VI.1 Introduction

The nonlinear Schrödinger (NS) equation governs a variety of phenomena, e.g., the self-focussing and self-modulation of plane waves, the propagation of heat pulses in solids, the propagation of a number of plasma waves e.g., Langmuir, Ion-Acoustic and Magnetosonic waves, etc. (Scott et al. 1973). The modified Korteweg-de Vries (KdV) equation on the other hand arises in the study of acoustic waves in anharmonic lattices, Alfvén waves in a collisionless plasma, etc. (Jeffrey and Kakutani 1972). Both these equations can be unified into the following generalized nonlinear dispersive wave equation (Hirota 1973)
where $\alpha$, $\beta$, $\gamma$ and $\delta$ are real constants. This equation, which is known as the Hirota equation, reduces to a NS equation for $\alpha = \gamma = 0$, and to a modified KdV equation for $\beta = \delta = 0$. For the case $\alpha \beta = \gamma \delta$, by using a rather heuristic approach, the exact envelope soliton solutions of Eq. (6.1) were obtained by Hirota (1973).

The propagation of waves, in one-dimensional nonlinear lattices e.g., continuum approximation of the Fermi-Pasta-Ulam problem (Zabusky equation) and in shallow water under gravity propagating in both directions (Toda 1975), are described by the nonlinear wave equation

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - 6 \frac{\partial^2 \phi}{\partial x^2} \frac{\partial^4 \phi}{\partial x^4} = 0,$$

(6.2)

where $\phi$, $x$ and $t$ are normalized to the quantities appropriate to the particular problem. This is the well known Boussinesq equation (Boussinesq 1872). The exact-$N$-soliton solution of this equation was also obtained by Hirota (1973).

Both the equations (6.1) and (6.2) are nonlinear and their plane wave solutions are dispersive, and hence they describe nonlinear dispersive media. As discussed in Chapter I, such media can give rise to modulational instability and consequent localized envelope states. In this chapter,
we study these properties of Eqs. (6.1) and (6.2) by using the KBM method.

VI.2 Nonlinear Schrödinger Equation

Let us first consider Eq. (6.1). To obtain the NS equation describing the systems governed by this equation, we use the KBM method as in Chapter II. The solution to Eq. (6.1) can then be written as

\[ \chi = \epsilon \chi_1(a, \bar{a}, \psi) + \epsilon^2 \chi_2(a, \bar{a}, \psi) + \ldots, \quad (6.3) \]

where \( \chi_1 \) is chosen to be the monochromatic plane wave given by

\[ \chi_1 = a e^{i \phi} (\psi) + \bar{a} e^{i \phi} (-i \psi). \quad (6.4) \]

Here \( a \) is the complex amplitude, \( \psi = kx - \omega t \) is the phase factor and \( \bar{a} \) is the complex conjugate of \( a \). In Eq. (6.3), \( \chi_1, \chi_2, \chi_3, \ldots \) are functions of \( x \) and \( t \) only through \( a, \bar{a} \) and \( \psi \). The complex amplitude \( a \) is a slowly varying function of \( x \) and \( t \) as given by Eq. (2.8).

On substituting Eq. (6.3) into Eq. (6.1), we get equations of different orders in \( \epsilon \). The equation to order \( \epsilon \) gives the linear dispersion relation, namely

\[ D(k, \omega) = \omega - \beta k^2 + \gamma k^3. \quad (6.5) \]

To order \( \epsilon^2 \), Eq. (6.1) contains terms proportional to
exp(±iψ) which give rise to resonant secularity. The condition for the removal of this secularity is

$$A_1 + V_g B_1 = 0,$$

(6.6)

where $V_g = 2\beta k - 3\gamma k^2$, is the group velocity of the plane waves. The secular free solution can then be written as

$$\chi_2 = b(a, \bar{a})\exp(\pm\psi) + \text{c.c.} + \chi_{20}(a, \bar{a}),$$

(6.7)

where $b(a, \bar{a})$ and $\chi_{20}(a, \bar{a})$ are constants with respect to $\psi$.

