CHAPTER III

EFFECT OF RANDOM INHOMOGENEITIES ON NONLINEAR ION-ACOUSTIC WAVES

III.1 Introduction

The ion-acoustic waves in a plasma give rise to interesting phenomena, viz., solitons, modulational instability, envelope solitons and envelope holes. Weakly nonlinear longwavelength ion-acoustic waves in a homogeneous plasma is governed by the KdV equation and admits soliton solutions (Washimi and Taniuti 1966). In the presence of a magnetic field both the slow and fast ion-acoustic modes are described by the KdV equation (Pokroev and Stepanov 1973, Tagare and Sharma 1976). The envelope characteristics of ion-acoustic waves are governed by the NS equation. In a collisionless system these waves are modulationally unstable
for $k > 1.47$. The modification of the spectrum of unstable waves due to collisions have been discussed by Buti (1976) and that due to electron inertia and ion temperature by Chan and Seshadri (1975).

The effect of random fluctuations is of great importance for systems like fluids and plasmas. Plasmas, in general are turbulent and consequently can be treated as random media. The propagation of nonlinear waves in these random media has been an active field of investigation. The effect of random fluctuations in plasmas is of special interest to the study of nonlinear dispersive waves because both the dispersion and dissipation of the system are affected. This influences the balance between the nonlinearity and the dispersion of the waves and consequently their stationary states, such as solitons, etc. In plasmas the nonlinear interaction among waves in the presence of random inhomogeneities has been studied by averaging the equations over the inhomogeneities (Tamoikin and Fainshtein 1972 and 1976). It has been shown that fluctuations can lead to the collapse of magnetohydrodynamic shock waves (Akhiezer et al. 1971).

The ion-acoustic waves in the presence of weak random inhomogeneities has been studied by Tamoikin and Fainshtein (1973). By averaging the relevant equations over the random inhomogeneities, they obtained a Korteweg-de Vries-Burgers type equation. A study of the stationary
solutions of this equation shows that the random electron density fluctuations introduce oscillations behind the shock-front. Here we study the envelope properties of the ion-acoustic waves in the presence of random inhomogeneities in the electron concentration. As in Chapter II we use the Krylov-Bogoliubov-Mitropolsky method. First we obtain the linear ion-acoustic modes in such a medium and then study the nonlinear states of these modes.

III.2 Plasma Equations in a Randomly Inhomogeneous Medium

Let us consider a plasma in which the electrons have average kinetic energy $T_e$ and the ions are cold. The ion density $n$, the fluid velocity $v$ and the electric potential $\varphi$ for such a plasma are described by the equations

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) = 0,$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \frac{e}{m} \frac{\partial \varphi}{\partial x}, \quad (3.1)$$

$$-\frac{\partial^2 \varphi}{\partial x^2} = 4\pi e(n_e - n).$$

The response of the electrons to the ion-acoustic waves is described by the Boltzmann distribution

$$n_e = n_0 \exp(-e\varphi/T_e), \quad (3.2)$$

where $n_0$ is the average number density.
In the presence of random density inhomogeneities $\delta n(x)$, the ion density $n$ can be expressed as

$$n = n_o + \delta n(x) + \langle \tilde{n}(x,t) \rangle + n'(x,t)$$

$$= n_o + \langle \tilde{n}(x,t) \rangle + n'(x,t), \quad (3.3)$$

where the brackets indicate averaging over $\delta n(x)$. Tilde denotes the average wave fluctuation, whereas prime denotes its deviation due to the wave. The equilibrium density $n_o$ is free of the wave fluctuations but has the random inhomogeneities embedded in it. Consequently,

$$\langle n_o \rangle = \langle n_o + \delta n(x) \rangle = n_o.$$

The electric potential $\varphi$ and the fluid velocity $v$ can similarly be written as

$$\varphi = \langle \tilde{\varphi}(x,t) \rangle + \varphi'(x,t),$$

$$v = \langle \tilde{v}(x,t) \rangle + v'(x,t). \quad (3.4)$$

On considering the density fluctuations to be weak, we define the small parameter $\mu$ as

