CHAPTER II

ANOMALOUS LOSS OF PARTICLES IN BASE-BALL II MIRROR EXPERIMENT

1. Introduction:

The present concept of simple mirror fusion reactor envisages the perpendicular injection of energetic neutral beams (500 MW) for the purpose of fueling the reactor plasma in a steady state magnetic field. A general problem associated with this kind of reactor is to develop a method of 'start up', that is to procure the initial plasma inside the machine. The aim of Livermore Base-Ball II experiment was to develop a method of start up which the can be extended to the reactor regimes.

In general there are following methods of procuring the initial plasma:

(i) Initially the transient target plasma is obtained inside the machine. This can be done in several ways like by laser irradiation of solid pellet, placed inside the machine (1,2) by passing cold plasma filament along the field lines (3) by using arc discharges, pulsed deuterium discharges (4) etc. or by injecting the plasma directly,
which is heated and trapped by HCRH (5). Further the density and temperature build up are obtained by shooting neutral beams perpendicular to the field lines.

(ii) This method consists in obtaining a build up from high vacuum (6,7). For this purpose neutral beams are injected into vacuum in mirror machine. The beam gets Lorentz ionised to provide a target plasma which is then heated by charge exchange and maintained by ionization of the injected beam.

In one of the experiments done on Base-Ball II machine the build up was obtained by the 2nd method; injecting powerful neutral beams (2 kW) in high vacuum (7). The observations indicated that in the process of density build up, beyond a certain threshold, ion-cyclotron oscillations were excited. The onset of the instability was observed to depend on beam geometry and instability signals are usually absent shortly after the beam injection ceases. The draw back of this technique is that anomalous losses increase as the injected beam current increases limiting the density to 3-5x10^9 cm^-3. The spatial and temporal distribution of anomalous losses is related to potential perturbation rotating at ion V_B drift frequency. In this Chapter we have studied in some detail the mechanism of this anomalous loss of particles from the trap. In this connection we identify the mode near V_B-drift frequency with the mode described by Varma (8) for simple mirror geometry. We find that this mode in the presence of ion-cyclotron fluctuations becomes nonlinearly unstable and exhibits a periodic bursting behaviour very similar to the periodic bursts of azimuthal potential perturbations observed in the experiment. We have calculated the nonlinear growth time which agrees
fairly well with the observed time of 50-100\(\mu\)s between the two bursts.

As the present investigation concerns Base-Ball II experiment it will be appropriate to briefly describe the experimental set up and observations. This is done in Section 1. In this section we have also discussed our reasons to identify the observed azimuthal potential perturbation with the Varma mode. Section 2 contains a brief description of the linear theory of the Varma mode, a description of Simon and Wing's nonlinear theory of the Varma mode, which, as we shall see, is important in the present context and the energy properties of the Varma mode. In Section 3 we have briefly discussed the model which we have chosen for the ion-cyclotron oscillations. Section 4 contains the non-linear instability theory of the Varma mode in the context of Base-Ball II mirror experiment, while in the last section we have summarized our results.

2. (i) Experiment:

The confining field of Base-Ball II experiment is a quadrupole magnetic well of depth 2.1 generated by a super conducting Base-Ball seam windings. This magnet has been operated at central fields upto 1.5 T. The mean diameter of the magnet is 1.2 m, resulting in a plasma radius of 0.1m and a volume of \(\approx 10\) lit. The mirror separation on axis is 90 cm. To prevent the loss of plasma ions by charge-exchange a high vacuum is maintained in the plasma region. In this experiment several square meters of cryogenic pumping surfaces, at temperatures down to 2\(^0\)K provide high speed pumping for all ambient gases. The base pressure is \(\approx 10^{-9}\) \(P_{\text{at}}\).

A single neutral beam of either hydrogen or deuterium atom was injected normal to the magnetic axis for a period of seconds. The beam energy was
usually in the range of 0.5 to 5 KeV with some measurement up to 20 KeV. Beam currents were adjustable with the maximum beam current ranging up to 0.1 A equivalent at 20 KeV.

(ii) Observations:

In the experiment collisional plasma regime was examined by injecting hydrogen atom at low energy (0.5 – 2 KeV) under good vacuum conditions. The observations showed that a stable plasma at a density of \( \lesssim 10^9 \text{ cm}^{-3} \) was obtained in which classical scattering losses dominated over the charge exchange losses by the ratio of 6 to 1. Along with a near classical confinement time the r-f activity indicated the presence of an ion-cyclotron instability. It was noticed that as the density was increased, the anomalous losses increase rapidly. The repetitive bursts of ions each lasting for 50-100 micro-seconds were observed. In the analysis of the end-loss signals, in a typical case with high current injection the loss due to instability exceeded the classical losses, including the charge exchange, by more than a factor of four. The plasma density which could be achieved with the available beam power was limited to 3 to \( 5 \times 10^9 \text{ cm}^{-3} \).

In the experiment low frequency oscillations at ion- drift frequency were also observed. The electrostatic probes displaced in azimuth around the central plasma region detected perturbations of plasma potential rotating generally in the direction of ion-drift. The signals were of two types. One was of a very small amplitude (perturbed potential of the order of 0.5 V) near the drift frequency. These types of
oscillations have been seen in stable plasmas as well. In experiment they were followed for as long as several minutes during the decay of the plasma. The other type of oscillations were also near \sqrt{B} frequency. These signals are large amplitude (20-50 V perturbed \( \psi \)) and exhibit a bursting behaviour. The amplitude rises to a full value in typically 50-100 \( \mu \)s seconds. Apparently the anomalous particles losses are due to these modes. Energetic bursts of protons and electrons are emitted from the rotating region of perturbed potential starting promptly with the potential increase in each bursts. The origin of this mode is not clear. If suitable density and temperature gradients are included in the fluid equations for min-B geometry, the characteristics of these modes can be probably reproduced. However, there are no apriori reasons to expect such gradients in the experiment. The authors of the experiments associate the stable small amplitude mode with the Varma mode. This may not be quite correct. It has been pointed out that the non-linear effects like wave-coupling through non-linear Landau damping plays an important role in the development of turbulence in the mirror plasma (9,10). Thus Varma mode which is linearly stable may not be so in nonlinear regime. In fact Simon and Weng (11) have shown that in the presence of other flute modes, in simple mirror geometry, the Varma mode becomes non-linearly explosive. They correlate this behaviour of the Varma mode with the observed instability in 'Alice' and 'Phoenix' (12,13). In Base-Ball II which has a magnetic well the Varma mode cannot be driven unstable by other flute modes, as all of them are linearly stable. But as we shall show here that in the presence of ion-cyclotron oscillations the Varma mode is again driven unstable. It exhibits aperiodic bursting behaviour. Hence it is
more likely that in the experimental observations the large amplitude signal at $\nabla B$-drift frequency which exhibits a periodic bursting behaviour, rather than the small amplitude which is stable, is Varma's mode.

