CHAPTER V

NON-LINEAR EXCITATION OF DRIFT WAVES

BY KINETIC ALFVEN WAVES

5.1 Introduction:

Over the years drift waves have been extensively studied because of the wide range of conditions under which they are unstable both in laboratory [1-7] and space plasmas [8]. Their presence is believed to contribute significantly to anomalous diffusion which is a principal factor in the confinement of plasmas in many laboratory experiments. These waves arise due to the inhomogeneity in the plasma density. They are driven unstable by the free energy associated with the spatial gradients, \( \nabla n \omega \). Since the presence of density gradients is a necessary feature of all magnetically confined plasmas, the drift waves were initially considered universal instabilities. In addition to containing spatial density gradients, a tokamak contains sheared magnetic fields. It is now established that the shear in the magnetic field lines plays an important role in the stability of the drift wave.
Since the pioneering work of Pearlstein and Berk [1] the instability of the collisionless drift waves in sheared magnetic fields has been the subject of numerous investigations. By recognising the existence of outgoing wave solutions, it was concluded on the basis of WKB analysis that there existed absolutely unstable collisionless drift waves. Subsequent work by Gladd-Horton [2], Liu et al [3] based on perturbative methods seemed to confirm the existence of the instability. In reference [4], the same differential equation representing the evolution of drift waves in sheared magnetic fields was solved by breaking up the spatial domain into inner and outer regions. Later Tsang et al [5] extended the work of reference [4] to obtain an improved eigen value equation for all even and odd radial eigen modes. As a result of their investigation there emerged the possibility that the perturbation theory form of the dispersion relation was inadequate because it could only be recovered in the limit in which small corrections could be important. In particular, the perturbative theory form was found to be more accurate for more strongly damped modes.

Recently Ross-Mahajan [6] and several others [7] retaining the full electron \( Z \) function showed that the drift waves in slab model were actually stable. They
confirmed the stability of the collisionless drift waves by numerical and analytical solutions of the appropriate second order differential equation for parameter range of interest in tokamak plasmas. It was pointed out that near the rational surface, the electron \( Z \) function varies rapidly and is poorly represented by its residue. Further more, away from the rational surface, where the residue becomes an accurate approximation a comparison of perturbation theory with numerical and improved analytical results of Tsang et al [5] indicates that the wave-electron interaction is somewhat less destabilising than is believed.

Although most of the drift waves are stabilised by magnetic shear, there remain several destabilising mechanisms even in a sheared magnetic field. These are force-free currents, toroidal effects, trapped electrons, and non-linear [9] effects among others. Of these the non-linear coupling of drift-waves, namely parametric [10] effects has been extensively studied. The principal motivation for the studies have been the understanding of wave phenomena, occurring in r.f. heating schemes.

The initial investigation of parametric excitation and stabilisation of drift waves was done by Fainberg Shapiro [12] who studied the stabilisation of collisionless and collisional drift instabilities, by high
frequency electric fields along a steady magnetic field. Later Amano et al [12] analysed the effects of r.f. electric field on the excitation and stabilisation of various collisional and collisionless drift waves. The effect of large amplitude spatially uniform dipole electric field, at the lower hybrid frequency on the drift waves in collisionless plasmas was investigated by Sundaram and Kaw [13]. It was shown that the lower hybrid waves could parametrically excite or suppress the drift waves. Subsequently Tripathi [14] included finite wave length effects and found that the drift wave spectrum was stabilised because of parametric coupling to lower hybrid waves. Antani-Kaup [15] have considered a three wave decay, involving the scattering of pump whistler from a drift wave. They found that the scattering would be mainly restricted to the forward direction and the drift wave has a large growth for parameters of interest.

In the context of Alfven wave heating scheme [16, 17] an important problem to investigate is the parametric interaction between kinetic Alfven waves and drift waves. In a tokamak plasma the kinetic Alfven waves have enhanced amplitude near the mode conversion surface and several non-linear processes are expected to take place. One such process, that of the parametric decay of kinetic Alfven wave into the acoustic wave has been investigated earlier [16].
We have studied the problem of non-linear interaction of kinetic Alfven waves with tearing modes in chapters II, III and IV. It was found that the tearing modes could be resonantly excited with large growth rates.

In this chapter we examine the question of an alternate channel of Alfven wave decay with particular reference to drift waves, since they are known to influence plasma confinement in tokamak devices.

We have studied the non-linear decay of the mode converted kinetic Alfven wave into another kinetic Alfven wave and a drift wave. We model the dynamics of the electrons and ions using kinetic equations, to retain the effects of shear and finite Larmor radius corrections.

Using quasineutrality condition and Ampere's law, we derive the coupled equations for the decay process. These equations turn out to be quite complicated and are not amenable to easy solutions.

Under a local approximation however, the differential operators simplify and reduce to algebraic expressions. The growth rates and thresholds for the decay process are calculated, and found to be comparable to that of the ion acoustic process obtained by Hasegawa-Chen \[16\]. The ratio of the growth rates for the two processes is proportional to \( \frac{\omega_\star}{k_\parallel c_s} \sim c(0) \) where \( \omega_\star \) is
the diamagnetic drift frequency and $C_A$ is the velocity of sound. We have demonstrated that the temperature gradient driven drift waves could also be parametrically excited. These long wave length modes are found to have larger growth rates.

These modes can jeopardise the heating efficiency by providing alternate channels of non-linear energy transfer as well as by their deleterious effect on plasma confinement.

We have in addition investigated the effects of the background inhomogeneity on the decay process. Near the mode conversion region ($\omega = k_A \nu_A$) where the wavelengths of the decay waves could become comparable to the background inhomogeneity scale lengths, the linear dielectrics of the decay waves are expanded linearly. Although the resulting equation in 'x' space is of a high order, in Fourier space it is only of second order and hence amenable to WKB analysis. Treating the inhomogeneity scale length as a perturbation on the homogeneous plasma, we establish the condition under which an absolute instability, which is a well behaved solution of the differential equation can occur near the mode conversion region. We find that for values of the pump amplitude above a threshold value, the drift waves can be parametrically excited with large growth rates.
We present the basic coupled equations for the parametric decay process in section (5.2) and obtain growth rates and threshold conditions from local approximation in section (5.3). Section (5.4) contains a discussion on the effects of background inhomogeneity and establishes the conditions for absolute instability. The concluding section summarises the results and discusses their relevance to present day tokamaks.

