METHOD OF STEEPEST DESCENTS

This method is applied to the approximate evaluation of integrals of the form

\[ I = \int_{a}^{b} F(x) e^{\rho f(x)} \, dx \quad (A-1) \]

where \( \rho \) is large, real and positive, and \( f(x) \) is analytic. One may write

\[ f(x) = \alpha + \beta \quad (A-2) \]

Separating its real and imaginary parts, \( \alpha \) and \( \beta \) both satisfy Laplace's equation, and the integrand will be large where \( \alpha \) is algebraically large. Hence the integral will be largest if the points \( a, b \) are so situated that a path connecting them must pass through regions where \( \alpha \) is large. Apart from the possible presence of a singularity, which will need special attention in any case, there will be advantages in choosing the path so that the large values of \( \alpha \) are concentrated in the shortest possible interval on it. \( \alpha \) can never have an absolute maximum, but it can have stationary points, where

\[ \frac{\partial \alpha}{\partial x} = \frac{\partial \alpha}{\partial y} = 0 \quad (A-3) \]

and these will also be stationary points of \( \beta \) and zeros of \( f'(x) \). These points are usually called saddle-points, or sometimes cols. Through any saddle-point it will be possible to draw at least two curves such that \( \alpha \) is constant along them. In sectors between these curves \( \alpha \) will be alternately greater.
and less than at the saddle-point itself. The sectors where $\alpha$ is greater may be called the hills, the others the valleys. If we wish to keep large values of $\alpha$ in as short a stretch of the path as possible we must avoid the hills and keep as far as possible to the valleys. If then the complex plane is marked out by the lines of $\alpha$ constant through all the saddle-points, and $a$ and $b$ lie in the same valley, our path must never go outside this valley; but if they are in different valleys the path must go through a saddle-point. In the latter case the integral will be much greater than in the former, and therefore most interest attaches to the case where $a$ and $b$ are in different valleys.

The direction of the path at any point, in the method of steepest descents, is chosen so that $|\frac{\partial \alpha}{\partial s}|$ is as great as possible. If $\Theta$ is the inclination of the path to the axis of $X$, we have

$$\frac{\partial \alpha}{\partial s} = \cos \Theta \frac{\partial \alpha}{\partial x} + \sin \Theta \frac{\partial \alpha}{\partial y}$$  \hspace{1cm} (A-4)

and if this is to be a numerical maximum for variations of $\Theta$

$$\Theta = - \sin \Theta \frac{\partial \alpha}{\partial x} + \cos \Theta \frac{\partial \alpha}{\partial y} = - \sin \Theta \frac{\partial \beta}{\partial y} - \cos \Theta \frac{\partial \beta}{\partial x} = - \frac{\partial \beta}{\partial s}$$  \hspace{1cm} (A-5)

Hence $\beta$ is constant along the path. Such a path is called a line of steepest descent. There will be one in each valley. In general $a$ and $b$ will not themselves be on lines of steepest descent, but can be joined to them by lines within the valleys.
Lines of steepest descent terminate only at singular points of \( f(\xi) \) or at infinity.

If \( \xi_o \) is a saddle-point \( f(\xi) \) near it can be expanded in the form

\[
f(\xi) = f(\xi_o) + \frac{1}{2} \left( \xi - \xi_o \right)^2 f''(\xi_o) + \ldots \tag{A.6}
\]

and the direction of the path will be such that \( (\xi - \xi_o)^2 f''(\xi) \) (\( \xi_o \)) is real and negative. If then we put

\[
f(\xi) - f(\xi_o) = -\frac{z^2}{2} \tag{A.7}
\]

and change the variable to \( z \) the integral takes the form considered in Watson's lemma and the existence of an asymptotic expansion in negative powers of \( y \) can be inferred. In practice, however, the inversion of series is usually troublesome, and if terms after the first are required they are usually found in some other way. For many purposes, however, the first term is sufficient, and can be obtained easily. We have

\[
I = e^{\rho f(\xi_o)} \int e^{-\gamma z^2} f(\xi) \frac{dz}{dz} d\xi \tag{A.8}
\]

But if we write for values of \( \xi \) on the path, with \( \gamma \) real and small

\[
(\xi - \xi_o) = \gamma e^{j\phi} \tag{A.9}
\]

\[
z^2 = -f''(\xi_o) \gamma^2 e^{2j\phi} \tag{A.10}
\]

since \( f''(\xi_o) e^{j\phi} \) is real and negative. Then

\[
\frac{dz}{d\xi} = \pm e^{-j\phi} |f''(\xi_o)|^{-\gamma} \tag{A.11}
\]

In the range \((-\pi, \pi)\) there are two possible choices for \( \alpha \), and they differ by \( \pi \). In any application of the method we
at this point to make an inspection of the behaviour of $\alpha$ and $\beta$ over the complex plane in order to decide the sense in which the path goes through the saddle-point. If we select the value of $\phi$ that makes $\gamma$ positive at points on the path after passing through $z_o$, we shall have to take the positive sign in (A-10), as $z$ goes from $-\infty$ to $+\infty$ on the path. Then by Watson's lemma the integral is given asymptotically by

$$I \sim \frac{F(z_0) \ e^{\rho f(z_0)}}{\sqrt{|\rho f''(z_0)|}} \ e^{\rho \int}$$

(A-12)

Since $\rho \text{exp}\{\rho f(z_0)^2\}$ will be large for all $n$ if $f(z_0)$ has a positive real part, we should strictly write (A-12) as

$$I \sim \frac{\sqrt{2\pi} F(z_0)}{\sqrt{|\rho f''(z_0)|}} \ e^{j\phi}$$

(A-13)

in order that Poincaré's definition shall be applicable. We shall, however, use the form (A-12) for convenience, with the understanding that where exponential factors are present in the approximation such a transposition is needed before the definition is applied.

To get from $a$ to $b$ it may be necessary to pass through two or more saddle-points, with possibly traverses in the valleys between lines of steepest descent between them. Then each saddle-point makes its contribution to the integral.