0 \leq x \leq \pi.
\]  

(5.26)

where \( p \) is positive integer, \( b \leq \pi, \ 0 < t_0 < t_1, \ldots < t_p < b, \ z_0(x) \in \mathbb{H} = L^2([0, \pi]), \)

\( K(x, y) \in L^2([0, \pi] \times [0, \pi]) \) and \( D_t^\alpha \) is Caputo’s fractional derivative of order \( 0 < \alpha < 1. \)

To study this system, we take \( \mathbb{H} = L^2([0, \pi]) \) and let \( A \) be the operator defined by \( Ay = y'' \) with domain \( D(A) = \{ y \in \mathbb{H} : y, y' \text{ are absolutely continuous} \ y'' \in \mathbb{H}, \ y(0) = y(\pi) = 0 \}. \) It is well known that \( A \) is the infinitesimal generator of an analytic semigroup \( (T(t))_{t \geq 0} \) in \( \mathbb{H}. \) Furthermore, \( A \) has discrete spectrum with eigenvalues of the form \( -n^2, \ n \in \mathbb{N} \) and corresponding normalized eigenfunctions are given by \( x_n(z) = \sqrt{\frac{2}{\pi}} \sin(nz). \) In addition \( \{ x_n : n \in \mathbb{N} \} \) is an orthonormal basis for \( \mathbb{H}, \)

\[
T(t)y = \sum_{n=1}^{\infty} e^{-n^2t}(y, x_n)x_n, \text{ for all } y \in \mathbb{H}, \text{ and every } t > 0.
\]

From these expressions it follows that \( (T(t))_{t \geq 0} \) is a uniformly bounded compact semigroup, so that \( R(\lambda, A) = (\lambda - A)^{-1} \) is a compact operator for all \( \lambda \in \rho(A) \) i.e. \( A \in \mathbb{K}^\alpha(\theta_0, \omega_0). \)
To represent the above fractional system (5.26) into the abstract form of (5.24), we introduce the functions $f : J \times \mathbb{H} \to \mathbb{H}$, $\sigma : J \times \mathbb{H} \to \mathcal{L}_Q^0$ and $g : \mathbb{H} \to \mathbb{H}$ by $f(t, z)(x) = h(t, z(x))$, $\sigma(t, z)(x) = \psi(t, z(x))$ and $g(w) = \sum_{i=0}^{n} K_{w}(t_i)$, where $K_{z}(z) = \int_{0}^{\tau} K(x, y)z(y)dy$. Thus, $f$, $\sigma$ and $g$ satisfy the assumption of Theorem 5.4.4. Hence, by Theorem 5.4.4 the system (5.26) has mild solution on $[0, T]$. 