CHAPTER IV

MODULATIONAL STABILITY OF OBLIQUELY PROPAGATING LANGMUIR WAVES IN COLLISIONAL PLASMAS

IV.1 Introduction

The study of the nonlinear properties of Langmuir waves is of increasing interest because of its relevance to the problems of the relativistic beam and laser heating of plasmas, and to the strong plasma turbulence theory (Mishikawa et al. 1975, and Morales and Lee 1975). The longtime behaviour of Langmuir waves is governed by the nonlinear Schrödinger (NS) equation and in a collisionless plasma they are found to be modulationally stable (Asano et al. 1969). Using the reductive perturbation technique, the NS equation for Langmuir waves has been derived from the isothermal plasma fluid equations by Asano et al. (1969) and
from the plasma kinetic equations by Ichikawa et al. (1972). In the latter work the effects of the resonant wave-particles were neglected. The resonant particles moving with the group velocity of the wave are expected to give rise to the nonlinear Landau damping. This interaction introduces a nonlocal-nonlinear term, which makes the Langmuir waves modulationally unstable (Ichikawa and Taniuti 1973). Starting from the adiabatic plasma fluid equations, Kakutani and Sugimoto (1974) derived the NS equation for these waves by using the KBM method. Moreover, Zakharov (1972) showed that the strong Langmuir turbulence is described by the NS equation coupled to a wave equation for the associated low frequency Ion-Acoustic oscillations. In this case, the Langmuir solitons, which normally do not interact among themselves, emit sound waves and coalesce together (Abdulloev et al. 1974).

The influence of collisions on the nonstationary evolution of the electroacoustic waves has been discussed by Gurovich and Karpman (1970) and on the Langmuir waves, when the perturbation propagates with near sound speed, by Karpman (1975b). The effect on the modulational stability and stationary states was not discussed by these authors. The effect of collisions on the modulational instability of ion-acoustic waves has been discussed by Buti (1976) by using the KBM method. The spectrum of the modulationally unstable ion-acoustic waves is found to be drastically modified by the
collisions.

The oblique modulation of the ion-acoustic waves has been discussed by Kako and Hasegawa (1975). The modulation-stably stable ion-acoustic waves are found to become unstable when modulated at an angle to the direction of the phase velocity.

The Langmuir oscillations are rapid processes and we can associate a fast space-time scale to it. Over these rapid variations are superimposed slow variations of the amplitudes and these are characterized by slow space-time scales. The presence of weak collisions in the system is not of much significance during the initial stages where the system is governed by the fast space-time scale and these collisions may be neglected in this regime. However it may be of interest to study the effects of collisions during the slow space-time scale.

In this chapter, we investigate the modulational stability of Langmuir waves in a weakly collisional plasma when modulated at an angle to the direction of the phase velocity. The KBM method is used for the present study.
Consider a two-dimensional warm collisional plasma with the ions providing the neutralizing background. If the electron fluid is taken to be isothermal and the wave propagates in the x-y plane, its dynamics is governed by the equations

\[
\begin{align*}
\frac{\partial n}{\partial t} + \nabla \cdot (n \vec{u}) &= 0, \\
\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + \frac{T_e}{mn} \nabla n + \frac{e}{m} \vec{E} + \nu \vec{u} &= 0, \\
\nabla \cdot \vec{E} - 4\pi e (n_e - n) &= 0, \\
\frac{\partial \vec{E}}{\partial t} - 4\pi e n \vec{u} &= 0,
\end{align*}
\]

(4.1)

where \( \nabla = (\partial/\partial x, \partial/\partial y) \), \( \vec{u} = (u_x, u_y) \) and \( \vec{E} = (E_x, E_y) \). In this set of equations the first three are the familiar continuity equation, momentum-balance equation and Poisson's equation respectively. The fourth equation is

\[
\nabla \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} \vec{J},
\]

with the magnetic field \( \vec{B} \) taken to be zero and the current density \( \vec{J} = ne \vec{u} \). The first and the last equations of the set (4.1) imply that the Poisson's equation holds good for all times, provided it is valid initially. Eqs. (4.1) can be expressed in the dimensionless form on normalizing the lengths to the Debye length \( \lambda_D \), the time to the inverse of electron plasma frequency, \( \omega_p^{-1} \), the electron number density \( n \) to the
average density $n_0$, the electron fluid velocity $\tilde{u}$ to the characteristic sound speed $c_s$, the electric field $\tilde{E}$ to the characteristic field $(T_e/e\lambda_D)$ and the collision frequency $\nu$ to $\omega_D$. We choose the first, second and fourth equations of the set (4.1) to describe the system in terms of the three functions $n$, $\tilde{u}$ and $\tilde{E}$. From these three equations, $\tilde{E}$ can be eliminated. So for a two-dimensional system our basic equations become