5.2 Basic equations for the decay process:

In this section, we shall derive the basic coupled equations describing the parametric decay of a mode converted kinetic Alfven wave into another kinetic Alfven wave and a drift wave. We choose a simple slab geometry with an equilibrium field, $\mathbf{B}_e = B_0 (\mathbf{e}_y + x \mathbf{e}_z)$. where $\mathbf{e}_y$ and $\mathbf{e}_z$ are unit vectors in the $y$ and $z$ directions and $L_s$ is the shear length. Background inhomogeneities in physical quantities such as density, temperature are assumed to be in the 'x' direction and have simple linear variations. On this equilibrium a self consistent pump wave $\phi_p(x)\mathbf{e}_y$ (the kinetic Alfven wave) of the form

$$\phi_p(x,t) = \phi_p \exp \left[ \frac{x}{L} \right] + c.c.$$  

is imposed where $(\omega_p, k)_{\parallel}$ satisfies the linear dispersion relation for the kinetic Alfven wave.

$$\omega^2 = k_\parallel^2 \gamma_{\omega}^2 (1 + k_\perp^2 e_\perp^2)$$  

... 5.1

... 5.2
We consider the non-linear coupling between the pump wave, \((\omega, \mathbf{k})\), the lower side-band \((\omega_-, \mathbf{k}_-)
\)
\[= (\omega - \omega_0, \mathbf{k} - \mathbf{k}_0)\]
and the low frequency drift wave \((\omega, \mathbf{k})\). Interactions with the upper side band mode are neglected as they are off resonant for the decay process. The pump field \(\phi_0\) is also assumed to be sufficiently weak to that only interactions upto order \(|\phi_0|^2\) need be kept.

For a low \(\beta\) plasma \((\beta \sim \sqrt{\gamma_i/m_i})\), the compressional perturbation of the magnetic field \(\nabla\) is negligible. We adopt the classic two potential representation for the electric field \([18]\).

\[
E_\parallel = -\nabla \phi, \quad E_\parallel = -\nabla \psi \quad \ldots \text{5.3}
\]

In adopting different potentials for the parallel and perpendicular electric perturbation, the shear in the magnetic field lines is taken into consideration but the compression of the field lines is neglected. From Maxwell's equations

\[
\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \quad \ldots \text{5.4}
\]

'\(x\)' and '\(y\)' components of equation (5.4) are given by

\[
\frac{\partial^2}{\partial z \partial y} (\phi - \psi) = -\frac{1}{c} \frac{\partial B_y}{\partial t} \quad \ldots \text{5.5}
\]

\[
-\frac{\partial^2}{\partial z \partial x} (\phi - \psi) = \frac{1}{c} \frac{\partial B_x}{\partial t} \quad \ldots \text{5.6}
\]
while the $\mathbf{j}^e$ component vanishes.

The coupled equations can then be derived from the quasineutrality condition and the parallel component of Ampere's law

$$\nabla \cdot \mathbf{J} = 0 \quad \text{and} \quad \nabla \times \mathbf{E} = 0$$

where subscripts 'e' and 'i' stand for electrons and ions. 'n' and 'j' are the number and current densities respectively. The superscripts (L) and (NL) stand for linear and non-linear contributions being proportional to the first and second order perturbations in wave amplitudes.

To calculate the above perturbed quantities, we model the dynamics of the plasma by kinetic equations. In the low frequency range, the finite Larmor radius of the electrons are unimportant and the electrons can be adequately described by the drift kinetic equation.

$$\frac{\partial n_e}{\partial t} + \mathbf{v}_e \cdot \nabla n_e = -\frac{e}{m_e} \nabla \cdot \mathbf{E}_z +$$

$$\left( \mathbf{v}_e \times \mathbf{E}_z \right) \cdot \mathbf{B}_z \frac{\partial n_e}{\partial z} = 0 \quad \ldots \quad 5.9$$
where
\[
\vec{v}_i = \vec{v}_e + \vec{v}_p + \vec{v}_b, \quad \vec{v}_e = c \frac{\hat{E} \times \hat{B}_0}{\hat{B}_0^2}, \\
\vec{v}_p = -\frac{m_e}{e B_0^2} \frac{d\hat{E}_0}{dt}, \quad \vec{v}_b = \frac{\hat{E}_0}{B_0}, \quad \frac{d\phi}{dt} = \frac{\partial}{\partial \varphi} (\vec{v} \cdot \vec{n})
\]

and \(\nu_e\) is the drift distribution function.

For the ions, we use the Vlasov description, in order to retain effects like Larmor radius, magnetic shear etc. which are important for drift waves.

\[
\frac{\partial n_i}{\partial t} + (\hat{\nabla} \cdot \vec{v}_i)n_i + \frac{e}{m_i} \left[ \hat{E} + \frac{\hat{E} \times \hat{B}}{c} \right] \frac{\partial \vec{v}_i}{\partial \vec{v}} = 0
\]

... 5.10

Writing \(\frac{\partial}{\partial t} = \frac{\partial}{\partial t}^{(P)} + \frac{\partial}{\partial t}^{(E)} + \frac{\partial}{\partial t}^{(N)}\) ... 5.11

(where superscript '0' refers to equilibrium quantities, and \(j = e, i\) using (5.9) and (5.10), we now calculate the first and second order perturbed number and current densities to be substituted in equations (5.7) and (5.8).

The perturbed quantities are assumed to be of the form \(Q = Q(x) \exp i [k_y y + k_z z - \omega t]\)

where 'Q' stands for any physical quantity. ... 5.12

The linear responses are straight forward to calculate. For the low frequency electrostatic drift waves, the first order density perturbation is then given by

\[
\frac{n_e}{n_0} = 1 + \frac{\omega - \omega_{ke}}{\omega_{ke}} \frac{\hat{E}}{k_{11} V_e} \left[ \frac{\omega_{ke}}{k_{11} V_e} \right] \hat{\phi}
\]

... 5.13
where \( \hat{\phi} = \frac{e \Phi}{T_e} \), \( k_n = k - \frac{k_y x}{T_e} \), \( \omega_{de} = -\frac{e^2}{\epsilon_0} \frac{k_y}{k} \)
\[ L_n^{-1} = \frac{d}{dx} \left( \frac{d \ln n_0}{dx} \right), \]

\[ \dot{V}_d = \frac{2 T_e}{m_d} \left( j = i, e \right), \quad \langle n \rangle \]
is the equilibrium number density and \( Z(x) \) is the plasma dispersion function.

