CHAPTER II

MODULATIONAL INSTABILITY OF ION-ACOUSTIC WAVES
IN A TWO-ELECTRON TEMPERATURE PLASMA

II.1 INTRODUCTION:

The ion-acoustic waves (IAW) in a plasma arise due to the restoring action of the electron thermal pressure on ion density perturbations. The properties of these waves are therefore functions of electron temperature. In the linear regime, the presence of a small fraction of cold electrons in a plasma of hot electrons and cold ions is found to affect the IAW characteristics appreciably (Jones et al. 1975). Recently Goswami and Buti (1976) have shown that due to decreased dispersion, which results due to the decrease in the Debye length in such a plasma, the ion-acoustic solitons have increased amplitude for a given soliton width.

In a plasma composed of cold ions and hot isothermal electrons, if a small fraction of cold isothermal electrons
is introduced, the electron velocity distribution can be represented by a superposition of two Maxwellians. This plasma is referred to as the two-electron temperature (TET) plasma. Such plasmas are not uncommon. The plasma produced by hot cathode discharge are exactly TET plasmas (Oleson and Found 1949, Jones et al. 1975). The plasmas of thermonuclear interest are generally turbulent and have high energy tails, e.g., the interaction of charged particles with localized fields give rise to highly populated super-thermal tails (Morales and Lee 1974). The nonlinear beam-plasma interaction results in high-energy tails. Computer simulations also show the formation of high energy tails (Sudan 1973). The plasma produced by the radio-frequency breakdown in the ELMO confinement device (Krall and Trivelpiece 1973) is also a TET plasma.

The envelope properties of the IAW in a TET plasma are discussed in this chapter. The Krylov-Bogoliubov-Mitropolsky (KBM) perturbation method is used to derive the nonlinear Schrödinger (NS) equation governing the envelope of these waves. The modulational stability of the IAW for different ratios of the densities of the cold and hot electrons and also of their temperatures is studied. The envelope solutions for the different physical states of the plasma are obtained.
II.2 Perturbation Scheme

Consider a one-dimensional plasma in which the electrons are divided into two groups—the hot component with density $n_h$ and temperature $T_h$, and the cold component with density $n_\perp$ and temperature $T_\perp$. We assume the electrons to be isothermal and neglect the effect of electron inertia. The propagation of IAW in this TET plasma can be described by the following fluid equations:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n u) = 0,$$  \hspace{1cm} (2.1a)

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - \frac{e}{M} E = 0,$$  \hspace{1cm} (2.1b)

$$\frac{\partial E}{\partial x} - 4\pi e (n - n_\perp n_h) = 0,$$  \hspace{1cm} (2.1c)

$$\frac{\partial n_\perp}{\partial x} + \frac{e}{T_\perp} n_\perp E = 0,$$  \hspace{1cm} (2.1d)

$$\frac{\partial n_h}{\partial x} + \frac{e}{T_h} n_h E = 0,$$  \hspace{1cm} (2.1e)

where $e$ and $M$ are the charge and the mass of a proton, $T_\perp$ is the average kinetic energy of the cold electrons and $T_h$ that of the hot electrons; $n$ and $E$ being the ion density and the electric field respectively.

The quantities describing the system, may be expanded around the unperturbed values as
The charge neutrality demands that

\[ n_0 = n_{t0} + n_{h0}. \]

Let us take a monochromatic plane wave for \( E_1 \), i.e.,

\[ E_1 = a \exp(i\psi) + \bar{a} \exp(-i\psi), \quad (2.3) \]

where \( a \) is the amplitude, \( \bar{a} \) its complex conjugate and

\[ \psi = kx - \omega t \]

is the phase; \( k \) being the wavenumber and \( \omega \) the frequency. The quantities other than the zero-order quantities in the expansion (2.2) depend on \( x \) and \( t \) only through \( a, \bar{a} \) and \( \psi \). On substituting Eq. (2.2) into Eq. (2.1), we obtain the \( \epsilon \)-order equations, whose solutions are

\[ n_t = \frac{ie}{M\omega}(a \exp(i\psi) - \bar{a} \exp(-i\psi)), \]

\[ n_1 = \frac{ie\eta_0 k}{M\omega^2}(a \exp(i\psi) - \bar{a} \exp(-i\psi)), \]

\[ n_{u1} = \frac{ie\eta_0}{k L}(a \exp(i\psi) - \bar{a} \exp(-i\psi)), \]