$$\mu = \frac{\left| \delta n(x) \right|}{n_o}.$$

Also for the weakly nonlinear system

$$\frac{e \langle \tilde{\varphi}_{\text{max}} \rangle}{T_e} \sim \frac{\langle \tilde{n} \rangle}{n_o} \sim \frac{\langle \tilde{v} \rangle}{C_s} \sim \mu^2,$$

where $C_s$ is the ion-acoustic velocity.
Bow on averaging equations (3.1) over the random inhomogeneities, the ion-acoustic waves are found to be described by the equation (Tamoikin and Fainshtein 1973),

$$\frac{\partial}{\partial t} \langle \Phi \rangle + C_s \left( 1 + \frac{e}{l_e} \langle \Phi \rangle \right) \frac{\partial}{\partial x} \langle \Phi \rangle + \frac{1}{2} \lambda_D^2 \frac{\partial^3}{\partial x^3} \langle \Phi \rangle$$

$$+ \alpha_1 \frac{\partial^3}{\partial x \partial t^2} \langle \Phi \rangle - \alpha_2 \frac{\partial^3}{\partial x^2 \partial t} \langle \Phi \rangle + \delta \frac{\partial^2}{\partial x \partial t} \langle \Phi \rangle = 0, \quad (3.5)$$

where $\omega_{pi}$ is the ion plasma frequency and $\lambda_D$ the Debye length. The parameters $\alpha_1$, $\alpha_2$ and $\delta$ are defined as

$$\alpha_1 = \frac{C_s}{3 \omega_{pi}^2} \left\langle \left( \frac{\delta n}{\bar{n}_o} \right)^2 \right\rangle \frac{\bar{L}}{\lambda_D^2}, \quad \alpha_2 = C_s \alpha_1,$$

and $\delta = \frac{1}{3} \bar{L} \left\langle \left( \frac{\delta n}{\bar{n}_o} \right)^2 \right\rangle$.

For a given correlation function $f(x)$ of the random density inhomogeneities, the quantities $\bar{L}$ and $\bar{L}^2$ are determined by

$$\bar{L} = \int_0^\infty f(x) dx \quad \text{and} \quad \bar{L}^2 = \int_0^\infty x f(x) dx.$$

Thus $\bar{L}$ and $\bar{L}^2$ are the integral scales of the inhomogeneities.

In general the random fluctuations can be represented by a Gaussian distribution (Landau and Lifshitz 1974). Hence we take

$$f(x) = \frac{1}{(2\pi \bar{L}^2)^{1/2}} \exp \left( - \frac{x^2}{2 \bar{L}^2} \right),$$

where $\bar{L}^2$ is the mean square scale-length of the random fluctuations. Then the integral scales defined above are
given by
\[ L = \int_0^\infty f(x)dx = \frac{1}{2} \quad \text{and} \quad \overline{L}^2 = \int_0^\infty x^2 f(x)dx = \frac{L}{2(8\pi)^{3/2}}. \]

On normalizing lengths by the Debye length \( \lambda_D \), the time by \( \omega_{pi}^{-1} \), the electric potential \( \varphi \) by the characteristic potential \( T_e/e \), \( a_1 \) by \( \lambda_D/\omega_{pi}^2 \), \( a_2 \) by \( \lambda_D^2/\omega_{pi}^1 \) and \( \delta \) by \( \lambda_D/\omega_{pi} \), equation (3.5) reduces to

\[
\frac{\partial}{\partial t} \langle \Phi \rangle + (1 + \langle \Phi \rangle) \frac{\partial}{\partial x} \langle \Phi \rangle + \frac{1}{2} \frac{\partial^2}{\partial x^2} \langle \Phi \rangle + \alpha_1 \frac{\partial^3}{\partial x^3} \langle \Phi \rangle - \alpha_2 \frac{\partial^3}{\partial x^3 \partial t} \langle \Phi \rangle + \delta \frac{\partial^2}{\partial x \partial t} \langle \Phi \rangle = 0. (3.6)
\]