The first order density perturbation for the ions is calculated from the linearised Vlasov equation (5.10).
For this purpose, the expressions for the velocities of the particles are substituted from the standard orbit equations.

Carrying out the integration over the unperturbed orbits, the density perturbation, given by
\[ n_i = \int_0^{\infty} d v_y \int_0^{\infty} d v_x \frac{\omega_{de}}{T_e} \]
is obtained.

\[ n_i = -n_0 \left[ H + \frac{\omega_{de}}{T_e} \right] \left[ 1 + \frac{\omega_{de}}{T_e} \right] \frac{d}{dx} \left[ \frac{d}{dx} n_i \right] \]
\[ \dot{z} = \frac{T_e}{T_i}, \quad b_i = k_y \omega_e. \]
Substituting equations (5.13) and (5.14) in equation (5.7), we get
\[ \varepsilon \hat{\phi} = 0 \]
\[ \varepsilon = \left[ \frac{\omega_{de}}{T_e} \left( \frac{d}{dx} \right) \left( \frac{d}{dx} \right)^2 \right] \]
where subscript 'D' is used to denote drift waves and
\[ \varepsilon = \left[ \frac{\omega_{de}}{T_e} \left( \frac{d}{dx} \right) \left( \frac{d}{dx} \right)^2 \right] + \]
where \( I_n = I_n(b_i) e^{-b_i} \), \( I_n \) is the modified Bessel function of order \( n \), and

\[
\Gamma_0'(b_i) = \frac{d\Gamma_0(b_i)}{db_i}
\]

Equation (5.15) the linear eigen mode equation for drift waves has been extensively discussed in literature. In order to obtain equation (5.15) we have used the quasineutrality condition, since for electrostatic modes, the parallel and perpendicular potentials \( \phi, \psi \) are identical, \( (\phi = \psi) \). For the lower side band (kinetic Alfven wave) electromagnetic mode, both equations (5.7) and (5.8) need to be used.

The linear density perturbations calculated from (5.9) and (5.10) for electrons and ions respectively are

\[
\begin{aligned}
&n_e(\omega_0, b) = n_0 \left\{ \frac{\omega_0}{k_y \omega_{pe,0}} \frac{k_y}{\omega_{pe,0}} \tilde{\phi}_{\omega_0} + \right. \\
&\left. (1 - \frac{\omega_0}{k_y \omega_{pe,0}}) \tilde{\psi}_{\omega_0} \right\}
\end{aligned}
\]

... 5.17
In equations (5.17) and (5.18) the subscripts \((\rightarrow, \leftarrow)\) stand for the side band mode and pump mode respectively. Further in arriving at the equations we have made use of the fact that \(k_{\parallel, \omega_0} \nu_\perp < \omega_{-\omega} < k_{\parallel, \omega_0} \nu_\parallel\).

The current density perturbations defined by

\[
\mathcal{J}_{(\rightarrow)} = \int_{-\infty}^{\infty} \frac{\nu_{(\rightarrow)}}{\hbar} \, d\nu_{(\rightarrow)} \quad \ldots \quad 5.19
\]

is given by

\[
\mathcal{J}_{(\rightarrow)} (\omega_{-\omega}) = -\frac{\omega_{-\omega}}{k_{\parallel, \omega_0}} \, e n_0 \left\{ \frac{\omega_\perp}{\omega_{-\omega}} \, \frac{k_{\parallel, \omega_0}}{\omega_{-\omega}} \, \hat{\Phi}_{-\omega} \right\}
\]

\[
+ \left( 1 - \frac{\omega_\perp}{\omega_{-\omega}} \, \frac{k_{\parallel, \omega_0}}{\omega_{-\omega}} \right) \psi_{-\omega} \} \quad \ldots \quad 5.20
\]

The parallel current is mainly carried by the massless electrons and hence the ion current density perturbation is neglected. Substituting equations (5.17) - (5.20) in equations (5.7) and (5.8), we obtain

\[
\frac{\epsilon_0}{\epsilon_\parallel} \frac{\partial^2 \phi}{\partial x^2} = 0 \quad \ldots \quad 5.21
\]

with \(\psi, \phi\) related through the equation

\[
\psi = \epsilon_\parallel \frac{\partial^2 \phi}{\partial x^2} \quad \ldots \quad 5.22
\]
\( \zeta_A \) is given by

\[
\zeta_A = \frac{\partial^2}{\partial x^2} + \frac{\omega^2}{(k_{\parallel} - \nu_A)^2} - 1 \quad \ldots \quad 5.23
\]

In the next order equations (5.15) and (5.21) get coupled through the pump potential \( \Phi_0 \).

For the drift waves, the important electron nonlinear terms in equation (5.9) are those that originate

\[
\left\{ \nabla \left[ \frac{\mathcal{V}_E(\omega, k_\perp)}{c} \times \mathcal{B}(\omega, k_\perp) \right] \cdot \mathcal{E}_y + \mathcal{O} \right\} \frac{\partial \zeta}{\partial \nu_y} - \nabla \cdot \left[ \mathcal{V}_E(\omega, k_\perp) + \mathcal{V}_B(\omega, k_\perp) \right] \mathcal{L}_e(k_\parallel, k_\perp) + \mathcal{O} \quad \ldots \quad 5.24
\]

For the ion response the leading contributions in equation (5.10) arise from

\[
\left\{ \frac{e}{m_i} \left[ \frac{\mathcal{E}(\omega, k_\perp)}{c} + \frac{\mathcal{V} \times \mathcal{B}(\omega, k_\perp)}{c} \right] \frac{\partial \zeta}{\partial \nu} \right. \\
\left. + \mathcal{O} \right\}
\]

and

\[
\left\{ \frac{e}{m_i c} \left[ \mathcal{V}_E(\omega, k_\parallel) \times \mathcal{B}_\perp(\omega, k_\perp) \right] \cdot \mathcal{E}_y \right\} \frac{\partial \zeta}{\partial \nu_y} \quad \ldots \quad 5.25
\]