\[ n_{h1} = \frac{ie\eta_{h0}}{k L}(a \exp(i\psi) - \bar{a} \exp(-i\psi)), \]

\[ n_{t2} = \ldots \]
From Eqs. (2.1a) - (2.1e) of order $\varepsilon$ we obtain,

$$\mathcal{L}(E_i) = 0,$$

where the operator $\mathcal{L}$ is defined as

$$\mathcal{L} = k^3 \frac{\partial}{\partial \psi^3} + \left\{ \frac{\omega_{pi}^2 k}{\omega^2} - \frac{4\pi e^2}{k} \left( \frac{n_{ei0}}{T_e} + \frac{n_{hi0}}{T_h} \right) \right\} \frac{\partial}{\partial \psi};$$

$$\omega_{pi} = (4\pi n_o e^2/M)^{1/2}$$ being the ion plasma frequency. If we define the effective temperature of the electrons as

$$T_{e^{\text{eff}}} = \frac{n_{e0} T_h T_e}{n_{e0} T_h + n_{h0} T_h},$$

and the corresponding effective Debye length $\lambda_{\text{Deff}}$ as

$$\lambda_{\text{Deff}} = \frac{T_{e^{\text{eff}}}}{4\pi n_o e^2},$$

then from Eqs. (2.3) and (2.5), we get the linear dispersion relation

$$D(k, \omega) \equiv -k + \frac{\omega_{pi}^2 k}{\omega^2} - \frac{1}{k \lambda_{\text{Deff}}^2} = 0.$$ (2.7)

The complex amplitude $a$ is a slowly varying function of $x$ and $t$ through the relations

$$\frac{\partial a}{\partial t} = \varepsilon A_1(\alpha, \bar{\alpha}) + \varepsilon^2 A_2(\alpha, \bar{\alpha}) + \cdots;$$

$$\frac{\partial a}{\partial x} = \varepsilon B_1(\alpha, \bar{\alpha}) + \varepsilon^2 B_2(\alpha, \bar{\alpha}) + \cdots,$$ (2.8)

and their complex conjugates. The functions $A_1, B_1; A_2, B_2; \cdots$
are yet unknown and are to be determined from the condition that the perturbation scheme envisaged by Eqs. (2.2) and (2.8) are free from secularities. This is the essence of the KBM method used here. This method belongs to the class of perturbation schemes based on the multiple space-time scales where the secularity arising from the fast scales are systematically annihilated (Jackson 1960, Bogoliubov and Mitropolsky 1961, Frieman 1963 and Sandri 1963).

Eqs. (2.1a) - (2.1e) to order $\varepsilon^2$, yield,

$$
\mathcal{L}(E_2) = \frac{2\varepsilon \omega^2}{M} \left( \frac{3k^2}{\omega_k^4} - \frac{1}{k^2 \nu_4^2} \right) \alpha^2 \exp(2i\Psi) 
+ \left( -\frac{\partial D}{\partial \omega} A_1 + \frac{\partial D}{\partial k} B_1 \right) \exp(i \Psi) + \text{c.c.}
$$

(2.9)

where

$$
\nu_4 = \frac{\epsilon_0 T_e^2 T_h^2}{M^2 (\epsilon_{10} T_h^2 + \epsilon_{10} T_e^2)}.
$$

In Eq. (2.9) the terms proportional to $\exp(i\Psi)$ give rise to resonant secularity because of the linear dispersion relation (2.7). This resonant secularity can be annihilated by the condition

$$
A_1 + \nu_4 B_1 = 0,
$$

(2.10)