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nux) + \frac{\partial}{\partial y}(nuy) = 0,$$  \hspace{1cm} (4.2)

$$\frac{\partial^2 u_x}{\partial t^2} + \frac{\partial}{\partial t}(u_x \frac{\partial u_x}{\partial x}) + \frac{\partial}{\partial t}(u_y \frac{\partial u_x}{\partial y}) \quad + \frac{\partial}{\partial t}\left(\frac{n}{\bar{n}} \frac{\partial n}{\partial x}\right) + nux + \epsilon \tilde{\nu} \frac{\partial u_x}{\partial t} = 0,$$  \hspace{1cm} (4.3)

$$\frac{\partial^2 u_y}{\partial t^2} + \frac{\partial}{\partial t}(u_x \frac{\partial u_y}{\partial x}) + \frac{\partial}{\partial t}(u_y \frac{\partial u_y}{\partial y}) \quad + \frac{\partial}{\partial t}\left(\frac{n}{\bar{n}} \frac{\partial n}{\partial y}\right) + nuy + \epsilon \tilde{\nu} \frac{\partial u_y}{\partial t} = 0,$$  \hspace{1cm} (4.4)

where $\tilde{\nu}$ is the scaled collision frequency, $\nu = \epsilon \tilde{\nu}$; this guarantees the collisions to be weak.

Considering a weakly nonlinear system we take the following expansions in powers of the small parameter $\epsilon$. 
and choose \( n_1 \) to be the monochromatic plane wave

\[ n_1 = \alpha \exp(i\psi) + \bar{\alpha} \exp(-i\psi), \]

where \( \psi = kx - \omega t \). The complex amplitude \( \alpha \) is a slowly varying function of \( x \) and \( t \), defined by Eq. (2.8), but does not depend on \( y \). Thus the system under consideration is not strictly two-dimensional. In Eqs. (4.5), \( n_1, u_{x1}, u_{y1}, n_2; \)
\( u_{x2}, u_{y2}, ... \) are functions of \( x \) and \( t \) only through \( \alpha, \bar{\alpha} \) and
\( \psi \); and are functions of \( y \) only through \( \psi \).

From Eqs. (4.2) - (4.4) to order \( \epsilon \) and from Eq. (4.6) we get

\[ U_{x1} = \frac{k_x \omega}{k^2} \alpha \exp(i\psi) + c.c., \]
\[ U_{y1} = \frac{k_y \omega}{k^2} \alpha \exp(i\psi) + c.c., \]

with the linear dispersion relation of the system

\[ D(k,\omega) \equiv -\omega^2 + k^2 + 1 = 0, \]

where \( k^2 = k_x^2 + k_y^2 \).

Now let us consider Eqs. (4.2) - (4.4) to order \( \epsilon^2 \). From these equations we eliminate \( u_{x2}, u_{y2} \) and then substitute, into the resulting equation, for \( n_1, n_2 \), \( u_{x1} \) and \( u_{y1} \),
along with the definitions given by Eq. (2.8). Then we get
\[
\frac{\partial^2 n_2}{\partial t^2} - \frac{\partial^2 n_2}{\partial x^2} - \frac{\partial^2 n_2}{\partial y^2} + n_2 = -2(2k^2 + 3)\exp(2i\psi) \\
+ (2i\omega A + i\tilde{\omega} A + 2i k_x B) \exp(i\psi) + c.c.
\]
(4.9)

The resonant secularity arising from the term with \(\exp(i\psi)\) and its complex conjugate can be removed by imposing the condition
\[
A_1 + V_{qk} B_1 + \frac{1}{2} \nabla A = 0,
\]
(4.10)
where the group velocity along the \(x\)-axis is defined as
\[
V_{qk} = -\frac{\partial D/\partial k_x}{\partial D/\partial \omega} = \frac{k_x}{\omega}.
\]