The last term is due to the parallel ponderomotive force \( \frac{e}{m_i c} (\mathcal{V}_E \times \mathcal{B}_\perp) \cdot \mathcal{E}_y \) acting on the ions. The expressions for \( \frac{n_i}{n_i^0} \), are given by
\[ \frac{N_{li}^{NL}}{N_{ci}^{NL}} = \frac{n_c}{\omega_{ci} |k_{ii}|} \left\{ \frac{\omega_{ci}}{k_y v_i} \left( \frac{k_0 - k_y}{\omega} \right) Z \left( \frac{\omega}{k_0 v_i} \right) \right\} + \left( \frac{k_0 - k_0}{\omega} \right) \left[ 1 + \frac{\omega}{k_{ii} v_i} Z \left( \frac{\omega}{k_{ii} v_i} \right) \right] \times \left[ k_y \frac{d}{dx} \left( \phi_0 \phi_i \right) - k_y \phi_0 \frac{d \phi_i}{dx} \right], \quad \ldots \quad 5.26 \]

\[ \frac{N_{li}^{NL}}{N_{ci}^{NL}} = \frac{n_c}{\omega_{ci} |k_{ii}|} \left\{ 1 + \frac{\omega - \omega_{ci}}{k_{ii} v_i} Z \left( \frac{\omega}{k_{ii} v_i} \right) \right\} \times \left( \frac{k_0 - k_0}{\omega} \right) \left[ k_y \frac{d}{dx} \left( \phi_0 \phi_i \right) - k_y \phi_0 \frac{d \phi_i}{dx} \right] \times \left[ \phi_i \frac{d}{dx} \left( \phi_0 - \phi_i \right) \phi_i \frac{d}{dx} \left( \phi_0 - \phi_i \right) \phi_i \right] \quad \ldots \quad 5.27 \]
Where some algebraic simplification has been effected through the usual approximations of \((k_{n}^{-1} \rho_{e}^{-2}) < < 1\)

\[ k_{n}^{-1} \nu_{e} < \omega_{-}; \nu_{e} < k_{n}^{-1} \nu_{e} \] for the kinetic Alfvén waves.

Substituting (5.26) and (5.27) in (5.7), the eigen mode equation for the drift wave that is driven by the mode coupling term proportional to the amplitude of the pump wave and the lower side-band is obtained.

\[
\frac{d \Phi_{0}}{d x} = \frac{C_{2}}{\omega_{e} k_{n}} \left[ k_{n}^{2} - \frac{k_{n}^{2}}{\omega^{2}} \right] \{ \frac{\rho_{d}}{\omega_{e}} \left[ k_{n} \frac{dy}{dx} (\Phi_{0} \Phi_{-}) - k_{y} \frac{d \Phi_{0}}{d x} \right] + \left[ 1 + \frac{(\omega - \omega_{e}) Z (\omega)}{\nu_{e} k_{n}} \right] \left[ k_{y} \frac{d \Phi_{0}}{d x} \right]
\]

\[
+ k_{y} \frac{d}{dx} \left[ (1 - \epsilon_{2} \frac{d^{2}}{dx^{2}}) \Phi_{-} \right] \left( 1 - \epsilon_{2} \frac{d^{2}}{dx^{2}} \right) \Phi_{0} \]

\[ \frac{d \Phi_{0}}{d x} = - k_{y} \left( \frac{d}{dx} \Phi_{-} - \epsilon_{2} \frac{d^{2} \Phi_{-}}{dx^{2}} \right) \]

\[ \left( \frac{d}{dx} \Phi_{0} - \epsilon_{2} \frac{d^{3} \Phi_{0}}{dx^{3}} \right) \] \[ ... 5.28 \]

Similarly, one needs to derive an equation for the lower side band that is coupled to the drift wave through the pump potential. For this, we need to calculate \(n_{d}^{+}(\omega, k_{-})\) and \(I_{2d}^{+}(\omega, k_{-})\).
For the electrons from the drift kinetic equation, the dominant non-linear contributions once again come from $\nabla_b(\omega, k) - \nabla_c(\omega, k)$ and $\nabla_e(\omega, k) \times \delta^s(\omega, k) \cdot \overline{E}$.

For ion density perturbations, $n_i^{NL}$ can be calculated from the equation of continuity,

$$\frac{d}{dt}(e n_i^{NL}) + \nabla \cdot J_{ii}^{NL} = 0 \quad \ldots 5.29$$

(The parallel ion current is negligible and can be ignored.) Where $J_{ii}^{NL}$ is composed of $\varepsilon [n_i^{''}(\omega) \nabla \hat{K}(\omega) + n_i^{'''}(\omega) \nabla \hat{K}(\omega)]$, ($\nabla \hat{K}(\omega)$, are the fluid drifts).

The second order electron and ion density perturbation for the side band mode obtained from equations (5.9) and (5.29) are given by

$$n_i^{NL}(\omega, \delta) = n_i \delta \frac{\Delta}{\Delta \omega} \left\{ - k y_0 \frac{\phi_0^*}{\omega} \frac{d}{dx}(\phi_a \phi_b) 
\right. 
+ k y_0 \phi_a \phi_b \frac{d \phi_a^*}{dx} 
- \frac{k y_0}{k y} \phi_a \phi_b \frac{d \phi_a^*}{dx} 
\left. \right\} \quad \ldots 5.30$$
We note that \( \nu_e(\omega) \) defined in equation (5.31) is proportional to \((\omega - \omega_{xe})\). For \( \omega \sim \omega_{xe} \) the drift frequency, \( \nu_e(\omega) \) is negligible compared to \( \frac{\nu_L}{\omega_{xe}} \). Previous analysis have revealed the same result \([16]\).

One needs to now obtain the electron current contribution \( \frac{\nu_L}{\omega_{xe}} \). This can be derived from the continuity equation

\[
\nabla \cdot \frac{\nu_L}{\omega_{xe}} - e \frac{\partial \nu_e}{\partial t} = 0 \quad \ldots \quad 5.32
\]

For ions \( J_{zi} \) can be neglected since the ion dynamics is mainly in the plane perpendicular to \( \vec{B}_0 \).