where
\[ V_g = -\frac{\partial D/\partial k}{\partial D/\partial \omega}, \]
is the group velocity of the plane waves. The secular-free solution of Eq. (2.9) is then given by
\[ E_2 = \frac{i e \omega^2}{3 M^2 \omega} \left( \frac{3 k}{\omega^4} - \frac{1}{k^3 \sqrt{\lambda}} \right) \alpha^2 \exp(2i\psi) \]
\[ + b(\alpha, \bar{\alpha}) \exp(i\psi) + c.c. + C_1(\alpha, \bar{\alpha}), \tag{2.11} \]
where \( b \) and \( C_1 \) are constants with respect to \( \psi \). Also we have from Eq. (2.1b) to order \( \epsilon^2 \),
\[ \frac{\partial U_2}{\partial \psi} = -\frac{i e^2 \omega^2}{3M^2 \omega} \left( \frac{3 k}{\omega^4} - \frac{1}{k^3 \sqrt{\lambda}} \right) \alpha^2 \exp(2i\psi) \]
\[ + \left( -\frac{e b}{M \omega} + \frac{ie A_1}{M \omega^2} \right) \exp(i\psi) + c.c. - \frac{e C_1}{M \omega}. \]
On integrating this equation we notice that the last term gives rise to secularity in the expression for \( U_2 \). Therefore, for the solution to be free of any secularity, we must take \( C_1 = 0 \); in which case, the secular-free solution is
\[ U_2 = -\frac{e^2 k}{6 M^2 \omega} \left\{ \omega^2 \left( \frac{3}{\omega^4} - \frac{1}{k^4 \sqrt{\lambda}} \right) + \frac{3}{\omega^2} \right\} \alpha^2 \exp(2i\psi) \]
\[ + \frac{e}{M \omega} \left( i b + \frac{1}{\omega} A_1 \right) \exp(i\psi) + c.c. + C_2(\alpha, \bar{\alpha}). \tag{2.12} \]
Similarly we get, from Eqs. (2.1a), (2.1d) and (2.1.e) to order $\varepsilon^2$,

$$\eta_2 = -\frac{n_0 e^2 k^2}{6 M^2 \omega^2} \left\{ \frac{\omega^2}{h} \left( \frac{3}{\omega^4} - \frac{1}{k^4 \omega^4} \right) + \frac{3}{\omega^2} \right\} \alpha^2 \exp(2i\psi)$$

$$+ \frac{n_0 e k}{M \omega^2} (ib + \frac{2}{\omega} A + \frac{1}{k} B_1) \exp(i\psi)$$

$$+ c.c. + C_3 (\alpha, \bar{\alpha}), \quad (2.13)$$

$$\eta_{12} = -\frac{n_0 e^2}{6} \left\{ \frac{\omega^2}{M T} \left( \frac{3}{\omega^4} - \frac{1}{k^4 \omega^4} \right) + \frac{3}{k^2 T^2} \right\} \alpha^2 \exp(2i\psi)$$

$$- \frac{e n_{10}}{T} \left( \frac{1}{k} B_1 - ib \right) \exp(i\psi) + c.c. + C_4 (\alpha, \bar{\alpha}), \quad (2.14)$$

$$\eta_{h2} = -\frac{n_{ho} e^2}{6} \left\{ \frac{\omega^2}{M T_h} \left( \frac{3}{\omega^4} - \frac{1}{k^4 \omega^4} \right) + \frac{3}{k^2 T_h^2} \right\} \alpha^2 \exp(2i\psi)$$

$$- \frac{e n_{ho}}{T_h} \left( \frac{1}{k} B_1 - ib \right) \exp(i\psi) + c.c. + C_5 (\alpha, \bar{\alpha}), \quad (2.15)$$

The constants $C_2, C_3, C_4$ and $C_5$ are determined from the conditions for the removal of secularities in Eq. (2.1.a)-(2.1.e) to order $\varepsilon^3$. These five conditions involve the four unknowns $C_2, C_3, C_4$ and $C_5$, and another constant that would occur in the expression for $E_3$. On eliminating the latter we are left with four equations in four unknowns. From these conditions we find
\[ C_2 = \frac{1}{M V_g} \left\{ -\frac{T_h}{n_{h0}} \frac{y_2}{y_i} + \frac{e^2}{M} \left( \frac{1}{\omega^2} - \frac{M}{T_h k^2} \right) \right\} \bar{a} \bar{a} + C_{20}, \quad (2.16) \]

\[ C_3 = -\left\{ \left( 1 + \frac{n_{h0} T_h}{T_l} \right) \frac{y_2}{y_i} - \frac{e^2 n_{h0}}{T_l k^2} \left( \frac{1}{T_l} - \frac{1}{T_h} \right) \right\} \bar{a} \bar{a} + C_{30}, \quad (2.17) \]

\[ C_4 = -\left\{ \frac{n_{h0} T_h}{T_l} \frac{y_2}{y_i} - \frac{e^2 n_{h0}}{T_l k^2} \left( \frac{1}{T_l} - \frac{1}{T_h} \right) \right\} \bar{a} \bar{a} + C_{40}, \quad (2.18) \]

and

\[ C_5 = -\frac{y_2}{y_i} \bar{a} \bar{a} + C_{50}, \quad (2.19) \]

where

\[ y_i = -M V_g^2 \left( 1 + \frac{n_{h0} T_h}{n_{h0} T_l} \right) + \frac{n_{h0} T_h}{n_{h0}}, \quad (2.20) \]

\[ y_2 = M V_g^2 \frac{e^2 n_{h0}}{T_l k^2} \left( \frac{1}{T_l} - \frac{1}{T_h} \right) + \frac{e^2 n_c}{M} \left\{ \left( \frac{1}{\omega^2} - \frac{M}{T_h k^2} \right) + \frac{2 k V_g}{\omega^3} \right\}. \quad (2.21) \]

\( C_{20}, C_{30}, C_{40} \) and \( C_{50} \) are independent of \( a, \bar{a} \) and \( \psi \), i.e., they are absolute constants.