**III.3 Linear Ion-Acoustic Waves**

The linear ion-acoustic waves can be represented by the plane waves

\[ \langle \Phi \rangle = \alpha \exp(i\psi) + c.c., \quad (3.7) \]

where \( \psi = kx - \omega t \) denotes the phase. The dispersion relation governing these waves is, from eq. (3.6),

\[ -\omega + k - \frac{1}{2} k^2 - \alpha_1 k \omega^2 - \alpha_2 k^2 \omega - i \delta k \omega = 0. \quad (3.8) \]

Among the contributions from the random inhomogeneities,
\( \alpha_1 \) and \( \alpha_2 \) terms are of the same order and contribute to the dispersion of the waves, whereas the \( \delta \) term is a dissipative term. The ratio of the dissipation to the dispersion term is \( \sim (k_1)^{-1} \) which is small for \( k_1 \gg 1 \); i.e., for wavelengths much smaller than the characteristic scalelength.

The dispersion relation (3.8) can be solved by writing \( \omega = \omega_r + i \omega_i \). The two modes of oscillations are given by

\[
\omega_{\pi 1} = k \left( 1 - \frac{1}{2}k^2 - \alpha_1 k^2 - \alpha_2 k^2 \right),
\]

(3.9a)

and

\[
\omega_{\pi 2} = -\frac{1}{\alpha_1 k} \left( 1 + \alpha_1 k^2 + \alpha_2 k^2 \right),
\]

(3.9b)

with the corresponding damping rates

\[
\omega_{\text{ri} 1} = -\delta k \left( 1 - \frac{1}{2}k^2 - 2\alpha_1 k^2 - 2\alpha_2 k^2 \right)
\]

and

\[
\omega_{\text{ri} 2} = -\frac{\delta}{\alpha_1} \left( 1 - \alpha_1 k^2 \right).
\]

The solution \( \omega_{\pi 1} \) is the ion-acoustic mode and is weakly damped by the random inhomogeneities. The other solution \( \omega_{\pi 2} \) is entirely due to the random inhomogeneities and is heavily damped with the damping rate \( \omega_{\text{ri} 2} \). This mode is similar to the usual thermal noise in a plasma due to the particle thermal motions.
III.4 Nonlinear Ion-Acoustic Waves

The envelope properties of ion-acoustic waves are governed by the NS equation. To study the effect of the random density fluctuations on these properties we apply the KBM method to (3.6). As shown above the contribution of the dissipative term is small compared to that of the dispersive terms, and hence we take $\delta$ to be of order $\varepsilon^2$, i.e.,

$$\delta = \varepsilon^2 \delta,$$

where $\varepsilon$ is the smallness parameter. For convenience we write $\phi = \langle \phi \rangle$ and make the expansion

$$\phi = \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \cdots,$$  \hfill (3.10)

where $\phi_1$ is the plane wave described by the dispersion relation

$$D(k,\omega) = -\omega + k - \frac{1}{2} k^2 - \alpha_1 k \omega^2 - \alpha_2 k^2 \omega = 0.$$  \hfill (3.11)

Since the mode given by Eq. (3.9b) is heavily damped, we will discuss the evolution of the ion-acoustic mode of Eq. (3.9a). The variations of the amplitude $a$ of the plane wave representing this mode are given by Eq. (2.8).