Carrying out these calculations and substituting the resulting expressions in equations (5.7) and (5.8), the non-linear dispersion relation for the side band mode is obtained

\[
\begin{align*}
\frac{C_A}{\omega_i} \frac{d^2 \phi}{dx^2} & = \frac{C_A}{\omega_i} \left[ \left( \frac{\omega}{k_{\parallel} V_B} \right)^2 \left( k_y \phi_z \frac{d^2 \phi_z}{dx^2} \frac{d \phi_z}{dx} \right. \\
& + \left. k_y \phi_z \frac{d^2 \phi_z}{dx^2} \right) + \phi_z \frac{d^2 \phi_z}{dx^2} \right] \\
& + k_y \phi_z \left( \frac{d^2 \phi_z}{dx^2} \right) + \frac{1}{\omega_{xe}^2} \left( k_y \phi_z \left( \frac{d^2 \phi_z}{dx^2} \right) \right) \quad \ldots \quad 5.33
\end{align*}
\]
where $\hat{\phi}_0^*$ is the complex conjugate of $\hat{\phi}$ and $'A'$ is given by

\[
\hat{A} = (\omega - \omega_{le}) \pm \left( \frac{\omega}{|k_n v_{le}|} \right) \left[ k_n v_e \frac{d\hat{\phi}_0}{dx} \left\{ \hat{\phi}_0^* \right\} + \left( 1 - \frac{k_3^2 \omega_{le}}{k_3^2 - \omega_e} \right) + \frac{\epsilon_s^2 k_3^2 \omega_e}{k_3^2 - \omega_e} \frac{d^2\hat{\phi}_0}{dx^2} \right] + k_n \frac{d\hat{\phi}_0}{dx} \left( 1 - \frac{k_3^2 \omega_{le}}{k_3^2 - \omega_e} \right) + \frac{\epsilon_s^2 k_3^2 \omega_e}{k_3^2 - \omega_e} \frac{d^3\hat{\phi}_0}{dx^3} \right] \]

Equations (5.28) and (5.33) constitute the general set of coupled equations between the kinetic Alfvén mode and the drift mode.

5.3 **Dispersion relation under local approximation** :

In this section, we shall examine the simplified versions of coupled equations (5.28) and (5.33) under the local approximation.

The local approximation is a major simplification of the drift wave problem and will be employed here without further proof. Extensive literature on this subject is available. The physical basis for the local approximation is that the mode is localised in a distance much less than the scale length of the gradients.
Choosing a Fourier wave field for the pump, the side band and the low frequency modes, equations (5.28) and (5.33) can be reduced to a set of algebraic equations.

\[ \overline{\varepsilon}_p \delta_\phi = \frac{i c^2}{\omega_c \omega_-} \left( \frac{k_{ll}^2}{\omega_-} - \frac{k_{ll}^2}{\omega_0} \right) \left[ (\vec{k} \times \vec{k}_s) \cdot \vec{e}_p \right] \overline{\varepsilon}_p + \]

\[ \left\{ 1 + \frac{\omega - \omega_{ve}}{k_{ll} \omega_{ve}} \right\} \left( \frac{\omega}{k_{ll} \omega_{ve}} \right) \frac{d(\delta\eta)}{d\phi} \left[ \delta_\phi \delta_\phi \right] \]

\[ \overline{\varepsilon}_A \delta\phi = \frac{-i c^2}{\omega_c \omega_- b_-} \left( \vec{k} \times \vec{k}_s \cdot \vec{e}_p \right) \left[ b_0 + b_0 \left( \frac{k_{ll} \omega_{ve}}{\omega_-} \right)^2 \right] \]

\[ + \frac{(\omega - \omega_{ve})}{|k_{ll} \omega_{ll}|} \left( \frac{\omega}{k_{ll} \omega_{ll}} \right) \left[ 1 - \frac{k_{ll} \omega_{ll} (1 + b_0)}{k_{ll} \omega_-} \right] \delta_\phi \delta_\phi \]

where \( b_{-0} = \left( k_{ll} \omega_{ve} \right)^2 \), \( d(\delta\eta) = b_- + b_0 + b_- b_0 \). \( \overline{\varepsilon}_A \) and \( \overline{\varepsilon}_p \) are given by

\[ \overline{\varepsilon}_A = 1 - (1 + b_-) \left( \frac{k_{ll} \omega_{ve}}{\omega_-} \right)^2 \]

\[ \overline{\varepsilon}_p = 1 + \tau + \frac{(\omega - \omega_{ve})}{|k_{ll} \omega_{ll}|} \left( \frac{\omega}{k_{ll} \omega_{ll}} \right) + \frac{(\omega - \omega_{ve})}{|k_{ll} \omega_{ll}|} \left( \frac{\omega}{k_{ll} \omega_{ll}} \right) \]

Eliminating \( \delta_\phi \) and \( \delta_\phi \) from equations (5.36) and (5.37) and taking \( |\omega_-| = \omega_0 \), the local dispersion relation can be written in the form
In reducing equation (5.40) to its simplified form, we have used the fact that $\mathcal{E}_d$ and $\mathcal{E}_A$ are nearly zero on the r.h.s. of equations (5.36) and (5.37). We shall now analyse equation (5.40) to obtain some simple estimates for the resonant decay instability. For the drift waves, we make use of the approximation, $b_i \ll 1$, $k_i V_i \ll |\omega| \ll k_i V_e$ so that $\mathcal{E}_d$ simplifies to

$$\mathcal{E}_d = 1 + \tau b_i - \frac{\omega V_e}{\omega} + \frac{i \sqrt{\tau}}{k_i V_e} (\omega - \omega V_e)$$  \hspace{1cm} \ldots 5.41$$

It may be recalled that the last term (being the inverse electron damping effect in equation (5.42)) is the source of the universal drift instability. We shall omit this term in equation (5.41) for resonant decay instability and set $\omega = \omega_k + i \gamma$, $\omega_\perp = -\omega_A + i \gamma$ where $\omega_A = \omega_e - \omega_k$ is the kinetic Alfvén wave frequency. Assuming $\gamma \ll \omega_k$, $\omega_A$ and Taylor expanding the dielectric functions $\mathcal{E}_A$ and $\mathcal{E}_d$ about $\omega_k, \omega_k$ respectively as

$$\mathcal{E}_A = \mathcal{E}_A (\omega_k) + (\omega - \omega_k) \frac{\partial \mathcal{E}_A}{\partial \omega} \bigg|_{\omega = \omega_k} \omega = \omega_k \quad \text{and}$$

$$\mathcal{E}_d = \mathcal{E}_d (\omega_k) + (\omega - \omega_k) \frac{\partial \mathcal{E}_d}{\partial \omega} \bigg|_{\omega = \omega_k} \omega = \omega_k$$

reduces to

$$(\tau + 1) (\tau + 1) = \frac{\omega V_e \omega_k}{2(1 + \tau b_i)^2} \left( \frac{C^2}{\omega_i, \omega_0} \right) \frac{b_0}{b_0} \left[ \frac{(k_x, k_y, \partial z)}{b_0} \right]$$