From Eqs. (2.1a) - (2.1e) to order \( \varepsilon^3 \), we obtain,
As indicated above the terms proportional to \( \exp(\mathrm{i} \Psi) \) on the right hand side of Eq. (2.22) will give rise to resonant secularity in the solution for \( E_3 \). The condition for the removal of this resonant secularity is found to be

\[
L(E_3) = -\frac{4\pi e}{\omega^2} \frac{\partial^2}{\partial t \partial x^2}(u_1 u_2 + u_2 u_1) + \frac{4\pi e \mu_0}{\omega^2} \frac{\partial^2}{\partial x^2}(u_1 u_2) \\
+ \frac{4\pi e^2}{k^2} \frac{\partial}{\partial x} \left\{ \left( \frac{\mu_1}{T_{\text{c}}} + \frac{\mu_{h1}}{T_{\text{h}}} \right) E_2 + \left( \frac{\mu_{l2}}{T_{\text{l}}} + \frac{\mu_{h2}}{T_{\text{h}}} \right) E_1 \right\} \\
+ \frac{1}{\epsilon} \left\{ -\frac{4\pi e}{\omega^2} \frac{\partial^2 \mu_2}{\partial t^2} - \frac{\omega_{\text{ex}}^2}{\omega^2} \frac{\partial E_2}{\partial x} - \frac{4\pi e}{\omega^2} \frac{\partial^2}{\partial t \partial x}(u_1 u_1) \right\} \\
+ \frac{4\pi e^2}{k^2} \frac{\partial}{\partial x} \left\{ \left( \frac{\mu_{l1}}{T_{\text{l}}} + \frac{\mu_{h1}}{T_{\text{h}}} \right) E_1 \right\} + \frac{4\pi e}{k^2} \frac{\partial \mu_2}{\partial x^2} \\
+ \frac{1}{\epsilon^2} \left\{ -\frac{\omega_{\text{ex}}}{\omega^2} \frac{\partial^2 E_1}{\partial x^2} + \frac{1}{k^2 \chi_{\text{eff}}^2} \frac{\partial E_1}{\partial x} - \frac{1}{k^2} \frac{\partial^3 E_1}{\partial x^3} \right\} \\
- \frac{4\pi e}{\omega^2} \frac{\partial^2 \mu_1}{\partial t^2} + \frac{4\pi e}{k^2} \frac{\partial \mu_1}{\partial x^2} \right\}. \tag{2.22}
\]

As indicated above the terms proportional to \( \exp(\mathrm{i} \Psi) \) on the right hand side of Eq. (2.22) will give rise to resonant secularity in the solution for \( E_3 \). The condition for the removal of this resonant secularity is found to be

\[
i(A_2 + \nu B_2) + P \left( \frac{\partial B'}{\partial \alpha} + \bar{B}' \frac{\partial B'}{\partial \bar{\alpha}} \right) \\
+ Q |\alpha|^2 \alpha + R \alpha = 0, \tag{2.23}
\]

where
\[ P = \frac{1}{2} \frac{dV_{30}}{d\kappa} = - \frac{3 M^2 \omega^5}{8 \pi n_e e^2 T_e \kappa^4}, \tag{2.24} \]

\[
Q = \frac{1}{\Delta D/\Delta \omega} \frac{e^2 \omega_k^3 k^3}{6 M^2 \omega^4} \left\{ \omega^2 \left( \frac{3}{k^4 \nu^4} - \frac{1}{k^4 \nu^4} \right) \left( 3 - \frac{\omega_k^4}{k^4 \nu^4} \right) + \frac{3 M^3 \omega_k^4}{n_0 k^6} \left( \frac{n_{\text{in}}}{T_l^3} + \frac{n_{\text{in}}}{T_h^3} \right) + \frac{15}{\omega^2} \left( \frac{2}{e^2 E} \frac{y_3 y_2}{y_1} + y_4 \right) \right\}, \tag{2.25} \]