On defining the operator

$$\mathcal{L} \equiv \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial^3}{\partial x^3} + \alpha_1 \frac{\partial^3}{\partial x^3 \partial t} - \alpha_2 \frac{\partial^3}{\partial x^2 \partial t^2},$$  \hfill (3.12)

and substituting the expansion (3.10) in Eq. (3.6), we get, to order $\varepsilon$,

$$\mathcal{L}(\phi) = 0.$$
This yields the dispersion relation (3.11). The equation to order $\varepsilon^2$ is

$$L(\phi_2) = - ik \alpha^2 e^{i(2\psi)} - \left( - \frac{\partial D}{\partial \omega} A_i + \frac{\partial D}{\partial \alpha} B_i \right)e^{i\psi},$$

(3.13)

The terms, proportional to $e^{i\psi}$ on the right hand side of Eq. (3.13) give rise to resonant singularity in the solution of $\phi_2$ and the condition for its removal is simply

$$A_i + V_g B_i = 0,$$

(3.14)

where $V_g$ is the group velocity, namely

$$V_g = - \frac{\partial D/\partial k}{\partial D/\partial \omega} = 1 - \frac{3}{2} k^2 - 3\alpha_1 k^2 - 3\alpha_2 k^2.$$  

(3.15)

The secular-free solution of Eq. (3.13) is

$$\phi_2 = \frac{k}{6(k - \omega)} \alpha^2 e^{i(2\psi)} + b(a, \bar{a}) e^{i\psi} + c.c. + c(a, \bar{a}),$$

(3.16)

where $b(a, \bar{a})$ and $c(a, \bar{a})$ are constants with respect to $\psi$.

To order $\varepsilon^3$, Eq. (3.6) can be written as

$$L(\phi_3) = - \frac{\partial}{\partial x} \left( \phi_2 \frac{\partial \phi_3}{\partial x} \right) \cdot \frac{\partial^2 \phi}{\partial x \partial t}$$

$$- \frac{1}{\varepsilon} \left( L(\phi_2) + \frac{\partial}{\partial x} \phi_2^2 \right) - \frac{1}{\varepsilon^2} L(\phi_1).$$

(3.17)

On collecting the $\psi$ independent terms in Eq. (3.17), we get the condition for the removal of secularity:
\[ \frac{\partial c}{\partial \alpha}(A_1 + B_1) + B_1 \bar{a} + c.c. = 0. \]

On using Eq. (3.14), this can be immediately integrated to give

\[ C = \frac{1}{V_g^{-1}} a \bar{a} + \beta, \]

where \( \beta \) is an absolute constant.

The terms proportional to \( \exp(\pm i \psi) \) on the right side of Eq. (3.17), however, give rise to the resonant secularity in the solution for \( \phi \). This resonant secularity can be removed by the condition

\[ i (A_2 + V_g B_2) + \frac{1}{2} \frac{d V_g}{d k} \left( B_1 \frac{\partial B_1}{\partial \alpha} + B_1 \frac{\partial B_1}{\partial \bar{a}} \right) \]

\[ + \frac{k}{\Delta} \left\{ \frac{k}{6(k-\omega)} + \frac{1}{V_g^{-1}} \right\} a \bar{a} + \frac{k (\beta - i \delta \omega)}{\Delta \omega} a \neq 0, \]

which on writing

\[ P = \frac{1}{2} \frac{d V_g}{d k} \]

\[ = -\frac{1}{2} \frac{\partial V_g}{\partial \omega} \left\{ 3k + 2 \alpha_2 \omega \right\}, \quad (3.19) \]

\[ + \frac{2}{\Delta \omega} \left( 2(\alpha_1 \omega + \alpha_2 k) + \frac{\alpha_1 k}{\Delta \omega} \right) \left( 1 - \frac{3}{2} k^2 - \alpha_1 \omega^2 - 2 \alpha_2 k \omega \right), \]

\[ Q = \frac{k}{\Delta} \left\{ \frac{k}{6(k-\omega)} + \frac{1}{V_g^{-1}} \right\}, \quad (3.20) \]

and
\[ R = \frac{k}{\partial D/\partial \omega} (\beta - i \delta \omega), \quad (3.21) \]

with \( \frac{\partial D}{\partial \omega} = -(1 + 2 \alpha_1 \kappa \omega + \alpha_2 k^2) \)

can be reduced to

\[ i(A_2 + V_g B'_2) + \frac{P}{\partial \gamma_2} B + \frac{\partial B'}{\partial \alpha} + Q |a|^2 a + Ra = 0. \quad (3.22) \]

On using the definitions given by Eqs. (2.29), (2.30) and (2.32), Eq. (3.22) reduces to

\[ i \frac{\partial \alpha}{\partial \gamma} + \frac{P}{\partial \xi^2} \frac{\partial^2 \alpha}{\partial \xi^2} + Q |a|^2 a + R \alpha = 0, \quad (3.23) \]

which is the nonlinear Schrödinger equation for the system under consideration.