\hspace{1cm} \ldots 5.42
where $\nu_A$ and $\nu_D$ represent the linear damping rates corresponding to kinetic Alfven waves and drift waves respectively. The threshold amplitude for the potential $|\Phi_0|$ can be obtained by setting $\gamma = 0$ in equation (5.43). The growth rate, well above the pump threshold value turns out to be

$$\gamma_\tau = \frac{\omega_c}{\sqrt{2}} \left( \frac{\omega_\tau}{\omega_c} \right)^{1/2} \left( \frac{B_{\parallel 0}}{B_0} \right)^{1/2} \frac{1}{(1 + e b c) \left( I + b_0 \right) \left( I + b_0 \right) \left( I + b_0 \right) \left( I + b_0 \right)}^{1/2}$$

...(5.45)

In deriving the expression for $\gamma_\tau$, we have used the relationship between the pump magnetic field, $B_{\parallel 0}$ and $\Phi_0$ namely,

$$B_{\parallel 0}^2 = \left( k_{\parallel 0} k_{\parallel 0} \right) \left( \frac{k_{\parallel 0}}{\omega_\tau} \right)^2 \left( 1 + b_0 \right)^2 \left( 1 + b_0 \right) \left( 1 + b_0 \right) \left( 1 + b_0 \right)$$

...(5.44)

'\theta' is the angle between the vectors $k_\parallel$ and $k_\parallel^*$ and $\Theta$ is the ratio of plasma pressure to magnetic field pressure. The growth rate $\gamma_\tau$ thus derived for resonant excitation of drift waves is found to be comparable in magnitude to the growth rate for excitation of ion acoustic waves as calculated by Hasegawa and Chen [16]. The ratio of the two growth rates can be readily calculated as

$$\frac{\omega_\tau}{k_{\parallel 0} c_s} \approx \alpha(\theta)$$

...(5.45)
In fact it is possible to obtain larger growth rates if one couples to other branches of drift modes e.g., a temperature gradient driven drift wave. The effect of temperature gradients is appropriately taken into account by modifying the equilibrium distribution functions [13].

For an equilibrium temperature gradient in the 'x' direction, $\xi^x$ is modified [13]

$$\bar{E}^x = 1 + b - \frac{\omega_{ke}}{\omega} - \frac{k_{\perp}}{m_i} \left\{ \frac{\omega_{ke}}{\omega} (1 + \gamma)^2 + 1 \right\} = 0$$

... 5.46

where $\gamma = \frac{d \ln T}{d \ln n_0}$. In equation (5.46) the electron Landau damping term has been neglected as its effect is unimportant for the macroscopic mode under consideration. For $\gamma \gg 1$, $\omega \gg \omega_{ke}$, $k_{\perp} \left( \frac{T}{m_i} \right)^{1/2}$, equation (5.46) reduces to

$$\bar{E}^x = 1 - \frac{k_{\perp}}{m_i} \frac{\omega_{ke}}{\omega} \gamma$$

... 5.47

Considering the non-linear decay of a kinetic Alfvén wave into a stable branch of the mode given by equation (5.47), we find the growth rate of the decay instability to be given by

$$\bar{T} = \frac{\omega_{ci}}{\sqrt{6}} \left| \frac{B_{\perp}}{B_0} \right| \left( \frac{k_{\perp}}{\omega_{ke} \gamma / m_i} \right)^{1/2} \phi_{\perp}^{1/2} \left( \frac{b_0}{1 + b_0} \right)^{1/2} \frac{\sin \theta}{(1 + b - 1/2) \left( 1 + b_0 \right)^{1/2}}$$

... 5.48
Since \( \omega_{dr} \sim \omega_{pe} n \gg k_B c_s \), this growth rate is much larger than that for ion acoustic waves or ordinary drift waves. The excitation of such macroscopic modes with large growth rates can pose a serious drawback to the efficacy of non-linear ion heating schemes using kinetic Alfven waves.

5.4 Decay instability in an inhomogenous medium:

The results of the previous section are based on the solution of a local dispersion relation where the effect of background inhomogeneities have not been taken into account (except for inclusion of the diamagnetic drift frequency). Further, near the mode conversion region \( (\omega = k_B v_a) \) the wave lengths of the decay waves could become comparable to the background inhomogeneity scale lengths, hence the spatial operators have large values and have to be retained. We now study the coupled differential equations (5.29) and (5.34) and analyse the stability properties of the solutions. The problem of interest is that of determining the nature of unstable waves (if they exist) supported by the system. A wave is said to be unstable if a complex \( \omega = \omega_r + i \omega_i \) with positive \( \omega_i \) is obtained from the dispersion relation, signifying growth in time of the disturbance. In an infinite system [19] a pulse disturbance that is initially of finite spatial extent may grow in time,
without limit at every point in space, or it may 'propagate along' the system so that its amplitude eventually decreases with time at any fixed point in space. The former is termed 'absolute instability' and the latter 'convective instability'. It is of course the former which is more dangerous because the distinguishing characteristics of the absolute instability is that it grows everywhere in space as a function of time. The convective instability on the other hand 'propagates along' the system as it grows in time, so that the disturbance eventually disappears if one stands at a fixed point. We wish to ascertain whether a growing solution, an absolute instability can be supported by the system formed by coupled equations (5.33), (5.28). The coupled equations are quite complex in view of the complicated spatial structure of the interacting waves in the region. To simplify the analysis somewhat, we shall drop some unimportant terms, (e.g. the Landau damping term ) neglect $\xi_\omega(\xi_\omega)$ on the right hand side of equation (5.28) and set $\frac{\omega}{k_\ell + \gamma_\omega} = 1$ on the right hand side of equation (5.33). With these simplifications and setting $X = \frac{\omega}{\gamma_\omega}$ equations (5.33) and (5.28) can be written in the form
Near the mode conversion region of the pump wave, the drift and the side band kinetic Alfven waves are also close to their respective resonance points. The functions \( g(x) \) and \( h(x) \) can therefore be Taylor expanded around these points and expressed as