The equation to order $\varepsilon^2$, which is obtained by using Eqs. (6.4) and (6.8) in Eq. (6.1), has two sources of secularities: the resonant secularity arising from $\exp(\pm\psi)$ terms, and the second one due to $\psi$ independent terms which become proportional to $\psi$ on integration. The condition for the removal of the latter secularity determines $\chi_{20}$ to be an absolute constant. The resonant secularity however is removed by the condition

$$i(A_2 + V_g B_2) + \frac{1}{2}\frac{dV_g}{d\lambda}(B_1 \frac{\partial B_1}{\partial a} + B_1 \frac{\partial B_1}{\partial \bar{a}}) - 3(\alpha k - \delta)|a|^2 \bar{a} = 0.$$

(6.8)

Defining

$$P = \frac{1}{2}\frac{dV_g}{d\lambda} = \beta - 3\gamma k$$

and $Q = -3(\alpha k - \delta)$, (6.9)

and using Eqs. (2.29) and (2.30), Eq. (6.8) can be written as

$$i\left(\frac{\partial a}{\partial \tau_2} + V_g \frac{\partial a}{\partial \chi_2}\right) + P_i \frac{\partial^2 a}{\partial \chi_1^2} + Q_1 |a|^2 a = 0.$$
This, on introducing the new variables as in Eq. (2.32), reduces to the familiar NS equation:

\[ i \frac{\partial \alpha}{\partial t} + P_1 \frac{\partial^2 \alpha}{\partial \tilde{z}^2} + Q_1 |\alpha|^2 \alpha = 0. \quad (6.10) \]

Exactly similar analysis is carried out for Eq. (6.2). The linear dispersion relation for the plane wave solutions in this case is given by

\[ D(k,\omega) \equiv -\omega^2 + k^2 - k^4. \]

To order \( \varepsilon^2 \), secular free solution of Eq. (6.2) is

\[ \phi_2 = \frac{2\alpha^2}{k^2} \exp(2i\psi) + c(a,\tilde{a}) \exp(i\psi) + \text{c.c.} + \phi_{20}(a,\tilde{a}). \]

From the secularity removal condition for Eq. (6.2) (to order \( \varepsilon^4 \)), \( \phi_{20}(a,\tilde{a}) \) is found to be

\[ \phi_{20} = \frac{12}{\sqrt{3} - 1} \alpha \tilde{a} + \mu, \]

where \( \mu \) is an absolute constant. As before, the condition for the removal of the resonant secularity in Eq. (6.2) to order \( \varepsilon^3 \) yields the NS equation.

\[ i \frac{\partial \alpha}{\partial t} + P_2 \frac{\partial^2 \alpha}{\partial \tilde{z}^2} + Q_2 |\alpha|^2 \alpha + R \alpha = 0, \quad (6.11) \]

with

\[ P_2 = \frac{1}{2} \frac{dV_3}{dk} = -\frac{k(3-2k^2)}{2k(1-k^2)}, \]

\[ Q_2 = \frac{3(12-17k^2+12k^4)}{\omega(3-4k^2)} \quad \text{and} \quad R = -b/k^2. \quad (6.12) \]
6.3 Modulational Instability and Envelope Solutions

The equations (6.10) and (6.11) describe how the amplitudes of the plane wave solutions of the equations (6.1) and (6.2) respectively will evolve according to their dispersion, determined by $p_i$ and nonlinearity, determined by $Q_i$ ($i = 1$ and 2). If $p_i Q_i > 0$, perturbations with $k < k_o = (2Q_i q_o / p_i)^{1/2}$ are unstable and grow with the maximum growth rate $\gamma = q_i p_o$ for $k = (q_i p_o / p_i)^{1/2}$, where $p_o = |\phi_o|^2$ is the initial intensity. However, if $p_i Q_i < 0$, perturbations of all wavelengths are stable. From Eq. (6.9), we find that

$$p_i Q_i = -3(\beta - 3\gamma k)(\alpha k - \delta).$$

So that modulational instability of Eq. (6.1) is decided by the values of $\alpha$, $\beta$, $\gamma$ and $\delta$. For $\alpha = \beta = 0$, when Eq. (6.1) becomes a NS equation itself, $p_i Q_i = 3 \beta \delta$; which in confirmation with earlier results is unstable for $\beta \delta > 0$. For $\beta = \delta = 0$, Eq. (6.1) reduces to the modified KdV equation with $p_i Q_i = 9\alpha \gamma k^2$; this unlike KdV equation is unstable provided $\alpha \gamma > 0$. For nonvanishing $\alpha$, $\beta$, $\gamma$, $\delta$ however, modulational instability can arise only if $k > k^*$ where

$$k^* = \left[ (\alpha \beta + 3\gamma \delta) \pm \left\{ (\alpha \beta + 3\gamma \delta)^2 - 12\alpha \beta \gamma \delta \right\}^{1/2} \right]/6\alpha \gamma.$$ 

For $k^*$ to be real we must have either $\alpha \beta \gamma \delta < 0$ or $\gamma \delta > 0$ with $(\alpha \beta + 3\gamma \delta)^2 > 12\alpha \beta \gamma \delta$.

If $\alpha = \delta = 0$, Eq. (6.1) is linear and dispersive and if $\beta = \gamma = 0$, Eq. (6.1) is nonlinear but dispersionless.
Evidently in both these cases we cannot obtain the NS equation.

In case of Eq. (6.2), Eq. (6.12) gives

\[ P_2 Q_2 = -\frac{3}{2} \frac{(3-2k^2)(12-17k^2+12k^4)}{(1-k^2)^2(3-4k^2)} = -\frac{3}{2} \frac{f(k)}{(1-k^2)^2(3-4k^2)^2} \]

where \( f(k) = 96k^8 - 352k^6 + 510k^4 - 369k^2 + 108 \). Thus Eq. (6.2) is modulational unstable for \( f(k) < 0 \). The critical wavenumber for modulational instability is found to be \( k_c = 0.866 \). Thus all waves with \( k > k_c \) are modulationally unstable.

Having settled the question of modulational stability of the equation (6.1) and (6.2), the corresponding envelope solutions are obtained immediately. If \( P_1 Q_1 > 0 \), i.e., the unstable case, the localized stationary solutions are the envelope solitons given by Eq. (2.48) with \( P_1 \) and \( Q_1 \) given by Eqs. (6.9) and (6.12). And if \( P_1 Q_1 < 0 \) (\( P_1 Q_1 = -|P_1 Q_1| \)), i.e., the stable case, the localized solutions are the envelope holes as given by Eq. (2.49).

Now the localized stationary solutions of Eqs. (6.1) and (6.2) may be easily obtained from these considerations. For example, in Eq. (6.1) the cases a) \( \alpha = \delta = 0 \), \( \beta \) and \( \gamma \) both positive, and b) \( \beta = \delta = 0 \), \( \alpha \) and \( \gamma \) both positive, admit envelope soliton solutions. In case of Eq. (6.2), the stationary solution is an envelope soliton for \( k > 0.866 \).
and an envelope hole otherwise.

VI.4 Conclusions and Discussion

We have shown that the amplitudes of the plane wave solutions of Eq. (6.1) are governed by the NS equation and that these plane waves can be modationally unstable. When \( \beta = \delta = 0 \) Eq. (6.1) reduces to the modified KdV equation, which has soliton solutions. And Eq. (6.10) with \( \beta = \delta = 0 \) describes the evolution of the amplitudes of the plane wave solutions of the modified KdV equation. The plane waves are modationally unstable for \( \alpha \gamma > 0 \) and stable otherwise, and give rise to envelope soliton or envelope hole solutions respectively.

In the case of the Boussinesq equation, plane waves with \( k > 0.886 \) are unstable against perturbations with \( k < \left( \frac{2Q_2 \Omega_0}{P_2} \right)^{1/2} \), and consequently give rise to envelope solitons. When one of these two conditions is violated, it is stable and hence has envelope hole solutions.

Equations (6.1) and (6.2) describe nonlinear dispersive media. In such media we can study two types of phenomena. One is the dynamics of the wave form itself and the other that of the wave envelope. Here we have studied the relation between these two phenomena.