The R-term in Eq. (3.23) can be eliminated by the transformation \( a \to a \exp (i R \gamma) \). It may be noted that the effect of the dissipative term is manifested through this term only. By appropriate choice of the constant \( p \), we can choose this damping to be negligible for the time scales of our interest. On keeping up to the first order terms only in \( a_1 \) and \( a_2 \) we find that

\[ P Q = -\frac{3}{2} (1 - 6 \alpha_1 k^2 - 5 \alpha_2 k^2), \quad (3.24) \]

so that \( PQ < 0 \). Consequently the ion-acoustic waves in the
presence of weak random inhomogeneities are stable against longwave modulations.

To discuss the envelope stationary states of the ion-acoustic waves, we express the complex amplitude $a$ as in Eq. (2.39). The stationary solution is the envelope hole given by Eq. (2.49), where $P$ and $Q$ are given by Eqs. (3.19) and (3.20) respectively. Thus the ion-acoustic wave may be represented asymptotically as

$$
\phi \sim P_{\tilde{a}} \left[ 1 - \tilde{a}^2 \text{sech}^2 \left( \sqrt{(QP)} P_{\tilde{a}} / 2 \right) \right] \times \exp \left\{ i \left( k \xi - \omega t + \sigma \right) \right\},
$$

where the phase change $\sigma$ is given by Eq. (2.49) with $P$ and $Q$ of Eqs. (3.19) and (3.20).

The width of the envelope hole is defined as

$$
\Delta = \frac{1}{P_{\tilde{a}}} \frac{|P|}{Q} \approx \frac{3k^2}{2P_{\tilde{a}}} \left\{ 1 + 2 \alpha_1 (2 - 3k^2) + \alpha_2 (4 - 7k^2) \right\}
$$

In the absence of the random inhomogeneities the width is simply

$$
\Delta_o = \frac{3k^2}{2P_{\tilde{a}}},
$$

so that the ratio of these two widths is

$$
\frac{\Delta}{\Delta_o} = 1 + 2 \alpha_1 (2 - 3k^2) + \alpha_2 (4 - 7k^2).
$$

Since both $\alpha_1$ and $\alpha_2$ are real positive, and $k^2 \ll 1$, 

$$
\Delta > \Delta_o,
$$

and hence the random inhomogeneities increase the width of the envelope hole state.
III.5 Discussion

The modulational instability of longwave ion-acoustic waves is not affected by the presence of the weak random inhomogeneities. However, the stationary envelope states of the waves are affected as depicted by Eq. (3.26), i.e. the inhomogeneities increase the width of the envelope state.

Physically this can be explained by looking into the process of formation of the envelope hole. The balance between the nonlinearity and dispersion leads to the formation of a hole of a certain width. As seen above, the random inhomogeneities contribute to the dispersion of the waves. Hence the balancing of the same nonlinearity against this increased dispersion will lead to the formation of an envelope hole with increased width.

In a homogeneous plasma the ion-acoustic waves with \( k \lambda_D > 1.47 \) are modulationally unstable (Kakutani and Sugimoto 1974) and the presence of collisions modifies the spectrum of the unstable waves (Buti 1976). The present analysis is restricted to the longwavelengths, \( k^2 \lambda_D^2 \ll 1 \), and for this region of the spectrum it is evident from Eq. (3.24) that ion-acoustic waves are modulationally stable in the presence as well as in the absence of random inhomogeneities.