\[
\begin{align*}
g(x) &= K_D (x - x_D) \quad ; \quad h(x) = K_A (x - x_A)
\end{align*}
\]

where \((x_D)\) and \((x_A)\) are the resonance points for the drift and kinetic Alfven waves respectively, \(K_D\) and \(K_A\) are typical inverse scale lengths of shear variation and density inhomogeneity respectively. Since at \(x_D\) and \(x_A\), the dispersion relation for the drift and Alfven waves are satisfied, \(g(x_D)\) and \(h(x_A)\) are set
equal to zero. The linear operators on the left band sides of equations (5.49) and (5.50) therefore indicate an Airy function kind of spatial behaviour for the daughter waves. A similar spatial structure also exists for the pump wave [16], which couples (5.49) and (5.50). To solve this coupled set is still quite formidable. For analytical simplification we adopt a plane wave model for the pump wave and study the spatial evolution of the daughter waves. We follow the method of White et al [20], for analysing the instability. We Fourier analyse the coupled equations (5.49) and (5.50) defining

$$\Psi_p = \int_{-\infty}^{\infty} e^{-ipx} \hat{\phi}_p(x) \, dx$$

$$\phi_p = \int_{-\infty}^{\infty} \hat{\phi}_p e^{ipx} \, dx \quad \ldots 5.54$$

$$\hat{\phi}_o = \phi_o \exp(-ik_0x)$$

Eliminating the variable $\phi_p$ we obtain a single second order equation in $\Psi_p$ as

$$\frac{d^2}{dp^2} \Psi_p + F^{(p)}(p) \Psi_p = 0 \quad \ldots 5.55$$
where \( \psi \) is related to \( \phi \) through the relation 
(5.57) and \( F^2(P) \) is given by,

\[
F^2(P) = \frac{1}{k_A^2} \left[ i \left( pk_A - (P + k_0) \alpha k_A \right) + \frac{(k_0^2 - p^2) k_A^2}{(k_0^2 + p^2)} \right] \\
- \frac{k_0^2 k_A^2}{2 [k_0 (P + k_0) - k k_0]^2} - \frac{1}{4} \left\{ i \left[ p^2 - \alpha (P + k_0)^2 \right] \\
+ \left. \frac{Q}{p^2 + k_0^2} - \frac{k_0 k_A}{(k_0 (P + k_0) - k k_0)} \right\}^2 \\
- \frac{\omega \theta}{(\omega + \omega\iota)} \frac{\omega\iota}{\omega} \left( \frac{\omega \theta}{\omega^2} \right)^2 \left\{ k k_0 \right. \\
\left. - k_0 (P + k_0) \right\}^2 \right\] 
\]

where

\[
Q = \left( \frac{\omega}{\omega + \omega\iota} \right)^2 - 1 - \left[ \frac{a \omega}{\omega + \omega\iota} \right] \left[ 1 + b - \frac{\omega\iota}{\omega^2} \right] 
\]

and \( a = \frac{k_A}{k_B} \) (treated as a constant in this analysis).
Solutions of equation (5.55) are given in terms the usual WKB expressions

\[ \Psi_\pm = F^{-\frac{1}{2}} \exp \left[ \pm i \int P \, dp \right] \] \hspace{1cm} \ldots \text{5.56}

We require that the solution be localised in k-space with finite extent of localisation, which implies the localisation of the Fourier transform solution in x-space. Such a localisation exists if the Bohr-Sommerfield quantisation condition [20] is satisfied.

\[ \int_{P_1}^{P_2} F \, dp = (\hbar + \frac{1}{2}) \pi \] \hspace{1cm} \ldots \text{5.59}

where \( P_1 \) and \( P_2 \) are called 'turning points', which are the roots of \( F^2(p) = 0 \). To solve for the exact solutions for the turning points is quite complicated; we shall therefore look for approximate solutions and eigen value condition by using a perturbation procedure. We treat \( L_n \) as a small parameter and seek corrections to the eigen values in successive orders of \( L_n \) by perturbation.

In the limit of a homogeneous plasma, \( L_n \rightarrow \infty (\kappa_n \rightarrow 0) \) the Bohr-Sommerfeld condition requires that the integral

\[ \int_{P_1}^{P_2} F \, dp \] be vanishingly small.
This implies that the two turning points must coalesce. In the limit of homogeneous plasmas, the turning points obtained by setting $F_n \to 0$ are the solutions of

$$P^2 - (P + k_o a)^2 + \left( \frac{\omega_c}{\omega_o} \right) \left( \frac{k_o^2}{k_o^2} \right) \left( P^2 + k_o^2 \right) = 0$$

From the above equation, calculation of the coalescence condition is quite straightforward and is given by

$$k_o^2 a^2 = (1 - a) \left[ \frac{8}{a} - \frac{\omega_c}{\omega_o} - a k_o^2 \right]$$

where

$$Q_1 = \frac{2 \omega_c^2 k_o \omega_c}{(a - 1)^2} \left[ k \left( \frac{3a + 1}{a} - 2a^2 - 2 \right) + k_o (1 - \frac{1}{a}) \right]$$

Solving for $Q = Q_c$ from equation (5.61) we get

$$Q_c = \frac{-k_o^2 a}{a - 1} + 2 \omega_c^2 \left( \frac{\omega_c}{\omega_o} \right) \left( \frac{k_o^2}{(a - 1)^2} \right) \left[ k \left( \frac{3a + 1}{a} \right) - 2a^2 - 2 \right] + k_o (1 - \frac{1}{a} - 2a)$$

and the value of the coalesced variable obtained from equation (5.60) is

$$P_c = \frac{k_o a}{a - 1}$$

$F_\infty^2$ can now be written as

$$F_\infty^2 = \frac{1}{4k_o^2} \left( P - P_{c2} \right) \left( P - P_{c1} \right) \left( P - P_{c3} \right)$$
Where $P_c$ is given by equation (5.64), and $P_{3,4}$ are given by

$$P_{3,4} = k_0 a \pm \sqrt{ k_0^2 a^2 - (1-a) (a^2 - a - a k_0^2)} \quad \cdots \quad 5.66$$

The introduction of a large inhomogeneity scale length $L_n$ produces a splitting of the coincident turning points at $P = P_c$. This can be readily investigated by examining the behaviour of $F^2(x)$ near $Q = Q_c$.