The secular-free solution of Eq. (4.9) then is
\[
\begin{align*}
\mathcal{N}_2 &= \frac{2}{3}(2k^2 + 3)\alpha^2 \exp(2i\psi) \\
&+ b(a, \bar{a}) \exp(i\psi) + c.c. + \alpha(a, \bar{a}),
\end{align*}
\]
(4.11)
where \(b(a, \bar{a})\) is complex function of \(a\) and \(\bar{a}\), but a constant with respect to \(\psi\). From Eqs. (4.2) - (4.4) to order \(\epsilon^2\) and Eq. (4.11), we get
\[
\mathcal{U}_{x,2} = \frac{k_x \omega (4k^2 + 3)}{3k^2} \alpha^2 \exp(2i\psi) \\
+ \left\{ - \frac{i (k_x + 2)A}{k^4} + \frac{(\omega - k_x \omega) A + k_x \omega B}{k^2} + \frac{i (k_x \omega^2) A}{k^4} \right\} \exp(i\psi) \\
+ c.c. - \frac{2k_x \omega}{k^2} \alpha \bar{a},
\]
(4.12a)
In order to determine \( \alpha \) we consider Eq. (4.2) to order \( \mathcal{C}^3 \). The condition for the removal of the secularity in this equation is

\[
\frac{\partial \alpha}{\partial a} A_1 + \frac{\partial \alpha}{\partial \bar{a}} \bar{A}_1 = 0,
\]

which implies that \( \alpha \) is an absolute constant.

Now let us consider Eqs. (4.2) - (4.4) to order \( \mathcal{C}^3 \). From these equations \( u_{x3} \) and \( u_{y3} \) can be eliminated and the expressions for \( n_1, u_{x1}, u_{y1}, n_2, u_{x2} \) and \( u_{y2} \) substituted. In the resulting equation for \( n_3 \), the coefficient of the \( \exp(i\psi) \) term gives rise to resonant secularity, which is removed by the condition

\[
\frac{\ell}{2} \left\{ \left( A_2 + V_9 x B_2 \right) + \frac{1}{2\omega} \left\{ \left( B_1 \frac{\partial B_1}{\partial a} + \bar{B}_1 \frac{\partial B_1}{\partial \bar{a}} \right) - \left( A_1 \frac{\partial A_1}{\partial a} + \bar{A}_1 \frac{\partial A_1}{\partial \bar{a}} \right) \right\} \right. \\
+ \frac{\ell}{2} \left\{ \frac{k^2 + 2}{k^4} A_1 + \frac{2k^2 \omega}{k^2} B_1 - \frac{1}{2} V_9 x \left( B_1 + \bar{A}_1 \frac{\partial B_1}{\partial a} + \bar{A}_1 \frac{\partial B_1}{\partial \bar{a}} \right) \right\} \\
- \frac{i\omega}{2} \left( \frac{\partial b}{\partial a} + \bar{A}_1 \frac{\partial b}{\partial \bar{a}} - b \right) - \frac{k^2}{6\omega} (8k^2 + 9)|\alpha|^2 \alpha \\
+ \frac{1}{2\omega} \left\{ \alpha + \frac{\omega^2}{k^2} \left( \frac{\omega^2}{k^2} - \frac{1}{4} \right) \right\} \alpha = 0.
\]

(4.13)
The function \( b(a, \bar{a}) \) occurring in (4.13) is the complementary function of the differential equation (4.9). In the present case this function cannot be determined uniquely and we choose

\[ b = a \frac{\partial b}{\partial a} + \bar{a} \frac{\partial b}{\partial \bar{a}}, \]

so that \( b = \text{constant} (a \bar{a})^{1/2} \). On using Eq. (4.10) into Eq. (4.13), we get,

\[
i (A_2 + V_{yB_2}) + \frac{1}{2\omega} (1 - V_{g\chi}) (B_1 \frac{\partial B_1}{\partial a} + \bar{B}_1 \frac{\partial \bar{B}_1}{\partial \bar{a}}) \]

\[-\frac{\gamma V_{g\chi}}{4\omega} (-B_1 + a \frac{\partial B_1}{\partial a} + \bar{a} \frac{\partial \bar{B}_1}{\partial \bar{a}}) \]

\[-\frac{k}{6\omega} (9 + 8k^2) |a|^2 \alpha + \frac{1}{2\omega} (\alpha + \frac{1}{4} \gamma^2) \alpha = 0. \quad (4.14)\]

From Eqs. (2.8) and (4.10) we get

\[ \alpha \frac{\partial B_1}{\partial a} + \bar{a} \frac{\partial \bar{B}_1}{\partial \bar{a}} = B_1, \]

so that with the definitions (2.29), (2.30) and (2.32), Eq. (4.14) becomes

\[ i \frac{\partial a}{\partial \tau} + P \frac{\partial^2 a}{\partial \xi^2} + Q |a|^2 a + Ra = 0, \quad (4.15)\]

where

\[ P = \frac{1}{2} \frac{dV_{g\chi}}{d\xi} = \frac{1 + k_y}{2\omega^3}, \quad Q = -\frac{k^2}{6\omega} (9 + 8k^2) \quad (4.16)\]

and

\[ R = \frac{1}{2\omega} (\alpha + \frac{1}{4} \gamma^2). \]
Eq. (4.15) is the NS equation describing the envelope of the Langmuir waves in the x-direction, i.e., at an angle to the direction of the phase velocity, in a collisional plasma. For the collisionless one-dimensional case, i.e., \( \gamma = 0 \) and \( k_x = k, k_y = 0 \), Eq. (4.15) reduces to that obtained by Asano et al. (1969). The R term is due to the integration constant and the collisions, and is absent in the work of Asano et al. (1969). This term is not of much significance and can be eliminated by the transformation \( a \rightarrow a \exp (iR \gamma) \).

### IV.3 Modulational Stability and Envelope Holes

The stability of the envelope of Langmuir waves against oblique longwave perturbations can be studied in the same manner as in Chapter II. We express \( a \) as in Eq. (2.39) and the resulting equations, i.e., Eqs. (2.40) - (2.41) are perturbed as in Eq. (2.42).

From Eq. (4.16) we find that

\[
PQ = - \frac{k^2 (1+k_y^2)(9+8k^2)}{12 \omega^2},
\]

so that for all \( k \), \( PQ < 0 \). As shown in Chapter II, modulations are unstable when \( PQ > 0 \). Hence the Langmuir waves in two-dimensions in a collisional plasma are stable against oblique modulations. The corresponding localized stationary
solutions are the envelope holes, Eq. (2.49), with P and Q given by Eq. (4.16). The collisions do not affect the shape of the envelope hole but the obliqueness of the modulations affects its width.

IV.4 Conclusions and Discussion

The Langmuir waves in a one-dimensional collisionless plasma are known to be modulationally stable (Asano et al., 1969). We find the collisions and obliqueness of the perturbation do not change this stability. Using the definitions given by Eq. (2.30), Eq. (4.10) can be written as

\[ \frac{\partial \alpha}{\partial t_1} + v_g \frac{\partial \alpha}{\partial x_1} + \frac{1}{2} \gamma \alpha = 0. \]

Transforming to a frame moving with velocity \( V_{gx} \) and then integrating, we get

\[ \alpha = \alpha(t_1 = 0) \exp\left( -\frac{1}{2} \gamma t_1 \right). \]

Thus the envelope, which propagates with the group velocity \( V_{gx} \), is damped by the collisions in this space-time scale. However, in the space-time scale defined by Eq. (2.32), it is seen from Eq. (4.15) that the collisions have insignificant effect.

The width of the envelope hole, given by Eq. (2.49), is defined as
\[ \Delta = \left( \frac{2}{q_1} \left| \frac{P}{Q} \right| \right)^{\frac{1}{2}} \]  

For the Langmuir waves modulated obliquely, from Eq. (4.16), we have

\[ \left| \frac{P}{Q} \right| = \frac{3(1 + k^2 \sin^2 \theta)}{k^2(\theta + 8k')} \]  

where \( \tan \theta = k_y/k_x \). If the wave is modulated parallel to the phase velocity, then

\[ \left| \frac{P}{Q} \right|_{\theta=0} = \frac{3}{k^2(\theta + 8k')} \]  

From Eqs. (4.17) - (4.19), it can be seen that the ratio of the widths for the oblique and parallel cases is

\[ \frac{\Delta(\theta)}{\Delta(\theta=0)} = \left(1 + k^2 \sin \theta \right)^{\frac{1}{2}} \]  

Thus the width of the envelope hole increases with the angle between the direction of propagation and modulation. This effect is due to the change in the dispersion of the system.