and

\[
R = \frac{1}{\Delta D/\Delta \omega} \frac{\omega_k^2 k^2}{\omega^3} \left\{ \frac{\omega}{n_0 k} c_{30} + 2 c_{20} \frac{M \omega^3}{n_0 k} \left( \frac{c_{40} + c_{80}}{T_l} \right) \right\}, \tag{2.26} \]

with

\[
y_3 = - \left( 1 + \frac{n_{\text{in}} T_l}{n_{\text{in}} T_h} \right) \frac{\omega}{n_0 k} - \frac{2 T_h}{M n_{\text{in}} V_q} + \frac{M \omega^3}{n_0 T_h} \left( 1 + \frac{n_{\text{in}} T_h^2}{n_{\text{in}} T_h^2} \right), \tag{2.27} \]

and

\[
y_4 = - \frac{e^2 n_{\text{in}}}{n_0 T_l} \left( \frac{1}{T_l} - \frac{1}{T_h} \right) \frac{\omega}{k^3} + \frac{2 e^2}{M^2 V_q} \left( \frac{1}{\omega^2} - \frac{M}{T_h k^2} \right) \]

\[
+ \frac{e^2 M n_{\text{in}}}{n_0 T_l^2} \left( \frac{1}{T_l} - \frac{1}{T_h} \right) \frac{\omega^3}{k^5}, \tag{2.28} \]

and \( y_1, y_2 \) are defined by Eqs. (2.20) - (2.21).
The slow variations of $a$ with respect to space and time are governed by the conditions (2.10) and (2.23). In Eq. (2.8), which defines the quantities $A_1, B_1, A_2, B_2, \ldots$, we can introduce multiple space and time scales to convert these conditions into differential equations governing the evolution of the amplitude $a$.

On defining the new space and time variables as
\[
t_2 = \varepsilon t_1, \quad t_1 = \varepsilon t,
\]
\[
x_2 = \varepsilon x_1, \quad x_1 = \varepsilon x,
\]
we can interpret the quantities $A_1, B_1, A_2, B_2$ as
\[
A_1 = \frac{\partial a}{\partial t_1}, \quad B_1 = \frac{\partial a}{\partial x_1},
\]
\[
A_2 = \frac{\partial a}{\partial t_2} - \frac{A_1}{\varepsilon}, \quad B_2 = \frac{\partial a}{\partial x_2} - \frac{B_1}{\varepsilon},
\]
\[
A_1 \frac{\partial A_1}{\partial a} + A_2 \frac{\partial A_1}{\partial a} = \frac{\partial^2 a}{\partial t_1^2}, \quad B_1 \frac{\partial B_1}{\partial a} + B_2 \frac{\partial B_1}{\partial a} = \frac{\partial^2 a}{\partial x_1^2}.
\]

With these definitions Eq. (2.10) becomes
\[
\frac{\partial a}{\partial t_1} + V_g \frac{\partial a}{\partial x_1} = 0.
\]

This equation indicates that in the slow scale $t_1$ and $x_1$, the amplitude $a$ propagates with the group velocity $V_g$ without
any change of form. Using the definitions given by Eq. (2.29), Eq. (2.23) can be rewritten as

\[ i \left( \frac{\partial \alpha}{\partial t_2} + V_\alpha \frac{\partial \alpha}{\partial x_2} \right) + p \frac{\partial^2 \alpha}{\partial x_2^2} + Q|\alpha|^2 \alpha + RA = 0. \] (2.31)

On using the coordinate transformation

\[ \xi = \epsilon (x - V_\alpha t) = x_1 - V_\alpha t_1 = \frac{1}{\epsilon} (x_2 - V_\alpha t_2), \]
\[ \gamma = t_2 = \epsilon t_1 = \epsilon^2 t. \] (2.32)

Eq. (2.31) becomes

\[ i \frac{\partial \alpha}{\partial \xi} + p \frac{\partial^2 \alpha}{\partial \xi^2} + Q|\alpha|^2 \alpha + R\alpha = 0, \] (2.33)

which is the Nonlinear Schrödinger equation governing the envelope of the IAW.

For the sake of convenience, we now introduce dimensionless quantities; length is normalized to the effective Debye length \( \lambda_{\text{Deff}} \), time to \( \omega^{-1}_{\text{pi}} \), velocities to the effective ion-acoustic velocity \( c_{\text{seff}} = (T_{\text{eff}}/M)^{1/2} \), electric field by \( (T_{\text{eff}}/e \lambda_{\text{Deff}}) \) and densities by the density of the hot electron component \( n_{\text{ho}} \). Also we define the ratios \( \alpha \) and \( \beta \) as

\[ \alpha = \frac{n_{\text{ho}}}{n_{\text{ho}}} \quad \text{and} \quad \beta = \frac{T_1}{T_\text{h}}. \]

In the rest of the chapter all the quantities are normalized as above. Eq. (2.24) - (2.26) then become

\[ P = -\frac{3 \beta^2}{2 (\alpha + \beta)^2} \frac{\omega^5}{k^4}. \] (2.34)
\[ Q = -\frac{(\alpha + \beta)\omega^3 (\sigma_1 + \sigma_2 k^2 + \sigma_3 k^4 + \sigma_4 k^6 + \sigma_5 k^8 + 3k^{10})}{12 \beta k^2 (3 + 3k^2 + k^4)} \tag{2.35} \]

and

\[ R = -\frac{1}{2} \omega c_{30} - \left\{ \frac{\beta (1+\alpha)}{\beta (1+\alpha)} \right\}^{1/2} k c_{20} + \frac{\beta (1+\alpha) \omega^3}{2 (\alpha + \beta) k^2} \left( c_{50} + \frac{c_{40}}{\beta} \right), \tag{2.36} \]

where

\[ \sigma_1 = 12 + \frac{6 \alpha (1-\beta) (1+\alpha)}{\beta (\alpha + \beta)} - \frac{12 (1+\alpha)(\alpha + \beta^2)}{(\alpha + \beta)^2} \]

\[ + \frac{6 \mu_1}{(\alpha + \beta)^2} + \frac{3 \mu_2}{\beta^2 (\alpha + \beta)^3} - \frac{3 (1+\alpha) \mu_3}{\beta^2 (\alpha + \beta)^3}, \]

\[ \sigma_2 = 51 + \frac{6 \alpha (1-\beta) (1+\alpha)}{\beta (\alpha + \beta)} - \frac{54 (1+\alpha)(\alpha + \beta^2)}{(\alpha + \beta)^2} \]

\[ + \frac{24 \mu_1}{(\alpha + \beta)^2} + \frac{9 \mu_2}{\beta^2 (\alpha + \beta)^3} + \frac{3 (1+\alpha)^2 (\alpha + \beta^2)^2}{(\alpha + \beta)^4}, \]

\[ \sigma_3 = 84 + \frac{36 \mu_1}{(\alpha + \beta)^2} - \frac{180 (1+\alpha)(\alpha + \beta^2)}{(\alpha + \beta)^2} \]

\[ + \frac{9 \mu_2}{\beta^2 (\alpha + \beta)^3} + \frac{(1+\alpha)^2 (\alpha + \beta^2)^2}{(\alpha + \beta)^4}. \]
\[ \sigma_4 = 66 + \frac{24\mu_1}{(\alpha+\beta)^2} + \frac{3\mu_2}{\beta^2(\alpha+\beta)^3} - \frac{30(1+\alpha)(\alpha+\beta^2)}{(\alpha+\beta)^2}, \]

\[ \sigma_5 = 24 + \frac{6\mu_1}{(\alpha+\beta)^2} - \frac{6(1+\alpha)(\alpha+\beta^2)}{(\alpha+\beta)^2}, \]

\[ \mu_1 = 2\alpha^2\beta - 4\alpha^2 + 2\alpha\beta - 2\beta^2 - 6\alpha\beta, \]

\[ \mu_2 = (1+\alpha)^2(\alpha+\beta)^2(\alpha+\beta^3) - 5(\alpha+\beta)^3\beta^2 \]

\[ + 2(1+\alpha)\beta^2(2\alpha^2\beta^2 + \alpha\beta^3 + 3\alpha^2 + 6\alpha\beta^2 + 3\beta^3 + \alpha), \]

and

\[ \mu_3 = \alpha^3(1+\alpha) + 2\alpha^2(1+\alpha)\beta + \alpha(1-\alpha)\beta^2 \]

\[ + \alpha^2(5+\alpha)\beta^3 + 2\alpha\beta^4 + \beta^5. \]

From Eqs. (2.34) - (2.35), we get,

\[ \mathcal{PQ} = \frac{\beta\omega^8\chi}{8(\alpha+\beta)k^6(3+3k^2+k^4)}, \quad (2.37) \]

with

\[ \chi = \sigma_1 + \sigma_2k^2 + \sigma_3k^4 + \sigma_4k^6 + \sigma_5k^8 + 3k^{10}. \quad (2.38) \]