Thus we write

$$F^2 = F_0^2 + \frac{dF^2}{dQ} \Delta Q + \frac{dF^2}{d\left(\frac{1}{L_n}\right)} (\frac{L_n}{L_n}) \quad \cdots \quad 5.67$$

If $P_3$ and $P_4$ in equation (5.65) are sufficiently far away from $P_c$, we can treat $(P-P_3)(P-P_4)$ as a constant. The resultant equation for $F^2$ is then a quadratic in $P$ and can be solved for the two turning points $P_{1,2}$ in the neighbourhood of $P_c$. We hence conclude that in the vicinity of $P = P_c$, the equation for $F^2$ assumes the form of a simple harmonic potential. In the region of interest (i.e., around $P = P_c$), we substitute $Q = Q_c, P = P_c$ everywhere in $F^2(P)$ in equation (5.65) except in the fast oscillating factor $(P-P_c)$.

We then find that equation (5.67) reduces to

$$F^2 = \frac{Q_1}{2k_0^2} \left[ a \nu^2 - a \nu^2 \right]$$
where

\[ q' = (a - 1) \Delta + \frac{i}{k_A} \left[ \frac{k_0 a}{Q_i} + \frac{(a - 1)}{2k_0} \left\{ \frac{2a}{a^2 + (a - 1)^2} + \frac{k_0}{(k_0 - k + ka)} \right\} \right] \]

\[ \Delta = (p - p_c) \]

Applying the quantisation condition between the split roots obtained by solving \( F^2 = 0 \) (equation (5.76)) the Bohr-Sommerfeld condition reduces to

\[ \left[ \frac{Q_i}{2k_A(a - 1)} \right]^\frac{1}{2} q'^2 = 2n + 1 \]

... 5.70

The expression for real and imaginary parts of obtained from the above dispersion relation is given by

\[ 1 + \frac{c b_i - \omega_{ke}}{\omega_R} = \frac{\omega_R + \omega_{ei}}{\omega_R a} \left[ \frac{k_0 a}{(a - 1) Q_i - Q_i} \right] \]

... 5.71

where

\[ \gamma = \omega_R (\omega_R + \omega_{ei}) \left[ \frac{k_0 a}{a \omega_{ke} k_A} (a - 1) Q_i + \frac{(a - 1)}{2k_0} \right] \]

\[ \left\{ \frac{2a}{a^2 + (a - 1)^2} + \frac{k_0}{(k_0 - k + ka)} \right\} \]

\[ + \sqrt{2} \left( \omega_R + \omega_{ke} \right) \frac{(2n + 1) k_A}{\omega_R a (a - 1)^{1/2} |Q_i|^{1/2}} \]

... 5.72
In deriving equations (5.71) and (5.72) we have assumed that $\gamma$ (the non-linear growth rate) is $<< \omega_R$, $\omega_A = \omega_0 - \omega_R$ and real \left( \frac{\omega}{k_{ii} - v_A} \right)^2 = 1$. Typically $k_A = \frac{1}{L_n}$ and $k_D = \left| \frac{k_{ii}}{k_{ii}} \right| \left( \frac{k_{ii}^2 v_T^2}{\omega} \right) = \frac{1}{L_n} \frac{k_{ii}^2 v_T^2}{\omega}$. For $\omega \sim \omega_R$, $k_{ii} v_T \ll 1$.

Further $L_s$ is much larger than $L_n$. It follows therefore that $K_D$ is much smaller than $K_A$. ($a = \frac{k_A}{k_D} \gg 1$).

For this value of $a$, we find from equation (5.62) that $Q_\perp$ is much less than zero. For $K_A > 0$ we observe that for certain threshold value ($|Q_\perp| > k_0^2$) of the pump field, $\gamma$ can be positive and therefore an absolute instability can exist. It must be noted that the threshold value for temporal growth of excited drift mode predicted from (5.72) should be treated as approximate since we have used a perturbative scheme, i.e. treating $L_n^{-1}$ as a perturbative parameter.

5.5 Summary

We have studied the parametric decay of a pump kinetic Alfven wave into a side band kinetic Alfven and a drift mode. The dynamics of the drift wave are sensitively dependent on the shear and finite Larmor radius effects. We have therefore used the kinetic equations to describe the motion of electrons and ions. Using quasineutrality condition and Ampere's law, the coupled equations for the decay process are derived. The coupled equations in 'x' space are quite complicated and are not amenable to easy solutions.
Under a local approximation, however, the differential operators reduce to algebraic expressions, and the dispersion relation could be readily obtained. We have calculated the threshold value for the decay process and the growth rate of the drift instability far above the threshold value. We find that the calculated growth rates are quite large and compete significantly with the growth rates of excited ion acoustic waves, calculated by Hasegawa-Chen [16]. The ratio of the growth rates in the two processes was shown to be
\[ \frac{\omega_d}{R_x c_s} \sim O(1) \]. We have demonstrated that the kinetic Alfven waves could couple to temperature gradient drift waves which have larger growth rates and longer wave lengths. In addition we have investigated the effects of the background inhomogeneity on the decay process. Near the mode conversion region when the wave lengths of the decay waves become comparable to the inhomogeneity scale lengths, the differential operators play a significant role. Expanding the linear dielectrics linearly around their resonant surfaces, we have obtained a second order differential equation in Fourier space. The equation is amenable to perturbative WKB analysis. Treating the inhomogeneity scale length \( L_a \) a perturbation parameter, we establish the condition under which an absolute drift instability can exist in the plasma. We have shown that for values of the pump amplitude above a certain
threshold value, the drift waves could be parametrically excited with large growth rates.

Our analysis has important application in Alfven wave heating schemes in laboratory plasmas. It has been shown by earlier investigations [16] that the excited kinetic Alfven wave has an enhanced amplitude at the mode conversion layer, which could lead to several non-linear processes. Hence a study of the non-linear properties of the kinetic Alfven wave is essential for better understanding of the propagation of the mode converted wave. In laboratory plasmas several experiments on Alfven wave heating have reported enhanced diffusion of particles [21] in addition to efficient heating. The diffusion of particles may have been caused by the excitation of drift waves. These modes could seriously jeopardise the heating efficiency by providing alternate channels of non-linear energy transfer, as well as by degrading plasma confinement.
References:


