Chapter 1

A New Class of Regular Weakly Closed Sets and Functions Using Grills

1.1 Introduction

It is found from literature that during recent years many topologists are interested in the study of generalized types of closed sets. For instance, a certain form of generalized closed sets was initiated by Levine [47], whereas the notion of regular weakly closed sets was studied by Wali [76]. Following the trend, we have introduced and investigated a kind of generalized closed sets, the definition being formulated in terms of grills. The concept of grill was first introduced by Choquet [11] in 1947. From subsequent investigations it is revealed that grills can be used as an extremely useful device for investigation of a number of topological problems, like extension of spaces, theory of proximity spaces and so on (see for instance, [9], [10], [68] for
This chapter contains six sections.

In section 2 of this chapter is to furnish some basic definitions and results of topological and grill topological spaces which have been used in this chapter.

In section 3 of this chapter, we have introduced and study a new class of generalized closed sets, termed $\mathcal{G}$-$rw$-closed, in terms of a given grill $\mathcal{G}$ on the ambient space, the definition having a close bearing to the above operator $\Phi$. This class of $\mathcal{G}$-$rw$-closed sets will be seen to properly contain the class of $rw$-closed sets as introduced in [76]. An explicit form of such a $\mathcal{G}$-$rw$-closed set is also obtained. Also we investigate the properties of the above mentioned closed sets. Further the concept is extended to derive some applications, of generalized $rw$-closed sets via Grills.

In section 4, we introduce and investigate the notion of regular weakly continuous functions in grill topological spaces. Also, we investigate its relationship with other functions. We also introduce $\mathcal{G}$-$rw$-closed functions and $\mathcal{G}$-$rw$-open functions in grill topological space and obtain certain characterizations of these functions.

As applications, some formulations of certain separation axioms in terms of $\mathcal{G}$-$rw$-closed sets and associated concepts will be established in section 5.

In section 6 of this chapter, in terms of grill $\mathcal{G}$, two operators $\Phi_{rw\mathcal{G}}$ and $\Psi_{rw\mathcal{G}}$ have been introduced and discussed in the space $(X, \tau^{rw}, \mathcal{G})$. Furthermore a new topology $\tau^{rw}_{\mathcal{G}}$ which is finer than $\tau_{\mathcal{G}}$ and $\tau^{rw}$ has been introduced.
1.2 Premilinaries

Throughout the Chapter, by a space $X$ we always mean a topological space $(X, \tau)$ with no separation properties assumed. If $A \subseteq X$, we shall adopt the usual notations $\text{int}(A)$ and $\text{cl}(A)$ respectively for the interior and closure of $A$ in $(X, \tau)$. Again, $\tau_g \cdot \text{cl}(A)$ and $\tau_g \cdot \text{int}(A)$ will respectively denote the closure and interior of $A$ in $(X, \tau_g)$. Similarly, whenever we say that a subset $A$ of a space $X$ is open (or closed), it will mean that $A$ is open (resp. closed) in $(X, \tau)$. For open and closed sets with respect to any other topology on $X$, e.g. $\tau_g$, we shall write $\tau_g$-open and $\tau_g$-closed. The collection of all open neighbourhoods of a point $x$ in $(X, \tau)$ will be denoted by $\tau(x)$.

Before entering into our work we recall the following definitions and results from various authors which are useful in the sequel of the thesis.

**Definition 1.2.1** A subset $A$ of a topological space $(X, \tau)$ is said to be **regular open** [63] if $A = \text{int}(\text{cl}(A))$ and **regular closed** [63] if $A = \text{cl}(\text{int}(A))$.

**Definition 1.2.2** A subset $A$ of a space $(X, \tau)$ is said to be **regular semiopen** [7] if there is a regular open set $U$ such that $U \subseteq A \subseteq \text{cl}(U)$.

The family of all regular semiopen sets of $X$ is denoted by $\text{RSO}(X)$.

**Definition 1.2.3** [75] A subset $A$ of a topological space $X$ is said to be **$\theta$-closed** if $A = \theta \text{cl}(A)$ where $\theta \text{cl}(A)$ is defined as $\theta \text{cl}(A) = \{x \in X/\text{cl}(U) \cap A \neq \phi$ for every $U \in \tau$ and $x \in U\}$.

**Definition 1.2.4** [75] A subset $A$ of $X$ is said to be **$\theta$-open** if $X \setminus A$ is $\theta$-closed.
Definition 1.2.5  [75] A subset $A$ of a topological space $X$ is said to be $\delta$-closed if $A = \delta cl(A)$ where $\delta cl(A)$ is defined as $\delta cl(A) = \{ x \in X/\text{int}cl(U) \cap A \neq \phi \text{ for every } U \in \tau \text{ and } x \in U \}.$

Definition 1.2.6  [75] A subset $A$ of $X$ is said to be $\delta$-open if $X\setminus A$ is $\delta$ closed.

Definition 1.2.7  [16] A subset $A$ of a topological space $X$ is said to be $\theta g$-closed if $\theta cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

Definition 1.2.8  [15] A subset $A$ of a topological space $X$ is said to be $\delta g$-closed if $\delta cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

Definition 1.2.9  [15, 16] A subset $A$ of $X$ is said to be $\theta g$-open ($\delta g$-open) if $X\setminus A$ is $\theta g$-closed ($\delta g$-closed) in $X$.

Definition 1.2.10 A subset of a topological space $(X, \tau)$ is called

(i) generalized closed set (briefly, $g$-closed) [47] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

(ii) regular weakly closed set (briefly, $rw$-closed) [76] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular semiopen in $X$.

The complements of the above mentioned closed set are their respective open sets.

The family of all $rw$-open (resp. $rw$-closed, closed, $g$-closed) sets of $X$ containing a point $x \in X$ is denoted by $RWO(X, x)$ (resp. $RWC(X, x)$, $C(X, x)$, $GC(X, x)$). The family of all $rw$-open (resp. $rw$-closed, closed, $g$-closed) sets of $X$ is denoted by $RWO(X)$ (resp. $RWC(X)$, $C(X)$, $GC(X)$).
In a topological space \((X, \tau)\), \(\tau^w\) is the collection of all \(rw\)-closed sets.

**Definition 1.2.11** Let \(A\) be a subset of a topological space \((X, \tau)\). The \(rw\)-closure of \(A\) is denoted as \(rw-clA\) and defined as intersection of all \(rw\)-closed sets containing \(A\) [76].

As a consequence of this definition, \(rw-clA\) consists of all point \(x\) such that every \(rw\)-closed set containing \(x\) intersects \(A\). In this context \(rw-clA\) may not be a \(rw\)-closed set in general but \(rw-clA = A\), if \(A \in \tau^w\).

**Definition 1.2.12** A function \(f : (X, \tau) \rightarrow (Y, \sigma)\) is said to be

(i) \(\theta\)-continuous [20] if \(f^{-1}(V)\) is \(\theta\)-closed set of \((X, \tau)\) for every closed set \(V\) of \((Y, \sigma)\).

(ii) \(\delta\)-continuous if \(f^{-1}(V)\) is \(\delta\)-closed set of \((X, \tau)\) for every closed set \(V\) of \((Y, \sigma)\).

(iii) \(\theta g\)-continuous if \(f^{-1}(V)\) is \(\theta g\)-closed set of \((X, \tau)\) for every closed set \(V\) of \((Y, \sigma)\).

(iv) \(\delta g\)-continuous if \(f^{-1}(V)\) is \(\delta g\)-closed set of \((X, \tau)\) for every closed set \(V\) of \((Y, \sigma)\).

(v) \(rw\)-continuous [76] if \(f^{-1}(V)\) is \(G\)-\(rw\)-closed set of \((X, \tau)\) for every closed set \(V\) of \((Y, \sigma)\).

**Definition 1.2.13** A function \(f : (X, \tau) \rightarrow (Y, \sigma)\) is said to be
(i) \( \theta \)-closed [49] if \( f(F) \) is \( \theta \)-closed set of \( (Y, \sigma) \) for every closed set \( F \) of \( (X, \tau) \).

(ii) \( \delta \)-closed [49] if \( f(F) \) is \( \delta \)-closed set of \( (Y, \sigma) \) for every closed set \( F \) of \( (X, \tau) \).

(iii) \( \theta g \)-closed if \( f(F) \) is \( \theta g \)-closed set of \( (Y, \sigma) \) for every closed set \( F \) of \( (X, \tau) \).

(iv) \( \delta g \)-closed if \( f(F) \) is \( \delta g \)-closed set of \( (Y, \sigma) \) for every closed set \( F \) of \( (X, \tau) \).

(v) \( rw \)-closed [76] if \( f(F) \) is \( \mathcal{G} \)-\( rw \)-closed set of \( (Y, \sigma) \) for every closed set \( F \) of \( (X, \tau) \).

The definition of grill goes as follows.

**Definition 1.2.14** [11] A nonempty collection \( \mathcal{G} \) of nonempty subsets of a topological space \( X \) is called a **grill** if

(i) \( A \in \mathcal{G} \) and \( A \subseteq B \subseteq X \implies B \in \mathcal{G} \), and (ii) \( A, B \subseteq X \) and \( A \cup B \in \mathcal{G} \implies A \in \mathcal{G} \) or \( B \in \mathcal{G} \).

If \( (X, \tau) \) is a topological space with a grill \( \mathcal{G} \) on \( X \). Then, we call it a grill topological space and denotes it by \( (X, \tau, \mathcal{G}) \) [58].

**Definition 1.2.15** [11] Let \( (X, \tau) \) be a topological space and \( \mathcal{G} \) be a grill on \( X \).

We define a mapping \( \Phi : \wp(X) \to \wp(X) \) (where \( \wp(X) \) stands for the power set of \( X \) ) denoted by \( \Phi_G(A, \tau) \) (for \( A \in \wp(X) \)) or \( \Phi_G(A) \) or simply \( \Phi(A) \), called the operator associated with the grill \( \mathcal{G} \) and the topology \( \tau \), and is defined by \( \Phi_G(A) = \{ x \in X : U \cap A \in \mathcal{G} \text{ for all open set } U \text{ containing } x \} \).
For any point \( x \) of a topological space \((X, \tau)\), we shall let \( \tau(x) \) to stand for the collection of all open neighbourhood of \( x \).

**Definition 1.2.16** [11] Let \( \mathcal{G} \) be grill on a space \( X \). We define a map \( \Psi : \wp(X) \to \wp(X) \), given by \( \Psi(A) = A \cup \Phi(A) \) for all \( A \in \wp(X) \).

**Definition 1.2.17** Definition of \( \tau_{\mathcal{G}} \) [59]: Corresponding to a grill \( \mathcal{G} \), on a topological space \((X, \tau)\) there exists a unique topology \( \tau_{\mathcal{G}} \) (say) on \( X \) given by \( \tau_{\mathcal{G}} = \{ U \subseteq X : \Psi(X \setminus U) = X \setminus U \} \) where for any \( A \subseteq X \), \( \Psi(A) = A \cup \Phi(A) = \tau_{\mathcal{G}} \)-\( cl(A) \).

**Definition 1.2.18** [] Let \((X, \tau)\) be a topological space and \( \mathcal{G} \) be a grill on \( X \). A subset \( A \) of \( X \) is said to be \( \tau_{\mathcal{G}} \)-**closed** (resp. \( \tau_{\mathcal{G}} \)-**dense in itself**) if \( \Psi(A) = A \) or equivalently if \( \Phi(A) \subseteq A \) (resp. \( A \subseteq \Phi(A) \)).

**Theorem 1.2.19** [58] Let \((X, \tau)\) be a topological space and \( \mathcal{G} \) be a grill on \( X \). Then for any \( A, B \subseteq X \) the following hold:
(a) \( A \subseteq B \Rightarrow \Phi(A) \subseteq \Phi(B) \).
(b) \( \Phi(A \cup B) = \Phi(A) \cup \Phi(B) \).
(c) \( \Phi(\Phi(A)) \subseteq \Phi(A) = cl(\Phi(A)) \subseteq cl(A) \), and hence \( \Phi(A) \) is closed in \((X, \tau)\), for all \( A \subseteq X \).

**Definition 1.2.20** A subset \( A \) of a space \((X, \tau, \mathcal{G})\) is said to be \( \mathcal{G} \)-**g-closed** [13] if \( \Phi(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open in \( X \).

The complement of a \( \mathcal{G} \)-g-closed set is called a \( \mathcal{G} \)-g-**open set** in \( X \).
**Definition 1.2.21** A function \( f : (X, \tau, \mathcal{G}) \to (Y, \sigma) \) is said to be \( \mathcal{G}_g \)-**continuous** if \( f^{-1}(V) \) is \( \mathcal{G}_g \)-closed set of \((X, \tau)\) for every closed set \( V \) of \((Y, \sigma)\).

**Definition 1.2.22** A function \( f : (X, \tau) \to (Y, \sigma, \mathcal{G}) \) is said to be \( \mathcal{G}_g \)-**closed** if \( f(F) \) is \( \mathcal{G}_g \)-closed set of \((Y, \sigma)\) for every closed set \( F \) of \((X, \tau)\).

**Lemma 1.2.23** Arbitrary union of grills on a set \( X \neq \phi \) is a grill on \( X \), but the intersection of two grills is not generally a grill [68].

### 1.3 \( rw \)-Closed Sets With Respect to a Grill

We begin by introducing a new class of regular weakly closed sets in terms of grills as follows.

**Definition 1.3.1** Let \((X, \tau)\) be a topological space and \( \mathcal{G} \) be a grill on \( X \). Then a subset \( A \) of \( X \) is said to be \( rw \)-**closed with respect to the grill** \( \mathcal{G} \) (\( \mathcal{G} - rw \)-closed, for short) if \( \Phi(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is regular semiopen in \( X \).

A subset \( A \) of \( X \) is said to be \( \mathcal{G} - rw \)-**open** if \( X \backslash A \) is \( \mathcal{G} - rw \)-closed.

**Proposition 1.3.2** For a topological space \((X, \tau)\) and a grill \( \mathcal{G} \) on \( X \), we obtain as follows.

(a) Every closed set in \( X \) is \( \mathcal{G} - rw \)-closed.

(b) For any subset \( A \) in \( X \), \( \Phi(A) \) is \( \mathcal{G} - rw \)-closed.

(c) Every \( \tau_{\mathcal{G}} \)-closed set is \( \mathcal{G} - rw \)-closed.

(d) Any non member of \( \mathcal{G} \) is \( \mathcal{G} - rw \)-closed.
(e) Every \( w \)-closed set is \( G \text{-}rw \)-closed.

(f) Every \( rw \)-closed set is \( G \text{-}rw \)-closed.

(g) Every \( \theta \)-closed set in \( X \) is \( G \text{-}rw \)-closed.

(h) Every \( \delta \)-closed set in \( X \) is \( G \text{-}rw \)-closed.

**Proof.** (a) Let \( A \) be a closed set then \( cl(A) = A \). Let \( U \) be any regular semiopen set in \( X \ni A \subseteq U \). Then \( \Phi(A) \subseteq cl(A) = A \subseteq U \) [by Theorem 1.2.19] \( \Rightarrow \Phi(A) \subseteq U \Rightarrow A \) is \( G \text{-}rw \)-closed.

(b) Let \( A \) be a subset in \( X \). Then \( \Phi(\Phi(A)) \subseteq \Phi(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is regular semiopen in \( X \) \( \Rightarrow \Phi(A) \) is \( G \text{-}rw \)-closed.

(c) Let \( A \) be a \( \tau_g \)-closed set then \( \tau_g \text{-}cl(A) = A \Rightarrow A \cup \Phi(A) = A \Rightarrow \Phi(A) \subseteq A \). Therefore \( \Phi(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is regular semiopen in \( X \Rightarrow A \) is \( G \text{-}rw \)-closed.

(d) Let \( A \notin G \) then \( \Phi(A) = \phi \Rightarrow A \) is \( G \text{-}rw \)-closed.

(e) Let \( A \) be a \( w \)-closed set and \( U \) be any regular semiopen set in \( X \ni A \subseteq U \) then \( cl(A) \subseteq U \), since every regular semiopen set is semiopen in \( X \). Then \( \Phi(A) \subseteq cl(A) \subseteq U \Rightarrow A \) is \( G \text{-}rw \)-closed. Thus every \( w \)-closed set is \( G \text{-}rw \)-closed.

(f) Let \( A \) be a \( rw \)-closed set and \( U \) be any regular semiopen set in \( X \ni A \subseteq U \) then \( cl(A) \subseteq U \), by Theorem 1.2.19 \( \Phi(A) \subseteq cl(A) \subseteq U \Rightarrow A \) is \( G \text{-}rw \)-closed. Thus every \( rw \)-closed set is \( G \text{-}rw \)-closed.

(g) Let \( A \) be a \( \theta \)-closed then \( A = \theta cl(A) \). Let \( U \) be a regular semiopen set in \( X \) such that \( A \subseteq U \) then by Theorem 1.2.19, \( \Phi(A) \subseteq cl(A) \subseteq \theta cl(A) = A \subseteq U \). Thus \( A \) is \( G \text{-}rw \)-closed.
(h) Let $A$ be a $\delta$-closed then $A = \delta cl(A)$. Let $U$ be a regular semiopen set in $X$ such that $A \subseteq U$ then by Theorem 1.2.19, $\Phi(A) \subseteq cl(A) \subseteq \delta cl(A) = A \subseteq U$. Thus $A$ is $G$-rw-closed.

The converse of the above proposition is not true in general as seen from the following examples.

**Example 1.3.3** Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $G = \{\{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, X\}$. Then $(X, \tau)$ is a topological space and $G$ is a grill on $X$. Then it is easy to verify that

(a) $\{a, b\}$ is not closed but is $G$-rw-closed.

(b) $\{a, b\}$ is not $\tau_G$-closed but is $G$-rw-closed.

(c) $\{c, d\}$ is not a grill but is $G$-rw-closed.

(d) $\{a, b\}$ is not $w$-closed but is $G$-rw-closed.

(e) $\{b\}$ is not $rw$-closed but is $G$-rw-closed.

**Example 1.3.4** Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b, c\}\}$ and $G = \{\{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, X\}$. Then $(X, \tau)$ is a topological space and $G$ is a grill on $X$. Then it is easy to verify that $\{a, b\}$ is not $\theta$-closed (resp. $\delta$-closed) but is $G$-rw-closed.

**Remark 1.3.5** The following Example shows that the concept of $G$-rw-closed sets and $G$-g-closed sets, $\theta G$-closed sets, $\delta G$-closed sets are independent of each other.

**Example 1.3.6** In the Example 1.3.3, the set $\{a, d\}$ is $G$-g-closed set, $\theta G$-closed set, $\delta G$-closed set but not a $G$-rw-closed set. Also the set $\{a, b\}$ is $G$-rw-closed set but not a $G$-g-closed set, not a $\theta G$-closed set, not a $\delta G$-closed set in $X$. 

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Remark 1.3.7 From the above discussions and known results we have the following implications. Here

$A \rightarrow B$ means $A$ implies $B$, but not conversely and

$A \leftrightarrow B$ means $A$ and $B$ are independent of each other.

Corresponding to any nonempty subset $A$ of $X$, a typical grill $[A]$ on $X$ was defined in [59] in the following manner.

Definition 1.3.8 Let $X$ be a space and $(\phi \neq) A \subseteq X$. Then $[A] = \{B \subseteq X : A \cap B \neq \phi\}$ is a grill on $X$, called the principal grill generated by $A$.

Proposition 1.3.9 In the case of principal grill $[X]$ generated by $X$, it is known [59] that $\tau = \tau_{[X]}$ so that any $[X]$-$rw$-closed set becomes simply a $rw$-closed set and vice-versa.

In what follows in this section, we derive certain characterizations and properties of $G$-$rw$-closed sets.

Theorem 1.3.10 Let $(X, \tau)$ be a topological space and $G$ be a grill on $X$. Then for a subset $A$ of $X$, the following are equivalent:

(a) $A$ is $G$-$rw$-closed.

(b) $\tau_{G-cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular semiopen.
(c) For all \( x \in \tau_{\mathcal{G}}-\text{cl}(A) \), \( \text{cl}\{\{x\}\} \cap A \neq \phi \).

(d) \( \tau_{\mathcal{G}}-\text{cl}(A) \setminus A \) contains no non-empty closed set of \( (X, \tau) \).

(e) \( \Phi(A) \setminus A \) contains no non-empty closed set of \( (X, \tau) \).

**Proof.** (a) \( \Rightarrow \) (b): Suppose \( A \) is \( \mathcal{G} \)-rw-closed set and \( A \subseteq U \) where \( U \) is regular semiopen in \( X \). Then \( \Phi(A) \subseteq U \Rightarrow A \cup \Phi(A) \subseteq U \Rightarrow \tau_{\mathcal{G}}-\text{cl}(A) \subseteq U \).

(b) \( \Rightarrow \) (c): Suppose \( x \in \tau_{\mathcal{G}}-\text{cl}(A) \). If \( \text{cl}\{\{x\}\} \cap A = \phi \), then \( A \subseteq X \setminus \text{cl}\{\{x\}\} \) and using (b), \( \tau_{\mathcal{G}}-\text{cl}(A) \subseteq X \setminus \text{cl}\{\{x\}\} \) which is a contradiction to our assumption that \( x \in \tau_{\mathcal{G}}-\text{cl}(A) \). Therefore \( \text{cl}\{\{x\}\} \cap A \neq \phi \).

(c) \( \Rightarrow \) (d): Suppose \( F \) is a closed set of \( (X, \tau) \) contained in \( \tau_{\mathcal{G}}-\text{cl}(A) \setminus A \) and \( x \in F \). Since \( F \cap A = \phi \), we have \( \text{cl}\{\{x\}\} \cap A = \phi \). Again since \( x \in \tau_{\mathcal{G}}-\text{cl}(A) \), by (c) we have \( \text{cl}\{\{x\}\} \cap A \neq \phi \), a contradiction. This proves (d).

(d) \( \Rightarrow \) (e): It follows from the fact that \( \Phi(A) \setminus A = \tau_{\mathcal{G}}-\text{cl}(A) \setminus A \).

(e) \( \Rightarrow \) (a): Suppose that \( A \subseteq U \) and \( U \) is regular semiopen in \( (X, \tau) \). Since \( \Phi(A) \) is closed (by Theorem 1.2.19) and \( \Phi(A) \cap (X \setminus U) \subseteq \Phi(A) \setminus A \) holds, \( \Phi(A) \cap (X \setminus U) \) is a closed set in \( (X, \tau) \) contained in \( \Phi(A) \setminus A \). Then by (e), \( \Phi(A) \cap (X \setminus U) = \phi \) which gives \( \Phi(A) \subseteq U \). Hence \( A \) is \( \mathcal{G} \)-rw-closed.

**Corollary 1.3.11** Let \( (X, \tau) \) be a \( T_1 \)-space and \( \mathcal{G} \) be a grill on \( X \). Then every \( \mathcal{G} \)-rw-closed set is \( \tau_{\mathcal{G}} \)-closed.

**Proof.** Follows from Theorem 1.3.10 ((a) \( \Rightarrow \) (c)).

**Corollary 1.3.12** Let \( (X, \tau) \) be a \( T_1 \)-space and \( \mathcal{G} \) be a grill on \( X \). Then \( A \subseteq X \) is \( \mathcal{G} \)-rw-closed iff \( A \) is \( \tau_{\mathcal{G}} \)-closed.
**Corollary 1.3.13** Let $\mathcal{G}$ be grill on a space $(X, \tau)$ and $A$ be a $\mathcal{G}$-rw-closed set. Then the following are equivalent:

(a) $A$ is $\tau_\mathcal{G}$-closed.

(b) $\tau_\mathcal{G} \cdot cl(A) \setminus A$ is closed in $(X, \tau)$.

(c) $\Phi(A) \setminus A$ is closed in $(X, \tau)$.

**Proof.** $(a) \Rightarrow (b)$ Let $A$ be $\tau_\mathcal{G}$-closed then $\tau_\mathcal{G} \cdot cl(A) \setminus A = \phi$ and so $\tau_\mathcal{G} \cdot cl(A) \setminus A$ is a closed set.

$(b) \Rightarrow (c)$ It is clear, since $\tau_\mathcal{G} \cdot cl(A) \setminus A = \Phi(A) \setminus A$.

$(c) \Rightarrow (a)$ Let $\Phi(A) \setminus A$ be closed in $(X, \tau)$ and $A$ is $\mathcal{G}$-rw-closed, then by Theorem 1.3.10, $\Phi(A) \setminus A = \phi$ and so $A$ is $\tau_\mathcal{G}$-closed. 

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**Lemma 1.3.14** [58] Let $(X, \tau)$ be a space and $\mathcal{G}$ be a grill on $X$. If $A(\subseteq X)$ is $\tau_\mathcal{G}$-dense in itself, then $\Phi(A) = cl(\Phi(A)) = \tau_\mathcal{G} \cdot cl(A) = cl(A)$.

**Theorem 1.3.15** Let $\mathcal{G}$ be a grill on a space $(X, \tau)$. If $A(\subseteq X)$ is $\tau_\mathcal{G}$-dense in itself and $\mathcal{G}$-rw-closed, then $A$ is rw-closed.

**Proof.** Follows at once from Lemma 1.3.14.

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**Corollary 1.3.16** For a grill $\mathcal{G}$ on a space $(X, \tau)$, let $A(\subseteq X)$ be $\tau_\mathcal{G}$-dense in itself. Then $A$ is $\mathcal{G}$-rw-closed iff it is rw-closed.

**Proof.** Follows from Proposition 1.3.2(f) and Theorem 1.3.15.

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**Theorem 1.3.17** For any grill $\mathcal{G}$ on a space $(X, \tau)$ the following are equivalent:

(a) Every subset of $X$ is $\mathcal{G}$-rw-closed.

(b) Every regular semiopen subset of $(X, \tau)$ is $\tau_\mathcal{G}$-closed.
**Proof.** (a) ⇒ (b): Let \( A \) be regular semiopen in \((X, \tau)\) then by (a), \( A \) is \( \mathcal{G} \)-\( rw \)-closed so that \( \Phi(A) \subseteq A \Rightarrow A \) is \( \tau_{\mathcal{G}} \)-closed.

(b) ⇒ (a): Let \( A \subseteq X \) and \( U \) be regular semiopen in \((X, \tau)\) such that \( A \subseteq U \). Then by (b), \( \Phi(U) \subseteq U \). Again \( A \subseteq U \Rightarrow \Phi(A) \subseteq \Phi(U) \) (by Theorem 1.2.19) \( \subseteq U \Rightarrow A \) is \( \mathcal{G} \)-\( rw \)-closed.

**Proposition 1.3.18** For any subset \( A \) of a space \((X, \tau)\) and a grill \( \mathcal{G} \) on \( X \), the following are equivalent:

(a) \( A \) is \( \mathcal{G} \)-\( rw \)-closed.

(b) \( A \cup (X \setminus \Phi(A)) \) is \( \mathcal{G} \)-\( rw \)-closed.

(b) \( \Phi(A) \setminus A \) is \( \mathcal{G} \)-\( rw \)-open.

**Proof.** (a) ⇒ (b): Let \( A \cup (X \setminus \Phi(A)) \subseteq U \), where \( U \) is regular semiopen in \( X \). Then \( X \setminus U \subseteq X \setminus (A \cup (X \setminus \Phi(A))) = \Phi(A) \setminus A \). Since \( A \) is \( \mathcal{G} \)-\( rw \)-closed, by Theorem 1.3.10, we have \( X \setminus U = \phi \), i.e., \( X = U \). Since \( X \) is the only regular semiopen set containing \( A \cup (X \setminus \Phi(A)) \), \( A \cup (X \setminus \Phi(A)) \) is \( \mathcal{G} \)-\( rw \)-closed.

(b) ⇒ (a): Suppose \( F \subseteq \Phi(A) \setminus A \) where \( F \) is regular semiclosed in \((X, \tau)\). Then \( A \cup (X \setminus \Phi(A)) \subseteq X \setminus F \) and so by (b), \( \Phi(A \cup (X \setminus \Phi(A))) \subseteq X \setminus F \Rightarrow \Phi(A) \cup \Phi(X \setminus \Phi(A)) \subseteq X \setminus F \Rightarrow F \subseteq X \setminus \Phi(A) \). Again, since \( F \subseteq \Phi(A) \) we have \( F = \phi \). Hence by Theorem 1.3.10, \( A \) is \( \mathcal{G} \)-\( rw \)-closed.

(b) ⇒ (c): Follows from the fact that \( X \setminus (\Phi(A) \setminus A) = A \cup (X \setminus \Phi(A)) \).

**Theorem 1.3.19** Let \((X, \tau)\) be a space, \( \mathcal{G} \) be a grill on \( X \) and \( A, B \) be subsets of \( X \) such that \( A \subseteq B \subseteq \tau_{\mathcal{G}} \)-\( cl(A) \). If \( A \) is \( \mathcal{G} \)-\( rw \)-closed, then \( B \) is \( \mathcal{G} \)-\( rw \)-closed.

**Proof.** Suppose \( B \subseteq U \), where \( U \) is regular semiopen in \( X \). Since \( A \) is \( \mathcal{G} \)-\( rw \)-
closed, $\Phi(A) \subseteq U \Rightarrow \tau_g \text{-} \text{cl}(A) \subseteq U$. Now, $A \subseteq B \subseteq \tau_g \text{-} \text{cl}(A) \Rightarrow \tau_g \text{-} \text{cl}(B) \subseteq \tau_g \text{-} \text{cl}(A)$. Thus $\tau_g \text{-} \text{cl}(B) \subseteq U$ and hence $B$ is $\mathcal{G} \text{-} \text{rw}-\text{closed}$. 

\textbf{Corollary 1.3.20} $\tau_g$-closure of every $\mathcal{G} \text{-} \text{rw}$-closed set is $\mathcal{G} \text{-} \text{rw}$-closed.

\textbf{Theorem 1.3.21} Let $\mathcal{G}$ be a grill on a space $(X, \tau)$ and $A, B$ be subsets of $X$ such that $A \subseteq B \subseteq \Phi(A)$. If $A$ is $\mathcal{G} \text{-} \text{rw}$-closed. Then $A$ and $B$ are $\text{rw}$-closed.

\textbf{Proof.} $A \subseteq B \subseteq \Phi(A) \Rightarrow A \subseteq B \subseteq \tau_g \text{-} \text{cl}(A)$, and hence by Theorem 1.3.19, $B$ is $\mathcal{G} \text{-} \text{rw}$-closed. Again, $A \subseteq B \subseteq \Phi(A) \Rightarrow \Phi(A) \subseteq \Phi(B) \subseteq \Phi(\Phi(A)) \subseteq \Phi(A)$ (by Theorem 1.2.19) $\Rightarrow \Phi(A) = \Phi(B)$. Thus $A$ and $B$ are $\tau_g$-dense in itself and hence by Theorem 1.3.15, $A$ and $B$ are $\text{rw}$-closed.

\textbf{Theorem 1.3.22} Let $\mathcal{G}$ be a grill on a space $(X, \tau)$. Then a subset $A$ of $X$ is $\mathcal{G} \text{-} \text{rw}$-open iff $F \subseteq \tau_g \text{-} \text{int}(A)$ whenever $F \subseteq A$ and $F$ is closed.

\textbf{Proof.} Let $A$ be $\mathcal{G} \text{-} \text{rw}$-open and $F \subseteq A$, where $F$ is closed in $(X, \tau)$. Then $X \setminus A \subseteq X \setminus F \Rightarrow \Phi(X \setminus A) \subseteq X \setminus F \Rightarrow \tau_g \text{-} \text{cl}(X \setminus A) \subseteq X \setminus F \Rightarrow F \subseteq \tau_g \text{-} \text{int}(A)$.

Conversely, $X \setminus A \subseteq U$ where $U$ is open in $(X, \tau) \Rightarrow X \setminus U \subseteq \tau_g \text{-} \text{int}(A) \Rightarrow \tau_g \text{-} \text{cl}(X \setminus A) \subseteq U$. Thus $X \setminus A$ is $\mathcal{G} \text{-} \text{rw}$-closed and hence $A$ is $\mathcal{G} \text{-} \text{rw}$-open.

\section{1.4 $\mathcal{G} \text{-} \text{rw}$-Continuous and $\mathcal{G} \text{-} \text{rw}$-Closed Functions}

\textbf{Definition 1.4.1} A function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ is said to be $\mathcal{G} \text{-} \text{rw}$-\textit{continuous} if $f^{-1}(V)$ is $\mathcal{G} \text{-} \text{rw}$-open for each $V \in \sigma$.

\textbf{Remark 1.4.2}
(i) Every continuous function (resp., $w$-continuous) is $rw$-continuous, but the converse is false as is shown in Examples 3.2.3 and 3.2.6 in [76].

(ii) Every $rw$-continuous function is $\mathcal{G}$-$rw$-continuous, but the converse is false as is shown in Example 1.4.3.

But the reverses of the above implications are false as is shown below.

**Example 1.4.3** In the Example 1.3.3, we define a function $f : (X, \tau, \mathcal{G}) \to (X, \tau)$ as follows: $f(a) = c$, $f(b) = d$, $f(c) = a$ and $f(d) = b$. Then it is easy to see that $f$ is $\mathcal{G}$-$rw$-continuous but not $rw$-continuous (in fact, $A = \{d\} \in \tau^c$ and $f^{-1}(\{d\}) = \{b\}$ is not $rw$-closed).

**Remark 1.4.4** Every $\theta$-continuous (resp., $\delta$-continuous) is $\mathcal{G}$-$rw$-continuous, but the converse is false as is shown in Example 1.4.5.

**Example 1.4.5** Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ and $\mathcal{G} = \{\{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, X\}$. Then the identity function $f : (X, \tau, \mathcal{G}) \to (X, \tau)$ is $\mathcal{G}$-$rw$-continuous but not $\theta$-continuous (resp., $\delta$-continuous) (in fact, $A = \{c\} \in \tau^c$ and $f^{-1}(\{c\}) = \{c\}$ is not $\theta$-closed (resp., $\delta$-closed).

**Remark 1.4.6** In the following Examples (1.4.7 and 1.4.8) show that $\mathcal{G}$-$rw$-continuous function and $\mathcal{G}$-$g$-continuous, $\theta g$-continuous, $\delta g$-continuous functions are independent.

**Example 1.4.7** In the Example 1.3.3, we define a function $f : (X, \tau, \mathcal{G}) \to (X, \tau)$ as follows: $f(a) = c$, $f(b) = b$, $f(c) = a$ and $f(d) = d$. Then the inverse image of
every closed set in $Y$ is $\mathcal{G}$-$g$-closed, $\theta g$-closed, $\delta g$-closed in $X$ and hence $f$ is $\mathcal{G}$-$g$-continuous, $\theta g$-continuous, $\delta g$-continuous. But $f$ is not $\mathcal{G}$-$rw$-continuous as the inverse image of the closed set $\{c, d\}$ is $\{a, d\}$ in $X$ which is not $\mathcal{G}$-$rw$-closed.

**Example 1.4.8** In the Example 1.3.3, we define a function $f : (X, \tau, \mathcal{G}) \to (X, \tau)$ as follows: $f(a) = c$, $f(b) = d$, $f(c) = a$ and $f(d) = b$. Then the inverse image of every closed set in $Y$ is $\mathcal{G}$-$rw$-closed in $X$ and hence $f$ is $\mathcal{G}$-$rw$-continuous.
Let $\{c, d\}$ is closed set in $Y$, $f^{-1}(\{c, d\}) = \{a, b\}$ is not $\mathcal{G}$-$g$-closed, $\theta g$-closed, $\delta g$-closed in $X$. Thus $f$ is not $\mathcal{G}$-$g$-continuous, $\theta g$-continuous, $\delta g$-continuous.

**Remark 1.4.9** From the above discussions and known results we have the following implications. Here
$A \to B$ means $A$ implies $B$, but not conversely and
$A \leftrightarrow B$ means $A$ and $B$ are independent of each other.

![Diagram]

**Theorem 1.4.10** For a function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$, the following are equivalent:

(a) $f$ is $\mathcal{G}$-$rw$-continuous.

(b) The inverse image of each closed subset of $Y$ is $\mathcal{G}$-$rw$ closed.
(c) For each \( x \in X \) and each \( V \in \sigma \) containing \( f(x) \), there exists a \( \mathcal{G} \)-\( rw \)-open set \( U \) containing \( x \) such that \( f(U) \subseteq V \).

**Proof.** (a) \( \Leftrightarrow \) (b): It is clear.

(a) \( \Rightarrow \) (c): Let \( V \in \sigma \) and \( f(x) \in V(x \in X) \). Then by (a), \( f^{-1}(V) \) is a \( \mathcal{G} \)-\( rw \)-open set containing \( x \). Taking \( f^{-1}(V) = U \), we have \( x \in U \) and \( f(U) \subseteq V \).

(c) \( \Rightarrow \) (a): Let \( V \) be any open set in \( Y \) and \( x \in f^{-1}(V) \). Then \( f(x) \in V \in \sigma \) and hence by (c), there exists a \( \mathcal{G} \)-\( rw \)-open set \( U \) containing \( x \) such that \( f(U) \subseteq V \). Now \( x \in U \subseteq \Psi(int(U)) \subseteq \Psi(int(f^{-1}(V))) \). This shows that \( f^{-1}(V) \subseteq \Psi(int(f^{-1}(V))) \). Thus \( f \) is \( \mathcal{G} \)-\( rw \)-continuous. \( \blacksquare \)

**Theorem 1.4.11** A function \( f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma) \) is \( \mathcal{G} \)-\( rw \)-continuous iff the graph function \( g : X \rightarrow X \times Y \), defined by \( g(x) = (x, f(x)) \), for each \( x \in X \), is \( \mathcal{G} \)-\( rw \)-continuous.

**Proof.** Suppose that \( f \) is \( \mathcal{G} \)-\( rw \)-continuous. Let \( x \in X \) and \( W \) be any open set in \( X \times Y \) containing \( g(x) \). Then there exist \( U \in \tau \) and \( V \in \sigma \) such that \( g(x) = (x, f(x)) \in U \times V \subseteq W \). Since \( f \) is \( \mathcal{G} \)-\( rw \)-continuous, there exists a \( \mathcal{G} \)-\( rw \)-open set \( G \) of \( X \) containing \( x \) such that \( f(G) \subseteq V \), \( G \cap U \) is \( \mathcal{G} \)-\( rw \)-open and \( g(G \cap U) \subseteq U \times V \subseteq W \). This shows that \( g \) is \( \mathcal{G} \)-\( rw \)-continuous.

Conversely, suppose that \( g \) is \( \mathcal{G} \)-\( rw \)-continuous. Let \( x \in X \) and \( V \) be any open set in \( Y \) containing \( f(x) \). Then \( X \times V \) is open in \( X \times Y \) and by \( \mathcal{G} \)-\( rw \)-continuity of \( g \), there exists a \( \mathcal{G} \)-\( rw \)-open set \( U \) containing \( x \) such that \( g(U) \subseteq X \times V \). Thus we have \( f(U) \subseteq V \) and hence \( f \) is \( \mathcal{G} \)-\( rw \)-continuous. \( \blacksquare \)

**Definition 1.4.12** Let \( (X, \tau) \) be a topological space and \( (Y, \sigma, \mathcal{G}) \) a grill topo-
logical space. A function \( f : (X, \tau) \rightarrow (Y, \sigma, \mathcal{G}) \) is said to be \( \mathcal{G} \text{-} rw \text{-open} \) (resp., \( \mathcal{G} \text{-}_rw\text{-closed} \)) if for each \( U \in \tau \) (resp., closed set \( U \) in \( (X, \tau) \)), \( f(U) \) is \( \mathcal{G} \text{-}rw\text{-open} \) (resp., \( \mathcal{G} \text{-}rw\text{-closed} \)) in \( (Y, \sigma, \mathcal{G}) \).

**Remark 1.4.13** (a) Every closed (resp., \( w \)-closed) function is \( rw \)-closed, but the converse is false as is shown in Examples 3.4.3 and 3.4.4 [76].

(b) Every \( rw \)-closed function is \( \mathcal{G} \text{-}rw\text{-closed} \), but the converse is false as is shown in Example 1.4.14.

**Example 1.4.14** Let \( X = \{a, b, c, d\} \), \( \tau = \{X, \phi, \{a, c, d\}\} \), \( \sigma = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \) and \( \mathcal{G} = \{\{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, X\} \). Then the identity function \( f : (X, \tau) \rightarrow (X, \sigma, \mathcal{G}) \) is \( \mathcal{G} \text{-}rw\text{-closed} \), but not \( rw \)-closed.

**Remark 1.4.15** Every \( \theta \)-closed (resp. \( \delta \)-closed) function is \( \mathcal{G} \text{-}rw\text{-closed} \), but the converse is false as is shown in Example 1.4.16.

**Example 1.4.16** Let \( X = \{a, b, c\} \), \( \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}\} \) and \( \mathcal{G} = \{\{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, X\} \). Then the identity function \( f : (X, \tau) \rightarrow (X, \tau, \mathcal{G}) \) is \( \mathcal{G} \text{-}rw\text{-closed} \) but it is not \( \theta \)-closed (resp., \( \delta \)-closed).

**Remark 1.4.17** In the following Examples (1.4.18 and 1.4.19) show that \( \mathcal{G} \text{-}rw\text{-closed} \) function and \( \mathcal{G} \text{-}g\text{-closed}, \ \theta g\text{-closed}, \ \delta g\text{-closed} \) functions are independent.

**Example 1.4.18** In the Example 1.3.3, we define a function \( f : (X, \tau) \rightarrow (X, \tau, \mathcal{G}) \) as follows: \( f(a) = c, \ f(b) = b, \ f(c) = a \) and \( f(d) = d \). Then this function is \( \mathcal{G} \text{-}g\text{-closed}, \ \theta g\text{-closed}, \ \delta g\text{-closed} \) but not \( \mathcal{G} \text{-}rw\text{-closed} \), as the image of the closed set \( \{c, d\} \) in \( X \) is \( \{a, d\} \) which is not \( \mathcal{G} \text{-}rw\text{-closed} \) in \( X \).
Example 1.4.19 In the Example 1.3.3, we define a function \( f : (X, \tau) \to (X, \tau, \mathcal{G}) \) as follows: \( f(a) = c, \ f(b) = d, \ f(c) = a \) and \( f(d) = b \). Then the image of every closed set in \( X \) is \( \mathcal{G} \)-\( rw \)-closed in \( X \) and hence \( f \) is \( \mathcal{G} \)-\( rw \)-closed function. Let \( \{c, d\} \) is closed set in \( X \), \( f(\{c, d\}) = \{a, b\} \) is not \( \mathcal{G} \)-\( g \)-closed, \( \theta g \)-closed, \( \delta g \)-closed in \( X \). Thus \( f \) is not \( \mathcal{G} \)-\( g \)-closed, \( \theta g \)-closed, \( \delta g \)-closed functions.

Remark 1.4.20 From the above discussions and known results we have the following implications. Here

\[ A \to B \] means \( A \) implies \( B \), but not conversely and

\[ A \leftrightarrow B \] means \( A \) and \( B \) are independent of each other.

\[
\begin{aligned}
\theta g \text{-closed function} & \quad \to \quad \mathcal{G} \text{-closed function} \\
\text{closed function} & \quad \to \quad w \text{-closed function} \\
\delta g \text{-closed function} & \quad \to \quad \mathcal{G} \text{-} \text{rw-closed function}
\end{aligned}
\]

Theorem 1.4.21 Let \( f : (X, \tau) \to (Y, \sigma, \mathcal{G}) \) be a \( \mathcal{G} \)-\( rw \)-open function. If \( V \) is any subset of \( Y \) and \( F \) is a closed subset of \( X \) containing \( f^{-1}(V) \), then there exists a \( \mathcal{G} \)-\( rw \)-open set \( H \) in \( (Y, \sigma, \mathcal{G}) \) containing \( V \) such that \( f^{-1}(H) \subseteq F \).

Proof. Suppose that \( f \) is \( \mathcal{G} \)-\( rw \)-open. Let \( V \) be any subset of \( Y \) and \( F \) be a closed subset of \( X \) containing \( f^{-1}(V) \). Then \( X \setminus F \) is open in \( (X, \tau) \) and hence by \( \mathcal{G} \)-\( rw \)-openness of \( f \), \( f(X \setminus F) \) is \( \mathcal{G} \)-open. Thus \( H = Y \setminus f(X \setminus F) \) is \( \mathcal{G} \)-\( rw \)-closed and consequently \( f^{-1}(V) \subseteq F \) implies that \( V \subseteq H \). Further we obtain that \( f^{-1}(H) \subseteq F \). \[ \blacksquare \]
Theorem 1.4.22 For any bijection \( f : (X, \tau) \to (Y, \sigma, \mathcal{G}) \) the following are equivalent:

(a) \( f^{-1} : (Y, \sigma, \mathcal{G}) \to (X, \tau) \) is \( \mathcal{G} \)-\( rw \)-continuous.

(b) \( f \) is \( \mathcal{G} \)-\( rw \)-open.

(c) \( f \) is \( \mathcal{G} \)-\( rw \)-closed.

Proof. Obvious.

1.5 Some Characterizations of Regular and Normal Spaces

As already proposed, this section is meant for deriving certain applications of the study in the last section; some characterizations of regular and normal spaces are achieved here in terms of the introduced concept of \( \mathcal{G} \)-\( rw \)-closed sets.

Theorem 1.5.1 Let \( X \) be a normal space and \( \mathcal{G} \) be a grill on \( X \) then for each pair of disjoint closed sets \( F \) and \( K \), there exist disjoint \( \mathcal{G} \)-\( rw \)-open sets \( U \) and \( V \) such that \( F \subseteq U \) and \( K \subseteq V \).

Proof. It is obvious, since every open set is \( \mathcal{G} \)-\( rw \)-open.

Theorem 1.5.2 Let \( X \) be a normal space and \( \mathcal{G} \)-be a grill on \( X \) then for each closed set \( F \) and any open set \( V \) containing \( F \), there exist a \( \mathcal{G} \)-\( rw \)-open set \( U \) such that \( F \subseteq U \subseteq \tau_{\mathcal{G}} - cl(U) \subseteq V \).

Proof. Let \( F \) be a closed set and \( V \) an open set in \( (X, \tau) \) such that \( F \subseteq V \). Then \( F \) and \( X \setminus V \) are disjoint closed sets. By Theorem 1.5.1, there exist disjoint \( \mathcal{G} \)-\( rw \)-
open sets $U$ and $W$ such that $F \subseteq U$ and $X\setminus V \subseteq W$. Since $W$ is $\mathcal{G}$-rw-open and $X\setminus V \subseteq W$ where $X\setminus V$ is closed, by Theorem 1.3.22 $X\setminus V \subseteq \tau\mathcal{G}$-int$(W)$. So $X\setminus V \subseteq \tau\mathcal{G}$-int$(W) \subseteq V$. Again, $U \cap W = \phi \Rightarrow U \cap \tau\mathcal{G}$-int$(W) = \phi$. Hence $\tau\mathcal{G}$-cl$(U) \subseteq X\setminus \tau\mathcal{G}$-int$(W) \subseteq V$. Thus $F \subseteq U \subseteq \tau\mathcal{G}$-cl$(U) \subseteq V$ where $U$ is a $\mathcal{G}$-rw-open set.

The following theorems gives characterizations of a normal space in terms of $rw$-open sets which are the consequence of Theorems 1.5.1, 1.5.2 and Proposition 1.3.9 if one takes $\mathcal{G} = [X]$.

**Theorem 1.5.3** Let $X$ be a normal space and $\mathcal{G}$ be a grill on $X$ then for each pair of disjoint closed sets $F$ and $K$, there exist disjoint $rw$-open sets $U$ and $V$ such that $F \subseteq U$ and $K \subseteq V$.

**Proof.** It is obvious, since every open set is $rw$-open [76].

**Theorem 1.5.4** Let $X$ be a normal space and $\mathcal{G}$ be a grill on $X$ then for each closed set $F$ any open set $V$ containing $F$, there exist a $rw$-open set $U$ such that $F \subseteq U \subseteq \tau\mathcal{G}$-cl$(U) \subseteq V$.

**Proof.** Proof is easy consequence from Theorem 1.5.2.

**Theorem 1.5.5** Let $X$ be regular and $\mathcal{G}$ be a grill on a space $(X, \tau)$. Then for each closed set $F$ and each $x \in X\setminus F$, there exist disjoint $\mathcal{G}$-rw-open sets $U$ and $V$ such that $x \in U$ and $F \subseteq V$.

**Proof.** The proof is immediate.
Theorem 1.5.6  Let $X$ be a regular space and $\mathcal{G}$ be a grill on a space $(X, \tau)$. Then for each regular semiopen set $V$ of $(X, \tau)$ and each point $x \in V$ there exist a $\mathcal{G}$-$rw$-open set $U$ such that $x \in U \subseteq \tau_\mathcal{G} - cl(U) \subseteq V$.

Proof. Let $V$ be any regular semiopen in $(X, \tau)$ containing a point $x$ of $X$. Then by Theorem 1.5.5, there exist disjoint $\mathcal{G}$-$rw$-open sets $U$ and $W$ such that $x \in U$ and $X \setminus V \subseteq W$. Now, $U \cap W = \phi$ implies $\tau_\mathcal{G} - cl(U) \subseteq X \setminus W \subseteq V$. Thus $x \in U \subseteq \tau_\mathcal{G} - cl(U) \subseteq V$. ■

The following theorems gives characterizations of a regular space in terms of $rw$-open sets which are the consequence of Theorems 1.5.5, 1.5.6 and Proposition 1.3.9 if one takes $\mathcal{G} = [X]$.

Theorem 1.5.7  Let $X$ be a regular and $\mathcal{G}$ be a grill on a space $(X, \tau)$. Then for each closed set $F$ and each $x \in X \setminus F$, there exist disjoint $rw$-open sets $U$ and $V$ such that $x \in U$ and $F \subseteq V$.

Proof. It is obvious, since every open set is $rw$-open [76]. ■

Theorem 1.5.8  Let $X$ be a regular space and $\mathcal{G}$ be a grill on a space $(X, \tau)$. Then for each regular semiopen set $V$ of $(X, \tau)$ and each point $x \in V$ there exist a $rw$-open set $U$ such that $x \in U \subseteq \tau_\mathcal{G} - cl(U) \subseteq V$.

Proof. Proof is easy consequence from Theorem 1.5.6. ■

1.6  Grill Regular Weakly Closed Space

In this section, we will introduce grill regular weakly closed space $(X, \tau^{rw}, \mathcal{G})$ by
replacing $\tau$ by $\tau^w$ of a grill topological space $(X, \tau, \mathcal{G})$. We will introduce the operators $\Phi_{rw\mathcal{G}}, \Psi_{rw\mathcal{G}}$ and discuss various properties of these operators.

**Definition 1.6.1** Let $(X, \tau^w, \mathcal{G})$ be a grill regular weakly closed space. We define a mapping $\Phi_{rw\mathcal{G}}: \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, denoted by $\Phi_{rw\mathcal{G}}(A, \tau^w)$ (for $A \in \mathcal{P}(X)$) or $\Phi_{rw\mathcal{G}}(A)$, called the operator associated with the grill $\mathcal{G}$ and $\tau^w$, and is defined by $\Phi_{rw\mathcal{G}}(A) = \{x \in X : U \cap A \in \mathcal{G} \forall \text{rw-closed set } U \text{ containing } x\}$.

In the next theorems, we discuss some of the properties of the $\Phi_{rw\mathcal{G}}$-operator.

**Theorem 1.6.2** Let $(X, \tau^w)$ be regular weakly closed space and let $\mathcal{G}$ and $\mathcal{H}$ be two grills on $X$. For a subset $A$ of $X$, the following statements are hold:

(i) $\mathcal{G} \subset \mathcal{H} \Rightarrow \Phi_{rw\mathcal{G}}(A) \subseteq \Phi_{rw\mathcal{H}}(A)$.

(ii) $\Phi_{rw(\mathcal{G} \cup \mathcal{H})}(A) = \Phi_{rw\mathcal{G}}(A) \cup \Phi_{rw\mathcal{H}}(A)$.

**Proof.** (i) Let $x \in \Phi_{rw\mathcal{G}}(A)$. Then, $(U \cap A) \in \mathcal{G}$, for every rw-closed set $U$ containing $x$. By hypothesis $\mathcal{G} \subset \mathcal{H}$, we get $(U \cap A) \in \mathcal{H}$, so $x \in \Phi_{rw\mathcal{H}}(A)$. Hence, $\Phi_{rw\mathcal{G}}(A) \subseteq \Phi_{rw\mathcal{H}}(A)$.

(ii) The inclusion $\Phi_{rw\mathcal{G}}(A) \cup \Phi_{rw\mathcal{H}}(A) \subseteq \Phi_{rw(\mathcal{G} \cup \mathcal{H})}(A)$ follows directly. Let $x \notin \Phi_{rw\mathcal{G}}(A) \cup \Phi_{rw\mathcal{H}}(A)$. Then $x \notin \Phi_{rw\mathcal{G}}(A)$ and $x \notin \Phi_{rw\mathcal{H}}(A)$. Hence, there are rw-closed sets $U_1, U_2$ containing $x$ such that $(U_1 \cap A) \notin \mathcal{G}$ and $(U_2 \cap A) \notin \mathcal{H}$. Let $U = U_1 \cap U_2$ rw-closed set containing $x$ such that $(U \cap A) \notin \mathcal{G}$ and $(U \cap A) \notin \mathcal{H}$. This follows that $(U \cap A) \notin \mathcal{G} \cup \mathcal{H}$ and so $x \notin \Phi_{rw(\mathcal{G} \cup \mathcal{H})}(A)$ i.e., $\Phi_{rw(\mathcal{G} \cup \mathcal{H})}(A) = \Phi_{rw\mathcal{G}}(A) \cup \Phi_{rw\mathcal{H}}(A)$.

**Theorem 1.6.3** Let $(X, \tau^w, \mathcal{G})$ be a grill regular weakly closed space. Then, for any $A, B \subseteq X$ the following hold:
(i) \( A \subseteq B (\subseteq X) \Rightarrow \Phi_{rwG}(A) \subseteq \Phi_{rwG}(B) \).

(ii) \( \Phi_{rwG}(A \cup B) = \Phi_{rwG}(A) \cup \Phi_{rwG}(B) \).

(iii) \( \Phi_{rwG}(A \cap B) \subseteq \Phi_{rwG}(A) \cap \Phi_{rwG}(B) \).

(iv) \( \Phi_{rwG}(A) \subseteq \Phi_{G}(A) \).

(v) \( \Phi_{rwG}(A) \subseteq rw-cl(A) \subseteq cl(A) \).

(vi) \( \Phi_{rwG}(A) = \phi, \text{ if } A \notin \mathcal{G} \).

(vii) \( \Phi_{rwG}(A) - \Phi_{rwG}(B) \subseteq \Phi_{rwG}(A - B) \).

(viii) \( \Phi_{rwG}(A \cup B) = \Phi_{rwG}(A) = \Phi_{rwG}(A - B), \text{ if } B \notin \mathcal{G} \).

(ix) \( \Phi_{rwG}(A) = \Phi_{rwG}(B), \text{ if } (A - B) \cup (B - A) \notin \mathcal{G} \).

**Proof** (i) Obvious.

(ii) In view of (i) it suffices to show that \( \Phi_{rwG}(A \cup B) \subseteq \Phi_{rwG}(A) \cup \Phi_{rwG}(B) \).

Suppose \( x \notin \Phi_{rwG}(A) \cup \Phi_{rwG}(B) \). Then, there are \( rw \)-closed sets \( U_1, U_2 \) containing \( x \) such that \( A \cap U_1 \notin \mathcal{G}, B \cap U_2 \notin \mathcal{G} \). Hence, \( (A \cap U_1) \cup (B \cap U_2) \notin \mathcal{G} \). Now \( (U_1 \cap U_2) \) is \( rw \)-closed set containing \( x \) and \( (A \cup B) \cap (U_1 \cap U_2) \notin \mathcal{G} \). Hence, \( x \notin \Phi_{rwG}(A \cup B) \).

So \( \Phi_{rwG}(A \cup B) \subseteq \Phi_{rwG}(A) \cup \Phi_{rwG}(B) \). Consequently, \( \Phi_{rwG}(A \cup B) = \Phi_{rwG}(A) \cup \Phi_{rwG}(B) \).

(iii) It is obvious in view of (i).

(iv) \( x \notin \Phi_{G}(A) \). Then, there is an open nbd. \( U \) containing \( x \) such that \( A \cap U \notin \mathcal{G} \). Since \( \tau \subseteq \tau^{rw} \), Then, \( U \) is \( rw \)-closed set containing \( x \) and \( A \cap U \notin \mathcal{G} \). Hence, \( x \notin \Phi_{rwG}(A) \) and so \( \Phi_{rwG}(A) \subseteq \Phi_{G}(A) \).

(v) \( x \in \Phi_{rwG}(A) \). Then, \( \forall rw \)-closed set \( U \) containing \( x \); \( A \cap U \in \mathcal{G} \). Since \( \phi \notin \mathcal{G} \). Then, \( U \cap A \neq \phi \) and so \( x \in rw-cl(A) \). Thus, \( \Phi_{rwG}(A) \subseteq rw-cl(A) \).

Moreover, since \( \tau \subseteq \tau^{rw} \). Then, \( rw-cl(A) \subseteq cl(A) \).
(vi) Obvious from the definition of the operator $\Phi_{rwG}$.

(vii) Let $A$ and $B$ be subsets of $X$ and $A = (A - B) \cup (A \cap B)$. Then, (ii) implies that, $\Phi_{rwG}(A) = \Phi_{rwG}(A - B) \cup \Phi_{rwG}(A \cap B)$, also (iii) implies that $\Phi_{rwG}(A \cap B) \subseteq \Phi_{rwG}(B)$. Consequently, $\Phi_{rwG}(A) - \Phi_{rwG}(B) = \Phi_{rwG}(A - B) - \Phi_{rwG}(B) \subseteq \Phi_{rwG}(A - B)$.

(viii) Let $B \notin G$. In view of (ii), (vi) and (vii) we have $\Phi_{rwG}(A \cup B) = \Phi_{rwG}(A) = \Phi_{rwG}(A - B)$.

(ix) Let $E = (A - B) \cup (B - A) \notin G$. Then, $A = (E - B) \cup (B - E)$. By using (ii) and (vii), $\Phi_{rwG}(A) = \Phi_{rwG}(E - B) \cup \Phi_{rwG}(B - E)$ which implies $\Phi_{rwG}(A) = \Phi_{rwG}(E - B) \cup \Phi_{rwG}(B) = \Phi_{rwG}(B \cup E) = \Phi_{rwG}(B)$. ■

The following example is to prove the opposite of (iii) of Theorem 1.6.3 does not satisfied.

Example 1.6.4 Let $X = \{a, b, c\}$, and $\tau = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$, $G = [X]$ i. e., principal grill generated by $X$. Take $A = \{a, b\}$, $B = \{a, c\}$. Then, $\Phi_{rwG}(A) = X$, $\Phi_{rwG}(B) = \{a, c\}$ and $\Phi_{rwG}(A \cap B) = \{a\}$.

Corollary 1.6.5 Let $(X, \tau^{rw}, G)$ be a grill regular weakly closed space and $A \subseteq X$. Then,

(i) $\phi \notin G \Rightarrow \Phi_{rwG}(\phi) = \phi$.

(ii) $\Phi_{rwG}(A) - \Phi_{rwG}(\Phi_{rwG}(A)) \subseteq \Phi_{rwG}(A - \Phi_{rwG}(A))$.

(iii) $\Phi_{G}(\Phi_{rwG}(A)) \subseteq \Phi_{G}(A)$, $\Phi_{rwG}(\Phi_{G}(A)) \subseteq \Phi_{G}(A)$ and $\Phi_{rwG}(\Phi_{rwG}(A)) \subseteq \Phi_{G}(A)$.

(iv) $\Phi_{rwG}(A) - \Phi_{rwG}(B) = \Phi_{rwG}(A - B) - \Phi_{rwG}(B)$.

(v) $A \cap \Phi_{rwG}(X) = \phi$, for every $rw$-closed set $A$ and $A \notin G$. 

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Proof. Follows from the Theorem 1.6.3.

Lemma 1.6.6 Let $(X, τ^{rw}, G)$ be a grill regular weakly closed space and $A ⊆ X$ be $rw$-closed set. Then, $Φ_{rwG}(A) ⊆ A$.

Proof. Let $x ∉ A$, $A$ be $rw$-closed set. Then, $(X - A)$ is a $rw$-closed set containing $x$ and $(X - A) ∩ A = φ ∉ G$. Consequently, $x ∉ Φ_{rwG}(A)$ and so $Φ_{rwG}(A) ⊆ A$.

Theorem 1.6.7 Let $(X, τ^{rw}, G)$ be a grill regular weakly closed space and $A$ be a $rw$-closed set. Then $Φ_{rwG}Φ_{rwG}(A) ⊆ Φ_{rwG}(A)$.

Proof. Let $x ∉ Φ_{rwG}(A)$. Then, $∃$ $rw$-closed set $U$ containing $x$ s. t. $(U ∩ A) ∉ G$. In view of Lemma 1.6.6. Hence, $∃$ $rw$-closed set $U$ containing $x$ s. t. $(U ∩ Φ_{rwG}(A)) ∉ G$. Consequently, $x ∉ Φ_{rwG}Φ_{rwG}(A)$ and so $Φ_{rwG}Φ_{rwG}(A) ⊆ Φ_{rwG}(A)$.

Theorem 1.6.8 Let $(X, τ^{rw}, G)$ be a grill regular weakly closed space and $A ⊆ X$. If $U$ is a $rw$-closed (resp., open) set containing $x$. Then, $U ∩ Φ_{rwG}(A) = U ∩ Φ_{rwG}(U ∩ A)$.

Proof. In view of (a) of Theorem 1.6.3 and $U ∩ A ⊆ A$. Then, $Φ_{rwG}(U ∩ A) ⊆ Φ_{rwG}(A)$ and so $U ∩ Φ_{rwG}(U ∩ A) ⊆ U ∩ Φ_{rwG}(A)$. Let $x ∉ Φ_{rwG}(U ∩ A)$ and $U$ is a $rw$-closed (resp., open) set containing $x$. Then, $∃$ $rw$-closed (resp., open) set $V$ containing $x$ such that $(V ∩ U ∩ A) ∉ G$. Then, $W = (V ∩ U)$ is a $rw$-closed (resp., open) set containing $x$ and $(W ∩ A) ∉ G$. Consequently, $x ∉ Φ_{rwG}(A)$ and so $U ∩ Φ_{rwG}(A) ⊆ U ∩ Φ_{rwG}(U ∩ A)$. Then, $U ∩ Φ_{rwG}(A) = U ∩ Φ_{rwG}(U ∩ A)$.
Lemma 1.6.9 Let $\tau_1^{rw}, \tau_2^{rw}$ be two collections of all $rw$-closed sets on $X$ generated by $\tau_1, \tau_2$ respectively with $\tau_1^{rw} \subseteq \tau_2^{rw}$. Then, for any grill $G$ on $X$ and any subset $A$ of $X$, $\Phi_{rwG}(A, \tau_2^{rw}) \subseteq \Phi_{rwG}(A, \tau_1^{rw})$.

Proof. Obvious. ■

Theorem 1.6.10 Let $(X, \tau^{rw}, G)$ be a grill regular weakly closed space. Then, a map $\Psi_{rwG} : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is given by $\Psi_{rwG}(A) = A \cup \Phi_{rwG}(A)$, satisfies Kuratowski closure axioms $\forall$ $rw$-closed set $A$.

Proof. (i) In view of Corollary 1.6.5 $\Psi_{rwG}(\phi) = \phi$.

(ii) It is obvious $A \subseteq \Psi_{rwG}(A)$, $\forall A \in \mathcal{P}(X)$.

(iii) In view of (ii) of Theorem 1.6.3 we have $\Psi_{rwG}(A \cup B) = \Psi_{rwG}(A) \cup \Psi_{rwG}(B)$, $\forall A, B \in \mathcal{P}(X)$.

(iv) Let $A$ be a $rw$-closed set. In view of Theorems 1.6.3 and 1.6.7, $\Psi_{rwG}(\Psi_{rwG}(A)) = \Psi_{rwG}(A \cup \Phi_{rwG}(A)) = A \cup \Phi_{rwG}(A \cup \Phi_{rwG}(A)) = A \cup \Phi_{rwG}(A \cup \Phi_{rwG}(\Phi_{rwG}(A))) = A \cup \Phi_{rwG}(A) = \Psi_{rwG}(A)$. ■

Corollary 1.6.11 Let $(X, \tau^{rw}, G)$ be a grill regular weakly closed space. $\forall A \in \mathcal{P}(X)$, then

(i) $\Psi_{rwG}(\Psi_{rwG}(A)) \subseteq \Psi_{G}(A)$.

(ii) $\Psi_{rwG}(A) \subseteq \Psi_{G}(A)$.

Proof. Obvious. ■

Definition 1.6.12 For a grill regular weakly closed space $(X, \tau^{rw}, G)$, there exists
a unique topology $\tau_{G}^{rw}$ on $X$ given by $\tau_{G}^{rw} = \{A \subseteq X : \Psi_{rwG}(X - A) = X - A\}$, for any $A \subseteq X$ and $\Psi_{rwG}(A) = \tau_{G}^{rw} - clA$. Moreover, $\Psi_{rwG}(A) \subseteq rw - cl(A)$.

Now, we will deal with certain properties concerning the topology $\tau_{G}^{rw}$.

**Theorem 1.6.13** For a grill regular weakly closed space $(X, \tau^{rw}, G)$. Then, $\tau_{G} \subseteq \tau_{G}^{rw}$ and $\tau^{rw} \subseteq \tau_{G}^{rw}$.

**Proof.** Let $A$ be $\tau_{G}$-closed. Then, $A = \Psi_{G}(A) = A \cup \Phi_{G}(A)$ i.e., $\Phi_{G}(A) \subseteq A$. By using (iv) of Theorem 1.6.3, we get $\Phi_{rwG}(A) \subseteq \Phi_{G}(A) \subseteq A$ and so $A = A \cup \Phi_{rwG}(A) = \Psi_{rwG}(A)$. Hence, $A$ is $\tau_{rwG}$-closed and $\tau_{G} \subseteq \tau_{G}^{rw}$. By using (v) of Theorem 1.6.3, we have $\Psi_{rwG}(A) \subseteq rw - cl(A)$. Hence, $\tau^{rw} \subseteq \tau_{G}^{rw}$. □

**Corollary 1.6.14** Let $(X, \tau^{rw}, G)$ be a grill regular weakly closed space and $A \subseteq X$.

(i) If $A$ is a $rw$-closed set, then $A$ is $\tau_{G}^{rw}$-closed set.

(ii) If $A \notin G$, then $A$ is $\tau_{G}^{rw}$-closed set.

**Theorem 1.6.15** Let $\tau_{1}^{rw}$, $\tau_{2}^{rw}$ be two collections of all $rw$-closed sets on $X$ generated by $\tau_{1}$, $\tau_{2}$ respectively with $\tau_{1}^{rw} \subseteq \tau_{2}^{rw}$. Then, for any grill $G$ on $X$ and any subset $A$ of $X$, $\tau_{1}^{rw} \subseteq \tau_{2}^{rw}$.

**Proof.** Since $\tau_{1}^{rw} \subseteq \tau_{2}^{rw}$ and by using Lemma 1.6.6. Then, $\Psi_{rwG}(A, \tau_{2}^{rw}) \subseteq \Psi_{rwG}(A, \tau_{1}^{rw})$. Hence $\tau_{2}^{rw} - clA \subseteq \tau_{1}^{rw} - clA$. Consequently, $\tau_{1}^{rw} \subseteq \tau_{2}^{rw}$. □

**Theorem 1.6.16** Let $(X, \tau^{rw})$ be a regular weakly closed space and let $G$ and $H$ be two grills on $X$ with $G \subseteq H$. Then, $\tau_{H}^{rw} \subseteq \tau_{G}^{rw}$. 

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**Proof.** Let $A$ be $\tau^r_{H}$-closed. Then, $A = \Psi_{rwH}(A) = A \cup \Phi_{rwH}(A)$ i.e., $\Phi_{rwH}(A) \subseteq A$. By using (i) of Theorem 1.6.3 $\Phi_{rwG}(A) \subseteq \Phi_{rwH}(A) \subseteq A$. Hence, $A = \Psi_{rwG}(A) = A \cup \Phi_{rwG}(A)$ and so $A$ is $\tau^r_{G}$-closed.

**Theorem 1.6.17** Let $(X, \tau^r, G)$ be a grill regular weakly closed space. Then, $\beta(\tau^r, G) = \{(V - A) : V \in \tau^r$ and $A \notin G\}$ is a base for $\tau^r_{G}$.

**Proof.** Let $U \in \tau^r_{G}$ and $x \in U$. Then, $(X - U)$ is $\tau^r_{G}$-closed so that $\Psi_{rwG}(X - U) = (X - U)$. Hence, $\Phi_{rwG}(X - U) \subseteq (X - U)$. Then, $x \notin \Phi_{rwG}(X - U)$ and so there exist $rw$-closed set $V$ and $x \in V$ such that $(X - U) \cap V \notin G$. Let $A = (X - U) \cap V$. Then, $x \notin A$ and $A \notin G$. Thus, $x \in (V - A) = V - ((X - U) \cap V) = V - (X - U) \subseteq U$, where $(V - A) \in \beta(\tau^r, G)$. Now we prove that $\beta(\tau^r, G)$ is closed under finite intersections. Let $(V_1 - A), (V_2 - B) \in \beta(\tau^r, G)$ i.e., $V_1, V_2 \in \tau^r$ and $A, B \notin G$. Then, $V_1 \cap V_2 \in \tau^r$ and $A \cup B \notin G$. Now, $(V_1 - A) \cap (V_2 - B) = (V_1 \cap V_2) - (A \cup B) \in \beta(\tau^r, G)$, proving ultimately that $\beta(\tau^r, G)$ is a base for $\tau^r_{G}$.

**Theorem 1.6.18** Let $(X, \tau^r, G)$ be a grill regular weakly closed space and $A \subseteq X$ be a $rw$-closed set. Then, $\Phi_{rwG}(A)$ is $\tau^r_{G}$-closed.

**Proof** Let $A \subseteq X$ be a $rw$-closed set and in view of Theorem 1.6.7 we have, $\Psi_{rwG}(\Phi_{rwG}(A)) = \Phi_{rwG}(A) \cup \Phi_{rwG}(\Phi_{rwG}(A)) = \Phi_{rwG}(A)$. Then, $\Phi_{rwG}(A)$ is $\tau^r_{G}$-closed.