In the limit \( \alpha \to 0 \), i.e., in the absence of the cold electron component, the expressions (2.34) - (2.38) reduce to those obtained by Kakutani and Suginoto (1974).
II.4 Modulational Instability and Envelope Solutions

The NS equation, i.e., Eq. (2.33) governs the evolution of the envelope of the plane IAW. In order to study the envelope behaviour of these waves we express the complex amplitude \( a \) in terms of two real functions \( \rho \) and \( \sigma \) (Hasegawa 1975), namely,

\[
\alpha = \rho^{1/2}(\xi, \tau) \exp \{ i \sigma(\xi, \tau) \}. \tag{2.39}
\]

Eq. (2.33) can then be separated into its real and imaginary parts as

\[
\frac{\partial \rho}{\partial \tau} + 2 \rho \frac{\partial}{\partial \xi} \left( \rho \frac{\partial \sigma}{\partial \xi} \right) = 0, \tag{2.40}
\]

and

\[
\frac{\partial \sigma}{\partial \tau} + \rho \left( \frac{\partial \sigma}{\partial \xi} \right)^2 + \frac{\rho}{4 \rho^2} \left( \frac{\partial \rho}{\partial \xi} \right)^2 - \frac{\rho}{2 \rho^2} \frac{\partial \rho}{\partial \xi}^2 - Q \rho = 0. \tag{2.41}
\]

In obtaining Eqs. (2.40) and (2.41), the \( R \)-term in Eq. (2.33) has been eliminated by using the transformation \( a \rightarrow a \exp(i R \tau) \).

Now if we linearize Eqs. (2.40) and (2.41) as

\[
\begin{pmatrix}
\rho \\
\sigma
\end{pmatrix} = \begin{pmatrix}
\rho_0 \\
\sigma_0
\end{pmatrix} + \begin{pmatrix}
\rho_1 \\
\sigma_1
\end{pmatrix} \exp \{ i (\kappa \xi - \Omega \tau) \}, \tag{2.42}
\]

we obtain the dispersion relation,

\[
\Omega^2 = \rho^2 \kappa^4 - 2 \rho Q \rho_0 \kappa^2 = (\rho \kappa^2 - Q \rho_0)^2 - Q \rho_0^2, \tag{2.43}
\]

which shows that there is no instability if \( PQ < 0 \). On the other hand if \( PQ > 0 \), the perturbations with wavenumber
$K < (2Q Q_o/P)^{1/2}$ are unstable and the mode with $K = (Q Q_o/P)^{1/2}$ grows fastest with the growth rate $Q Q_o$.

From Eq. (2.37) it is clear that $PQ > 0$ only if $\chi > 0$. For different values of $\alpha$ and $\beta$, the critical wavenumbers $k_c$ for modulational instability can then be obtained from the equation $\chi = 0$. The variation of the critical wavenumber squared, $k_c^2$, with the different values of $\alpha$ and $\beta$ is depicted in the Table.

The localized stationary solutions of the NS equation in the form given by Eqs. (2.40) and (2.41) for the modulationally stable and unstable cases can be obtained as follows (Hasegawa 1975). For a localized solution, i.e., with a single hump for example, we require that $\phi = |a|^2$ be bounded between the extremum value $\phi_s$ and the asymptotic value $\phi_D$.

For stationary $\phi$, i.e., $\partial \phi/\partial \tau = 0$, Eq. (2.40) can be integrated to give

$$\phi \frac{\partial \sigma}{\partial \xi} = C(\tau), \quad (2.44)$$

$C(\tau)$ being a function of $\tau$ alone. Moreover Eq. (2.41) can be rewritten as

$$\frac{\partial^2 \sigma}{\partial \tau^2} + P \left( \frac{\partial \sigma}{\partial \xi} \right)^2 = \frac{1}{4 \partial \phi} \left[ 2Q \phi^2 + \frac{P}{\partial} \left( \frac{\partial \phi}{\partial \xi} \right)^2 \right] = \Phi(\xi) \quad (2.45)$$

From Eqs. (2.44) and (2.45) we get

$$\frac{d^2 C}{d \tau^2} \left/ \frac{d C}{d \tau} \right. = \frac{1}{\phi^2} \frac{d \phi}{d \xi} = \text{const.}$$
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*indicates modulational stability for all $k$. 
Since for a localized solution, we cannot have \( \xi^2 \frac{d\varphi}{d\xi} = \text{const.} \), we conclude that
\[ G(\tau) = \text{const.} = C_1, \text{ say.} \]
Eq. (2.44) can then be integrated to give
\[ \sigma = \int \frac{C_1}{\varphi} d\xi + A(\tau), \]
with \( A(\tau) \) as constant of integration. Since \( \partial \sigma / \partial \xi \) is a function of \( \xi \) only, it follows from Eq. (2.45) that \( \partial \sigma / \partial \tau \) is a function of \( \xi \) only. Consequently
\[ \frac{\partial \sigma}{\partial \tau} = \frac{dA}{d\tau} = \Lambda, \text{say}, \]
and then
\[ \sigma = \int \frac{C_1}{\varphi} d\xi + \Lambda \tau. \] (2.46)

On substituting Eq. (2.46) into Eq. (2.45) we get
\[ \left( \frac{d\sigma}{d\xi} \right)^2 = -\frac{2Q}{P} \varphi^3 + \frac{4\Lambda}{P} \varphi^2 + \frac{C_2}{P} \varphi - \frac{4}{P} C_1^2. \] (2.47)

If \( PQ > 0 \), i.e., when the waves are modulationally unstable, Eqs. (2.46) - (2.47) can be integrated to give the following localized solution:
\[ \varphi = \varphi_s \text{sech}^2 \left( \frac{(Q \varphi_s \sqrt{2}) \xi}{2P} \right), \quad \text{with } \varphi_s = \frac{2\Lambda}{Q}, \quad (2.48) \]
This is an envelope soliton. If \( PQ < 0 \), i.e., when the waves are modulationally stable, Eqs. (2.46) - (2.47) can be integrated to give
\[ \varphi = \varphi_s \left[ 1 - \tilde{\alpha}^2 \text{sech}^2 \left( \left( \frac{|PQ| \varphi_s \sqrt{2}}{2P^2} \right) \tilde{\alpha} \xi \right) \right], \]
\[ \sigma = \text{sin}^{-1} \left\{ \frac{\tilde{\alpha} \text{tanh} \left( \left( |PQ| \varphi_s / 2P^2 \right) \sqrt{2} \tilde{\alpha} \xi \right)}{\left[ 1 - \tilde{\alpha}^2 \text{sech}^2 \left( \left( |PQ| \varphi_s / 2P^2 \right) \sqrt{2} \tilde{\alpha} \xi \right) \right]^{1/2}} + \Lambda \tau + \frac{Q}{\varphi_s} \xi, \right\}. \]
where
\[ \tilde{\alpha}^2 = \frac{\tilde{\rho} - \rho_s}{\rho_s} \leq 1. \]  
(2.50)

The solution (2.49) is an envelope hole and represents a region of depletion in the wave intensity. The depth of depletion or modulation is given by \( \tilde{\alpha} \), as defined by Eq. (2.50).

II.5 Conclusions and Discussion

If the electrons in a plasma can be divided into hot and cold groups, the behavior of the ion-acoustic waves are drastically changed by the relative abundance of the two groups and their temperatures. We have discussed the properties of such a plasma by considering the cold electrons to be a fraction of the hot electrons, and for various ratios of their temperatures. The variation of the critical wave-number \( k_0^2 \) with \( \alpha \) and \( \beta \) are shown in the Table. When \( \alpha = 0 \), i.e., the cold electron component is absent, \( k_0^2 = 2.163 \). For nonzero \( \alpha \), the waves are stable for all \( k \) until a critical value of \( \beta \) is reached. Below this critical \( \beta \), the presence of the cold electron component stabilizes the wave. This critical \( \beta \) increases with \( \alpha \). For a given \( \alpha \), as \( \beta \) increases, \( k_0^2 \) reaches a maximum and then decreases slowly.

As discussed in Chapter I, the modulational instability and consequent envelope states are due to the balance
of the dispersion and nonlinearity. As seen from Eqs. (2.34) and (2.35), the dispersion, given by $P$, and the nonlinearity, given by $Q$, are functions of $\alpha$ and $\beta$. The sign of the nonlinear term $Q$ can change for different values of $k$. The functional dependence of $P$ and $Q$ on $\alpha$ and $\beta$ are different and hence the dispersion and nonlinearity vary differently. Consequently, for a given $\alpha$ and $\beta$, the critical wavenumber for modulational instability changes. Similarly the characteristics of the envelope states also change.