3. The Varma Mode:

Experiments on 'Alice' and 'Phoenix' (12,13) have indicated that plasma is capable of collective behaviour even at such low densities as $10^8$ cm$^{-3}$. As the plasma is built up particle by particle from almost zero density by injecting neutrals atoms, the system becomes unstable at the density of about $10^9$. It was noted that in addition to the usual flute modes, a density independent mode was seen both in 'Alice' as well as 'Phoenix'. Post (12) has given a theory for the phenomenon observed in 'Alice' which is worked out for plane geometry and approximate identification with the cylindrical geometry of the mirror machine is made by introducing appropriate boundary conditions. His theory, which amounts to using an equivalent gravitational acceleration 'g' for the force due to the inhomogeneous magnetic field and is worked out for $f$ -function velocity distribution predicts a two branch curve for the frequency versus density upto a critical density where the two branches merge. While Kadomtsev (14) has discussed the problem for a mirror machine self-consistently using an equation for transverse motion averaged over the line of force. However, the use of 'effective gravity' may give results which may not be wholly correct. In certain circumstances, the use of an effective gravity not only gives results which are quantitatively incorrect, but also suppresses some information which is contained in the actual magnetic gradient force ($=-\mu \nabla B$) which depends upon the velocity of the particle. Varma (8)
Improved the theoretical treatment by relaxing the constraint of constant gravity. By allowing for variation of $\mu$ and using a closure equation for $\mu$ of the type

$$\frac{d\mu}{dt} = 0 \quad (1)$$

along with the usual set of fluid equation, it was shown that dispersion relation for the flute mode is of the form

$$|\gamma| = -\left(\frac{\omega_{pi}^2}{\omega^2} \right) \frac{1}{k''} \left( \frac{1}{\omega'' + k''\omega_{bi}} \right) \left[ \frac{2}{|\omega''/\sigma''| + 2} \right]$$

$$- \frac{\omega_{pe}^2}{\omega^2} \frac{1}{k''} \left[ \frac{2}{\omega''} \right]$$

$$\quad (2)$$

where $(\omega'', k'')$ are the frequency and wave numbers of flute mode, $\sigma'' = -\frac{1}{B} \frac{\partial B}{\partial z}$ is the magnetic field gradient, $\sigma'' = \frac{\partial}{\partial z} \frac{\partial}{\partial x}$ is the density gradient, $|\omega_{bi}| = \frac{1}{2} \sqrt{\frac{U_{lo}^2}{\omega^2}}$ is the $\nabla B$-drift, $U_{lo}$ is the velocity with which $N^+-B$ particles are injected etc. Equation (2) is cubic in $\omega$ rather than the usual quadratic.

Defining

$$\nu = \frac{\omega''}{m \omega_i} \quad \omega_i = |\omega_{bi}|/\gamma_0$$

and

$$\eta = \frac{\omega_{pe}^2 \gamma_0}{|\omega_{bi}| m^2 |\omega_{bi}|}$$

[$\gamma_0$ is the radius of plasma]

we can put equation (2) in a more tractable form as

$$\nu^3 + 2\nu + \nu - \eta \nu - \nu = 0 \quad (3)$$
In the limit \(|\gamma | \ll 1\), this can be solved to give the three roots as

\[ \gamma = -1 \]
\[ \gamma = -\frac{1}{2} \pm \frac{1}{2} \sqrt{1 + 4 \alpha} \]

\( \gamma = -1 \) is the root which can be identified with the density independent mode observed in 'Alice' and 'Phoenix'.

Later Simon and Weng (11) examined the nonlinear stability of the Varma mode in the presence of the other flute modes. They found that if the injection current is increased beyond a threshold one of the lower modes starts growing. Because of the nonlinear coupling it triggers an explosive growth of the Varma mode and shuts off itself. The Varma mode then grows on its own until inhibited by some conservation process. This behaviour was identified by them with the sudden explosion of the amplitude signal when the injection current was increased beyond the threshold. In min-B geometry such a coupling is not possible as all the modes are linearly stable.

The energy of the Varma mode in weakly dissipative dielectric media is given by

\[ W = \varepsilon_0 \gamma \frac{2}{\gamma} \left\langle \frac{E''}{E''} \right\rangle \]

(4)

where \( E'' \) is the electric field amplitude of the Varma mode and \( \varepsilon'' \) is the dielectric constant given by

\[ \varepsilon'' = 1 - \frac{\alpha}{\gamma (\gamma + 1)} - \frac{\alpha_0 \gamma_0}{(\gamma + 1)^2} \]

(5)
In order to calculate the energy of the Varma mode we will have to retain terms of the order $\sigma \gamma_0$ while solving for roots from equation (2) (In obtaining equation (3) they were neglected). Thus including the first order correction in $\sigma \gamma_0$ to the density independent root we have

$$\gamma = \sqrt{1 - \sigma \gamma_0}$$

(6)

Using this value of $\gamma$ in equation (5), we have the energy of Varma mode as

$$\omega = \varepsilon_o \frac{3q_i}{|\sigma \gamma_0|^2} \frac{<E'^2>}{2}$$

(7)

Now as $q > 0$, $\omega$ is $> 0$. Hence the Varma mode is always a positive energy mode.

4. Model for Ion Cyclotron Oscillations:

In this section we will try to model the linear properties of ion-cyclotron oscillations observed in the experiment. In the mechanism of excitation of these oscillations electron Landau damping plays a very important role (12). In mirror machines, because of Loss Cone in velocity space a mirror confined plasma develops a positive ambipolar potential $\Phi_M$. In this case the electrons are retained electrostatically having a truncated Maxwellian distribution extending up to an energy corresponding to the positive potential $\Phi_M$. The anisotropic distribution tries to drive the ion-Berstein mode with finite $\kappa_{||}$ unstable. But in situations involving freely streaming electrons, strong Landau damping occurs when $\omega/|\kappa_{||}|$ matches the electron thermal velocity. The frequency of the $n$th
harmonic is given by
\[ \omega = \alpha - 2\beta = \mu \rho \frac{k_n}{k_z} \left( \frac{k_n}{k_z} \right) \]

Hence the resonance condition for Landau damping requires
\[ -i \mu^2 \omega = -i \mu^2 \left( \frac{\omega^2}{k_z^2} \right) \]

when the electrons are held in by an electrostatic potential, as in the present experiment, then in the central region of the plasma, there exists a range of possible electron energies given by
\[ \frac{1}{2} m v_{\text{th}}^2 \leq \epsilon \phi_{\text{m}} \]

From these considerations it is clear that threshold for the instability is the point when the density becomes so that
\[ \frac{1}{2} m \left( \frac{\omega^2}{k_z^2} \right) > \epsilon \phi_{\text{m}} \]

\[ \frac{\omega^2}{k_z^2} > \left( \frac{k_n}{k_z} \right)^2 \frac{\epsilon \phi_{\text{m}}}{T_i} \]

where \( a_1 \) is the ion gyro radius and \( T_i \) is the ion temperature. This equation suggests that with increase in the electron temperature and the concomittant increase in \( \phi_{\text{m}} \) (\( \approx 3 T_e \)), the density threshold should increase linearly. This agrees well with the observed increase of \( \frac{\omega^2}{\omega_i^2} \) with \( \epsilon \phi_{\text{m}}/v_{\text{th}} \) in Phoenix II, Base-Ball I & II etc. (7,12).

To find the energy characteristics of these oscillations we write the general dispersion relation for electrostatic mode as (15)
\[ \xi = 1 - \frac{2 \pi \omega_{pe}^2 k_{||}^2}{k^2 \omega^2} \left( \sum_n \frac{J_n^2}{\theta} \left[ k_{||} \nu_{||} \frac{\partial \nu_{||}}{\partial \nu_{||}^2} + \nu_{\perp}^2 \frac{\partial \nu_{\perp}^2}{\partial \nu_{\perp}^2} \right] \right) \left( k_{||} \nu_{||} - \omega - \nu_{\perp} \Omega \right) \]

\[ - \frac{2 \pi \omega_{pe}^2}{k^2} \left( \sum_n \frac{J_n^2}{\theta} \left[ k_{||} \nu_{||} \frac{\partial \nu_{||}}{\partial \nu_{||}^2} + \nu_{\perp}^2 \frac{\partial \nu_{\perp}^2}{\partial \nu_{\perp}^2} \right] \right) \left( k_{||} \nu_{||} - \omega - \nu_{\perp} \Omega \right) \]

(11)

where \( \int_{-\infty}^{\infty} \) and \( \int_{-\infty}^{\infty} \) are the zero order electron and ion distribution functions respectively. In the plasma formed by neutral beam injection the electrons are generally cold (\( T_e \ll \approx \) few eV) hence we may use cold electron approximation i.e. \( k^2 \nu_{c}^2 \ll 1 \). In the ion term we may neglect the term \( \frac{\partial f_{io}}{\partial \nu_{||}^2} \) as compared to \( \frac{\partial f_{eo}}{\partial \nu_{||}^2} \) for the reason that in plasma formed from neutral beam injection there is a large temperature anisotropy i.e. \( T_{||} \gg T_{\perp} \). Under this approximation equation (11) becomes

\[ \xi = 1 - \frac{\omega_{pe}^2 k_{||}^2}{k^2 \omega^2} - \frac{2 \pi \omega_{pe}^2}{k^2} \left( \sum_n \frac{J_n^2}{\theta} \left[ k_{||} \nu_{||} \frac{\partial \nu_{||}}{\partial \nu_{||}^2} + \nu_{\perp}^2 \frac{\partial \nu_{\perp}^2}{\partial \nu_{\perp}^2} \right] \right) \left( k_{||} \nu_{||} - \omega - \nu_{\perp} \Omega \right) \]

(12)

where \( \omega \approx \omega_{pe} k_{||} \approx \nu_{\perp} \Omega \). It can be shown that for the \( n^{th} \) harmonic only the \( n^{th} \) term in the summation will contribute significantly.

Hence we retain only the \( n^{th} \) term in the summation. The condition for the instability for the \( n^{th} \) mode from the linear theory is
To find out the energy characteristics of the $n^{th}$ harmonic we differentiate $E$ with respect to $\omega$ to get

$$\frac{\partial E}{\partial \omega} = 2 \frac{\omega_{pe}^2 k_{ni}^2}{k^2 \omega^3} - \frac{e_{hi} \omega_{pi}^2}{k^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J_{n} \frac{\partial f_{10}}{\partial V_{1}^{2}} dV_{1}^{2} dV_{1}^{1}$$

(14)

For given $\omega$ and $k$, $\partial E/\partial \omega$ can be evaluated. In order to evaluate the energy of the wave, let us take $\omega$ to be positive so that the energy of the wave is given by sign of $\frac{\partial E}{\partial \omega}$ which can be evaluated for a given value of $\omega$ and K from equation (14). We may evaluate it for some typical values of $\omega$ and K. Accordingly typically for $\omega \approx \omega_{pe} k_{ni}/k - \Omega_{i}$ the first term is $\sim \frac{1}{\Omega_{i}}$, while the lowest bound on second term is $\sim \frac{1}{k^2} \lambda_{ni}^2 \Omega_{i}$. In the experimental observations $k < Q_{i}$ was typically in the range 1 to 1.5. This yields $k^2 \lambda_{ni}^2 \approx 0.4$ to 0.5 $< 1$. Hence the waves observed in the experiment were negative energy waves.

5. Non-linear Stability of the Varca Mode;

In this section we investigate the nonlinear instability of the Varca-mode in the framework of 'weak turbulence theory'. The validity of the application of this theory requires that $W = \frac{\langle E^2 \rangle}{\langle E \rangle^2} / \gamma_{kT} / kT$ be $< 1$ where $\langle E^2 \rangle / \gamma_{kT}$ is the average field energy density and $\gamma_{kT}$
is the thermal energy density of particles. In the experiment (7) the amplitude of the Varms mode ranges from 25 to 500 V in which case for an ion temperature of 1 KeV, \( \omega \approx 1/20 < 1 \). Under this condition the perturbation expansion of \( f \) and \( E \) are made in terms of \( \omega \) as the smallness parameter

\[
\left\{ f \right\} = \left\{ f_0 \right\} + \omega \left\{ f_1 \right\} + \omega^2 \left\{ f_2 \right\} + \cdots
\]

\[
\left\{ E_k \right\} = \omega \left\{ E_k^{(i)} \right\} + \omega^2 \left\{ E_k^{(ii)} \right\} + \cdots
\]

(15)

where we assume that the equilibrium distribution function gives a weak instability. Then using Vlasov-Maxwell's equation once the solution is obtained to a certain order (in this case to the third order), appropriate statistical averages using random phase approximation are performed over spatially uniform ensemble to obtain a set of coupled equation for the spectral energy density. Based on this approach, the comprehensive treatment of the theories of plasma turbulence have been developed by a number of authors (16,17,18,19). Our aim in this Chapter is to obtain in explicit form, the matrix elements for nonlinear Landau damping of the Varms mode and the ion-cyclotron mode. This will be done using Porkolab and Change formalism (20) based on expansions of Vlasov's equation and methods of characteristics. The resonance condition for such an interaction is given by

\[
(\omega - \omega'' - (\kappa_{11} - \kappa_{11}'')) U_{ij} = \lambda \beta \omega^2 i
\]

(16)

where \( (\omega, \kappa') \), \( (\omega'', \kappa'') \) are the frequency and the wave
number of ion cyclotron and the Varma mode respectively. It should be noted that selection rules of non-resonant interaction like non-linear Landau damping are more easily satisfied than those for resonant interaction and hence they are the dominant processes in the development of turbulence. In fact it can be shown that in the matrix element of non-linear Landau damping, there are two terms of opposite sign. One of the terms represents space charge effects (i.e. scattering from dressed particle) while the other term represents the ponderomotive force due to the beat wave (four wave scatter from the bare particle). Normally, these terms cancel making this interaction ineffective i.e. terms of order $O(k^2 \lambda_D^{-2})$ cancel (the first surviving term is $O(k^4 \lambda_D^{-1} \omega_i \omega_0)$) except for short wave-length modes in the absence of magnetic field or for waves travelling perpendicular to the magnetic field. The present case belongs to the later type.

We start from the Vlasov-Maxwell's equations which for electrostatic waves are

$$\frac{\partial f_k}{\partial t} + \nabla \cdot \frac{\partial f_k}{\partial \vec{x}} + \frac{q}{m} \left( \nabla \times \vec{E} \right)_k + \frac{q}{m} \frac{\partial f_k}{\partial \vec{v}} + \frac{q}{m} \vec{E}_k \cdot \frac{\partial f_k}{\partial \vec{v}} = -\frac{q}{m} \sum_{\kappa \neq \kappa} \vec{E}_{\kappa - \kappa} \cdot \frac{\partial f_k}{\partial \vec{v}}$$

$$- (17)$$

$$i \vec{k} \cdot \vec{E}_k = A \pi \sum_{i,e} q_i \int d^3 \psi \frac{\delta f_k}{\delta \psi}$$

$$- (18)$$

$$\vec{E}_k = - \nabla \phi_k = -i \vec{k} \phi_k$$

$$- (19)$$
The notations are standard. Here all quantities are expanded in Fourier series namely

\[ \widetilde{F}(\vec{x}, t) = \sum_k \widetilde{E}_k \, e^{i(k \cdot \vec{x} - \omega_k t)} \]  

(20)

where the \( \widetilde{f}_k \) and \( \widetilde{E}_k \) are the Fourier components of the distribution function \( f \) and electric field \( \mathcal{E} \) defined through

\[ f(x, t) = \langle \rangle + \sum_{k' \neq 0} \int_{k' \neq 0} \, e^{i(k' \cdot \vec{x} - \omega_{k'} t)} \]

(21)

\( K = 0 \) term has been treated separately according to the usual quasi-linear treatment. Thus \( \int_{k=0} f(x, t) = \langle \rangle \) varies slowly in time. In this work we are interested in mode coupling term on the right hand side of equation (17). In the following we shall make use of the reality condition

\[ \omega_{-k} = -\omega_k, \quad \psi_{-k} = \psi_k^* \]

As stated before, we expand \( \psi_k \) and \( f_k \) in a perturbation expansion. Since we are mainly interested in nonlinear Landau damping we will need perturbation solution valid up to fourth order in \( \psi_k \). The iterative solution of equation (17) to equation (19) can then be written to first, second and third order in \( \psi_k \) as follows.

In first order we have:

\[ \mathcal{E}(\omega, \vec{k}) \, \psi^{(1)}_k (t) = 0 \]

(22)

where \( \mathcal{E}(\omega, \vec{k}) \) is the quasi-linear dielectric function given later.

To second order we have:
To third order:

\[
\mathcal{E}(\omega, \mathbf{k}) \Phi^{(3)}_k(t) = - \sum_j \sum_{k'} \frac{4\pi q^2 n_j}{k^2 m_j^2} \int d^3v \int_0^t dt \int_0^t dt' \int_0^t dt'' \int_0^t dt'''
\]

\[
\phi^{(2)}_{k-k'}(t) \phi^{(2)}_{k-k'}(t') \phi^{(2)}_{k-k'}(t'')
\]

\[
+ \frac{\partial}{\partial \sqrt{V(t)}} G_{k-k'}(t) \left( \int d\mathbf{t} \right) \frac{\partial}{\partial \sqrt{V(t)}} G_{k-k'}(t') \left( \int d\mathbf{t} \right) \frac{\partial}{\partial \sqrt{V(t)}} G_{k-k'}(t'') \left( \int d\mathbf{t} \right)
\]

\[
\Phi^{(3)}_{k-k'} \Phi^{(3)}_{k-k'} \Phi^{(3)}_{k-k'}
\]

\[
\left( \int d\mathbf{t} \right) \frac{\partial}{\partial \sqrt{V(t)}} G_{k-k'}(t) \left( \int d\mathbf{t} \right) \frac{\partial}{\partial \sqrt{V(t)}} G_{k-k'}(t') \left( \int d\mathbf{t} \right) \frac{\partial}{\partial \sqrt{V(t)}} G_{k-k'}(t'') \left( \int d\mathbf{t} \right)
\]

\[
\Phi^{(3)}_{k-k'} \Phi^{(3)}_{k-k'} \Phi^{(3)}_{k-k'}
\]
where $G_k(t)$ is the Green's function for the propagation along the exact orbits, $M$ is the particle mass, $n_0$ is the equilibrium density. In the weak turbulence approximation, we have

$$G_k(t) = \mathcal{E}$$

(25)

where $\mathcal{X}'$ designates the unperturbed particle orbits. We recall that in equation (24) there are two types of terms, namely those associated with the resonant mode-mode coupling (i.e. $\mathcal{E}(\omega, \kappa) = 0$) and those due to non-resonant wave-wave scattering (i.e. $\mathcal{E}(\omega, \kappa) \neq 0$). The latter also are also called 'virtual waves' or 'quasi-modes'. To obtain a wave-kinetic equation valid up to fourth order in $C_{\kappa}$, we substitute the appropriate expression for virtual waves $\Phi^{(2)}_{\kappa}$ and $\Phi^{(2)}_{\kappa-k'}$ from equation (23) in equation (24). The last term of equation (24) is the so-called four-wave-scattering term. This term also contributes to the non-linear Landau damping through the resonance condition mentioned before. In the random phase-approximation, equation (23) and equation (24) contribute to the same order in $\Phi_{\kappa}$ in the wave-kinetic equation which is constructed from the wave equation

$$\mathcal{E}(\omega, \kappa) \Phi_{\kappa}(t) = \mathcal{E}(\omega, \kappa) \left[ \Phi_{\kappa}^{(2)}(t) + \Phi_{\kappa-k'}^{(2)}(t) \right]$$

(26)

Utilizing the random phase approximation expanding $\mathcal{E}(\omega, \kappa)$ in the usual manner and transforming to time-co-ordinates, equation (26) becomes (Appendix E)
\[
\frac{\partial \Phi_{ke}(t)}{\partial \omega} = \gamma_k \frac{\partial \zeta(\omega, k)}{\partial \omega} \Phi_{ke} + \sum_{k'} \left\{ \text{A}_{kk', k'} \Phi_{ke} \Phi_{k'c} + \text{B}_{kk', k'} \Phi_k \Phi_{k'}^{\dagger} \right\}^2 + \text{C}_{kk', k'} \Phi_k \Phi_{k'}^{\dagger} \right\}^2 + \text{D}_{kk', k'} \Phi_k \Phi_{k'}^{\dagger} \right\}
\]

(27)

where \( \gamma_k \) is the quasi-linear growth rate, \( A \) is the resonant mode coupling coefficient given by the right hand side of equation (23) and the terms \( B, C \) and \( D \) are the contribution from equation (24).

We are here interested in obtaining compact expression for the latter terms. Using previous equations, we obtain the following expressions:

\[
\text{B}_{kk', k'} = -\sum \frac{16 \pi^2 \rho \zeta_n}{M_k^2} \frac{1}{(k - k')^2} \left[ \omega - \omega', \vec{r} - \vec{r}' \right] \\
\times \left\{ \int d^2 \varphi \int d \xi \int d \xi' \frac{G_{k-k'}(\xi)}{G_{k-k'}(\xi')} \left[ \vec{k} \cdot \frac{\partial}{\partial \xi} G_{k-k'}(\xi') \right] + \vec{k'} \cdot \frac{\partial}{\partial \xi} G_{k-k'}(\xi') \right\}
\]

(28)
\[ C_{k', k-k'} = - \sum_{j} \frac{16 \pi^2 q^6 \eta_0^2}{m_j^4 \ell^2 k_{j}^2 k_{j}^2} G_{k', k} \times \left[ \left( \frac{\partial}{\partial V(c)} \right) G_k(k') + \frac{\partial}{\partial V(c')} G_{-k-k'}(k') \cdot \frac{\partial}{\partial V(c')} \right] \]

\[ \times \left\{ \int d^3r \int d^3r' \int d^3r'' \right. \left[ G_k(k') G_{-k-k'}(k') \cdot \frac{\partial}{\partial V(c')} \right] \right. \]

\[ \left\{ G_k(k') \cdot \frac{\partial}{\partial V(c')} G_{-k-k'}(k') \cdot \frac{\partial}{\partial V(c')} \right\} \]
where we normalized so that \[ \int d^3u \, g(u) = 1 \] or
\[ \int_{-\infty}^{\infty} d\alpha\int_{-\infty}^{\infty} d\beta \, f_\alpha (\beta) = 1 \]
(where \( \mathcal{F} \equiv f_\alpha \)). In order to perform the indicated integrations, we use the cylindrical coordinates in velocity space with the magnetic field in Z-direction. In particular, we have \( \mathbf{B} = [0, 0, B] \),
\[ \mathbf{K} = [k_z, k_\theta, k_\gamma], \quad \mathbf{K}' = [k'_z, k'_\theta, k'_\gamma] \]
and we put \( k - k' = k'' \)
\[ \omega - \omega' = \omega'' \]. For orbits we have
\[ \mathbf{V}(\tau) = \left\{ u_\perp \cos \left[ -\Omega \tau + \Theta(\tau) \right], u_\parallel \sin \left[ -\Omega \tau + \Theta(\tau) \right], u_{11} \right\} \]
\[ y' = y + \frac{u_\parallel}{\omega} \left[ \cos \left( -\Omega \tau + \Theta \right) - \cos \Theta \right] \]
\[ z' = \left[ z - u_{11} \tau \right] \]
\[ (31) \]
and hence
\[ \mathbf{V}(\tau) \frac{\partial}{\partial V(\tau)} = k_z \sin \left( -\Omega \tau + \Theta \right) \frac{\partial}{\partial V_\perp} + \frac{k_\parallel \cos \left( -\Omega \tau + \Theta \right)}{V_\perp} \frac{\partial}{\partial \Theta} \]
\[ + k_{11} \frac{\partial}{\partial V_{11}} \]
\[ (32) \]
where \( \Omega = qB/mc \) is the cyclotron frequency. For simplicity we assume co-linear propagation in the direction perpendicular to \( \mathbf{B} \). Note that in the foregoing \( \Theta(\tau) = \Theta \), \( \Theta(\tau') = (\Theta + \Omega \tau) \), \( \Theta(\tau'') = \Theta + \Omega \left( \tau + \tau' \right) \)
and so the integration of \( C_\mathbf{K} \) and \( \mathbf{V} \) over \( \tau, \tau' \) and \( \tau'' \) are inter-related. This is a consequence of the time dependent orbits of particles in the magnetic field. Following the method outlined by Porkolab and Chang...
we can perform these tedious integrations to get explicit expressions for \( B_{k', k''}, C_{k', k''} \) and \( D_{k', k''} \). This is done in Appendix A.

In order to obtain the wave-kinetic equation for the mode \((\omega, k)\) we multiply equation (27) by \( \phi_k^{*} \) and average over the initial phases and obtain

\[
\frac{\partial N_k}{\partial t} = 2 \gamma_k N_k + \frac{2}{k''} \left\{ \frac{C_{k', k''}}{\omega_k} N_{k', k''} + \text{Re} \left[ \omega_k \phi_{k'} \phi_k^{*} \right] \right\} \tag{33}
\]

In equation (33) the resonant mode-mode coupling contribution has been dropped as this process is not important here. \( L_{k', k''} \) is the coupling co-efficient given by

\[
L_{k', k''} = \frac{i}{8\pi} \frac{1}{\omega_k^{*}} \left| \frac{\partial E}{\partial \omega} \right| \left| \frac{\partial E''}{\partial \omega''} \right| \left| k'' \right|^2 \text{Im} \left[ B_{k', k''} + C_{k', k''} + D_{k', k''} \right] \tag{34}
\]

where \( N_k = \frac{k''}{8\pi} \left| \phi_k \right|^2 \frac{\partial E}{\partial \omega} \) and \( S_k = g_{km} \left[ \frac{\partial E}{\partial \omega} \right] \)

It is understood that \( \left| \frac{\partial E}{\partial \omega} \right| = \frac{\partial E}{\partial \omega} \left| \omega_k \right| \).

Using expressions for \( B_{k', k''}, C_{k', k''} \) and \( D_{k', k''} \) from the Appendix A we can write \( L_{k', k''} \) in the useful form as:
\[ L_{k,k''}^{(m)} = \sum_j \frac{4\pi \omega_p^2 m \gamma_j}{|\omega_j|^2 k^2 k''^2 m_j m_{j'}} \int_{-\infty}^\infty \int_{-\infty}^\infty \] 

\[ \frac{2F_0}{\omega_{k''}} \delta \left( \omega - m \Omega_j - k'' \omega_j \right) x \left| \sum_j \frac{K_{k} K_{k''} J_{p}(x) J_{p-m}(x'')}{(\omega - \omega_j - k'' \omega_j)^2 - \Omega_j^2} \right|^2 \]

where \( H_{k,k'',k'} \) and \( E'(\omega',\omega'') \) are given by the following equations:

\[ H_{k,k'',k'} = \sum_l \sum_s \int_{-\infty}^\infty \int_{-\infty}^\infty \left[ \frac{\omega - \omega_j}{\omega'' - (s-l) \Omega_j - k'' \omega_j} + \frac{1 - \omega_j}{\omega'' - (s-l) \Omega_j - k'' \omega_j} \right] \]

where

\[ \gamma_{m,p}(\omega) = J_m(k^p \omega/\Omega) J_{p-m}(k''^p \omega/\Omega) \]

and

\[ E(\omega,k) = 1 + \sum_j \frac{\omega_p^2}{k^2} \sum_{m=-\infty}^{\infty} \int_{-\infty}^\infty \int_{-\infty}^\infty \left[ \frac{\omega - \omega_j}{\omega - m \Omega_j - k'' \omega_j} \right] \]

\[ \times \left[ \frac{m \Omega_j}{k} \frac{\partial F_0}{\partial \omega_j} + k'' \frac{\partial F_0}{\partial \omega_j} \right] \]

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(35)

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(36)

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(37)
As explained in Appendix B it should be noted that we shall be interested mainly in the imaginary term associated with the resonance condition (16) and hence the poles associated with \( \omega \) and \( \omega' \) will be neglected. Thus in the foregoing expressions we split the integrals with \( \omega \) poles in the form:

\[
(\omega' - m\omega_j - k_{i1} - \omega_{i1})^{-1} = P(\omega' - m\omega_j - k_{i1} - \omega_{i1})^{-1} - i\pi \delta(\omega' - m\omega_j - k_{i1} - \omega_{i1})
\]

where \( P \) denotes the principle value part.

Finally we call attention to the importance of symmetry relation which will be needed shortly

\[
L_{k', k}^{(-mn)} = -L_{k', k''}^{(mn)}
\]  

(38)

Thus relation can be easily obtained from equation (35). Thus using equation (33) we can write the kinetic wave equation for \( (\omega_k', k) \) mode as

\[
\frac{\partial N_{k'}}{\partial t} = \gamma_k N_{k'} + \sum_{k''} \sum_m S_{k''} L_{k', k''}^{(mn)} N_{k''} N_{k'}
\]

(39)

where \( L_{k', k''} \) is given by equation (35).

Proceeding in a similar manner and using equation (38), we can derive a kinetic wave equation for \( (\omega_k'', k'') \) modes as

\[
\frac{\partial N_{k''}}{\partial t} = -\gamma_{k''} N_{k''} - \sum_k \sum_m S_{k''} L_{k, k''}^{(mn)} N_k N_{k''}
\]

(40)

Now we identify \( (\omega_k', k) \) modes with ion-cyclotron oscillations described in Section 5 and \( (\omega_k'', k'') \) modes with the Varma mode described...
In Section 3. From equation (35) it is easy to see that the matrix element $L_{k, k''}^{(m)}$ depends upon $\mathcal{K}_{k} = (\mathcal{K}_{k} - \mathcal{K}_{k''})$ which remains finite as long as one of the modes has finite $|\mathcal{K}_{k}|$. Hence even if we assume a perfect Varma flute mode, the nonlinear instability will not be significantly affected. However, a finite $|\mathcal{K}_{k}|$ in Varma mode may give rise to damping due to electrons which is given by the first term on the right-hand side of equation (40).

To proceed further let us know find out the sign of matrix element $L_{k, k''}^{(m)}$. Accordingly we have

$$L_{k, k''}^{(m)} = \frac{4\pi \omega_{bi}}{2\Omega_1} \frac{\partial^2}{\partial w''} \int_0^\infty \frac{k_i k'' M_{11} \eta_0}{k^2} \exp \left( - \frac{\omega - \omega_{2i} - k_i \omega_0}{k^2} \right)$$

$$\times \left( \sum_{\substack{p \neq 0 \neq x}} \frac{k_p \mathcal{K}_k \mathcal{K}_k''}{\mathcal{K}_p (\omega) \mathcal{K}_p' (\omega') \mathcal{K}_k' (\omega'')} \right)$$

$$\times \exp \left( - \frac{\omega - \omega_{2i} - k_i \omega_0}{k^2} \right)$$

$$- \frac{\omega_{2i}^2}{k^2} \left( \frac{\mathcal{K}_p (\omega') \mathcal{K}_k' (\omega'')}{\mathcal{K}_k' (\omega)} \right)^2$$

$$= - \cdots (41)$$

It should be noted that in equation (41) summation over species and $m$ has already been performed. Since in the experiment under discussion

$$\omega = \omega_{2i} \left[ -10 \text{ MHz \text{ range}} \right], \quad \omega'' = \omega_{11} \text{ MHz范围}$$

and modes are nearly flute (i.e., $\mathcal{K}_{k} = \omega$).
\( \omega' \approx \Omega_1 \), in which case because of \( S \)-function the only finite term would be \( m = 1 \) in the ion species. Now as \((\omega', \Omega')\) is a damped mode (quasi-mode) of the system \( C'(\omega', \Omega') > 0 \) in which case we may neglect the second term as compared to the first under the integral sign in equation (41). Hence equation (41) becomes

\[
L_{k_2 k''}^{(1)} = \frac{4 \pi (\omega')^4 \Omega_i}{\left| \frac{\partial \epsilon}{\partial (\omega')} \right| \left| \frac{\partial \epsilon''}{\partial (\omega'')} \right|} \int_0^\infty dU_{11} \delta \left( \omega' - \omega_{11} \Omega_{11} - \Omega_{1} \right) \times \left\{ \frac{k_2 k'' \int_{p-1} \left( \omega'' \right)}{(\omega - \omega_{11} \Omega_{11} - \Omega_{1})^2} \right\}^2
\]

(42)

In summation \( p = 2 \) will have dominant contribution : \( \omega_{11} = \Omega_1 \) and

\( k_{11} = 0 \). For \( J_2 (x''') \) we may use the Bessel's identity in which case the matrix element \( L_{k_2 k''}^{(1)} \) finally becomes

\[
L_{k_2 k''}^{(1)} = \frac{4 \pi (\omega')^4 \Omega_i}{\left| \frac{\partial \epsilon}{\partial (\omega')} \right| \left| \frac{\partial \epsilon''}{\partial (\omega'')} \right|} \int_0^\infty dU_{11} c \frac{4 J_1 (x) J_1 (x''') - J_3 (x) J_3 (x''')}{x}
\]

(43)

where \( x = k_{2} U_{11} / \Omega_{11} \); \( x''' = k_{2}'' U_{11} / \Omega_{1} \);
In equation (43), it is to see that as $\mathcal{J}_0$ and $\mathcal{J}_1$ are out of phase by nearly 90° the $\mathcal{J}_0 \mathcal{J}_1$ will give negligible contribution to the $d\mathcal{V}_\perp$ integral. While for distribution appropriate to plasma formed by neutral beam injection in vacuum, i.e.

$$\mathcal{F}_{01}(U) = \frac{1}{2} \frac{\mathcal{F}_0(U_\perp)}{U_{10}} e^{-\frac{U_{10}^2}{\mathcal{F}_0(U_\perp)}}$$

(44)

the term $\chi_1^2 = \frac{k^2 U_\perp^2}{\Omega_i^2}$ tends to weight the positive (rising) portion of $\mathcal{F}_0(U_\perp)$ over the negative (falling) portion of $\mathcal{F}_0(U_\perp)$. Hence the integral in equation (41) is positive and hence $\mathcal{L}_{k''}^\prime$ is $> 0$ for the case under consideration. In Section 4 we have already shown that for the observed ion-cyclotron mode $\partial \mathcal{E}/\partial \Omega < 0$ for positive $\omega$. Thus $\xi_{k''} < 0$ in equation (39). Let us now find out $\xi_{k''}$. We recall that $\xi_{k''}$ represents the sign of $\partial \mathcal{E}/\partial \mathcal{E}_{k''}$ at $\gamma = \omega/\gamma_{m_i} = -1$. Then

$$\frac{\partial \mathcal{E}}{\partial \omega''} = \left(\frac{\partial \mathcal{E}}{\partial \gamma}\right) \frac{1}{\gamma_{m_i}}$$

(45)

From equation (5) we have

$$\frac{\partial \mathcal{E}}{\partial \gamma} = -\frac{3\gamma}{|\sigma \gamma_c|^2}$$

(46)

Then for positive $m$, $\partial \mathcal{E}/\partial \omega''$ bears a negative sign. Hence $\xi_{k''}$ in equation (40) is $< 0$. Hence the wave-kinetic equation (39) and equation (40) become

$$\frac{\partial N_k}{\partial t} = \left[ \gamma_k - \sum_{k'} l_{kk'} N_{k''} \right] N_k$$

(47)
Plot of energy densities of Varma mode (N₁) and ion-cyclotron mode (Nₖ) with time.

Fig. 1
Thus coupling gives rise to the following very interesting situation: As the beam injection current is increased beyond a certain critical density the electron Landau damping vanishes and the ion-cyclotron oscillations are linearly excited as represented by the first term in equation (46). Thus \( N_{\kappa} \) grows and when \( N_{\kappa} L_{\kappa}^{(1)} \) becomes larger than \( \gamma_{\kappa} \), then Varma mode is nonlinearly excited. Hence \( N_{\kappa} \) grows and when \( N_{\kappa} L_{\kappa}^{(1)} \) becomes larger than \( \gamma_{\kappa} \) the ion-cyclotron oscillations are nonlinearly damped, and hence \( N_{\kappa} \) starts decreasing and when \( \gamma_{\kappa} > L_{\kappa}^{(1)} N_{\kappa} \) Varma mode also becomes damped and starts decreasing. Thus we see that because of the coupling the energy of Varma mode and ion-cyclotron mode bursts periodically (Fig.1).

This is quite similar to the periodic bursts of Varma-mode and ion-cyclotron mode observed in Base-Ball II. We will now estimate the nonlinear time-scale of the bursts and compare it with the experimental value. Here we consider the time between bursts to be a meaningful measure of the time for the wave amplitude to grow from a very small to a very large level. In the model this time is given by the nonlinear growth time which is the inverse of the nonlinear growth-rate \( \gamma_{N\kappa} \). From equation (42) and equation (48) \( \gamma_{N\kappa} \) for Varma mode is given by

\[
\frac{\partial N_{\kappa}}{\partial t} = \left[ -\gamma_{\kappa} + \sum_{l} L_{\kappa l}^{(1)} N_{l} \right] N_{\kappa} \tag{48}
\]
\[ \gamma_{N_L}^{+} = \frac{4\pi W_{B_1}}{|\omega|} \sum \frac{\omega_i N_i}{k^{2} k_i^{2}} \]
From equation (49), we may write \( \gamma_{k''NL} \) as

\[
\gamma_{k''NL} = \frac{4\pi \frac{W_{pi}}{\Omega_2 i}}{\frac{2E''}{\gamma \omega''}} \times \frac{1}{\Omega_2 i} W
\]  

(51)

where \( W = \sum_k N_k \Omega_2 i / n_0 \gamma m_i U_{10} \) i.e. the ratio of energy in waves to that in particles. For Basic-Ball II parameters i.e. \( n_p = 4 \times 10^9 \text{ cm}^{-3} \) (\( n_p \) is the plasma density), \( T_i = 2 \text{ KeV} \) (\( T_i \) is the ion temperature), \( \omega_{pi} = 10^7 \text{ Hz} \), \( U_{H-1} = 3.09 \times 10^7 \text{ cm/sec} \), \( \sigma = 1/50 \text{ cm}^{-1} \), \( \sigma' = 1/5 \text{ cm}^{-1} \), \( \Omega_2 i = 10^7 \text{ Hz} \), we have \( \nu_{D1} = 2 \times 10^6 \text{ cm/sec} \). To our knowledge there is no experimental data on the wave number of low frequency fields. Typically for azimuthal mode number equal to 1 or 2 we expect \( k'' \approx 1 \). For these parameters

\[
\frac{2E''}{\gamma \omega''} = \frac{1.25 \times 10^2}{\Omega_2 i}
\]

(52)

It should be noted that energy of ion-cyclotron oscillations \( W \) also fluctuates. Hence to get the order of time scale of nonlinear interaction we may choose a modest average value of \( W = \sqrt{m_e/m_i} \) under the weak-turbulence approximation. Substituting for various terms on the right hand side of equation (51), we get the order of nonlinear interaction time scale of Varma mode as
For $\Omega_i = 10^7$ Hz, $\omega_{k'' NL}$ turns out to be $\approx 40 \mu s$ which agrees fairly well with the observed bursting time of 50-100 $\mu$s for Varma mode.

This periodic bursting of these oscillations periodically increases the scattering of particles into the loss cone. From the loss cone particles are lost thereby giving rise to bursts of particles.

7. Discussion and Conclusions:

The model developed in the preceding pages we have shown that the stability of the Varma mode is affected in the presence of the ion-cyclotron modes observed in the experiment. It no longer remains stable but exhibits a periodic bursting instability on the time scale of $\approx 400 \Omega_i^{-1}$ ($\approx 40 \mu s$). Hence the identification made previously by O. \( \text{O} \) of a small amplitude in KHz frequency range with the Varma mode may not be correct. Rather the large amplitude wave exhibiting a bursting instability on the time scale of 500-100 $\mu$s is more likely to be the Varma mode. The small amplitude mode may be one of the other two flute modes whose coupling efficiency to ion-cyclotron fluctuations on account of its low fluctuation level may be small (i.e. the matrix element $\left| \langle \mu \downarrow \mid k, j \rangle \right|^2$ of this coupling may be quite small as compared with the matrix element for coupling between Varma mode and the ion-cyclotron mode. It should be noted that the nonlinear mechanism in the present experiment i.e. BASE-BALL II and the previous ones like 'Alice' or 'Phoenix' etc. are slightly different. As has been pointed out by Simon and
Weng, in 'Alice' and 'Phoenix' explosive instability of the Varna mode is responsible for the observations. While in the present case it is the periodic bursting instability responsible for the anomalous losses.
By substituting the orbits given by equation (31) and equation (32) into equation (28) and equation (29), the time integration can be performed to give the following compact expression for $[B_{k''} + C_{k'''}]$:

$$B_{k,k''} + C_{k,k''} = \sum \frac{\omega_{p0} H_{k,k''}^{'}, H_{k',k''}, k^2 k''^2 \in (\omega', \omega)}{4\pi \eta_0 M_0 k^2 k''^2}$$

where

$$H_{k,k''} = \sum \sum_{\ell} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{y}_{\ell, \ell} (U_{1})$$

$$\times \left[ k_{1} \left\{ k_{1} Z_{s, \ell} (k') + k_{1}'' Z_{s, \ell} (k'') \right\} \right]$$

$$+ k_{1} \left\{ k_{1} Z_{s, \ell} (k') + k_{1}'' Z_{s, \ell} (k'') \right\}$$

and
\[ H_{k', k, -k''} = -\sum_{\eta} \sum_{\pi'} \int_{-\infty}^{\infty} d\eta_{\pi} \int_{-\infty}^{\infty} d\eta_{\pi'} \tilde{\mathcal{Y}}_{\eta, \pi'}(\eta_{\pi}) \]

\[ \times \left\{ \frac{k_{i}^{\prime} \left[ k_{i}^{\prime\prime} Z_{\eta} (k) - k_{i} Z_{\eta - n} (k') \right]}{(\omega - \eta - \Omega_{j} - n_{\eta} \Omega_{j})^{2} - \Omega_{j}^{2}} \right\} \]

\[ + \frac{k_{i}^{\prime} \left[ k_{i}^{\prime\prime} Z_{\eta} (k) - k_{i} Z_{\eta - n} (k') \right]}{(\omega - \eta - \Omega_{j} - n_{\eta} \Omega_{j})^{2} - \Omega_{j}^{2}} \]

\[ - \left( \omega - \eta - \Omega_{j} - n_{\eta} \Omega_{j} \right) \right\} \]

where \( s, l, n, \) and \( p \) are all possible integers, \( \Gamma \) and \( \omega \), and

\[ (\omega_{\eta})^{2} = \frac{n_{\eta}^{2} \Omega_{j}^{2} / n_{\eta} \Omega_{j}}. \]

Here we have defined

\[ \tilde{\mathcal{Y}}_{\eta, \pi'}(\eta_{\pi}) = J_{\eta}(k_{i}^{\prime} \eta_{\pi}) J_{\eta - n}(k_{i}^{\prime\prime} \eta_{\pi} \Omega_{j}) J_{\eta}(k_{i} \eta_{\pi} \Omega_{j}) \]

\[ Z_{\eta}(k) = \frac{\eta \Omega_{j}^{2} F_{\eta} + k_{i} \Omega_{j} F_{\eta}}{(\omega - \eta - \Omega_{j} - k_{i} \eta \Omega_{j})} \]

\[ Z_{\eta}(k) = Z_{\eta} (-k \rightarrow k', \omega \rightarrow \omega') \]

\[ (4) \]

\[ F_{\pi}(\eta) = F_{\eta}(\eta_{\pi}) \left( \pi_{\eta} \right) \]

\[ (5) \]

and
\[ E(\omega, \kappa) = 1 + \sum_{j} \frac{(\omega_{j} \lambda)^{2}}{k^{2}} \sum_{\eta} \int_{0}^{\infty} \left( \frac{d\tau_{\eta}}{d\tau_{1}} \right) \frac{\overline{F_{\eta \mu}}}{d\tau_{1}} + \kappa_{\eta} \frac{\overline{F_{\eta \mu}}}{d\tau_{1}} \right] 
\]

\[ \times \frac{J_{n}^{\frac{2}{3}} \left( \frac{\kappa_{1} \nu_{j}}{\omega_{j} \omega_{j} \omega_{j}} \right)}{(\omega - \eta \omega_{j} - \kappa_{\eta} \omega_{\eta})} \left[ \frac{\eta^{\frac{2}{3}} \nu_{j}}{\nu_{j}} + \kappa_{\eta} \frac{\overline{F_{\eta \mu}}}{d\tau_{1}} \right] \]

And of course as \((\omega', \kappa)\) is a quasi-mode \( E'(\omega', \kappa) \neq 0 \).

Equation (1) gives the contribution to the scattering from the shielding cloud which is characteristic of plasma. It is also called the nonlinear scattering term. As a remark in passing we note that resonant mode coupling coefficient in equation (27) is given by

\[ A_{k, k', k'} = - \sum_{j} \frac{\omega_{j}^{2}}{k_{j}^{2} \tau_{j}} \left[ H_{k, k', k'} \right] \]

The derivation of four-wave coupling coefficient is as follows: It may be written in the following form:

\[ D_{k, k''} = \sum_{j} \frac{\omega_{j}^{4}}{4 \pi \eta_{j} \eta_{j} \eta_{j} \eta_{j} k_{j}^{2}} \]

where

\[ T_{k, k'} = \sum_{\eta} \sum_{b} \sum_{c} \int_{0}^{\infty} \left( \frac{d\tau_{\eta}}{d\tau_{1}} \right) \int_{0}^{\infty} \left( \frac{d\tau_{b}}{d\tau_{1}} \right) \int_{0}^{\infty} \left( \frac{d\tau_{c}}{d\tau_{1}} \right) \int_{0}^{\infty} \left( \frac{d\tau_{c}}{d\tau_{1}} \right) \left( \frac{\kappa_{1} \nu_{j}}{\omega_{j}} \right) \left( \frac{\nu_{j}}{\omega_{j}} \right) \left( \frac{\nu_{j}}{\omega_{j}} \right) \left( \frac{\nu_{j}}{\omega_{j}} \right) \]

Here \( n, b, \) and \( c \) are all integers in the range \((-\infty, \infty)\) and

\[ W(\nu_{j}) = J_{n} \left( \frac{\kappa_{1} \nu_{j}}{\omega_{j}} \right) J_{b} \left( \frac{\kappa_{1} \nu_{j}}{\omega_{j}} \right) J_{b-c} \left( \frac{\kappa_{1} \nu_{j}}{\omega_{j}} \right) J_{n-c} \left( \frac{\kappa_{1} \nu_{j}}{\omega_{j}} \right) \]
and

\[
\mathbf{D} = \begin{bmatrix}
0 & 0 & 0 \\
0 & \mathbf{D}_{\perp,\perp} & \mathbf{D}_{\perp,11} \\
0 & \mathbf{D}_{11,\perp} & \mathbf{D}_{11,11}
\end{bmatrix}
\]  

(11)

so that

\[
k \cdot \mathbf{D} \cdot \mathbf{k}'' = k_1 k_2'' \mathbf{D}_{\perp,\perp} + k_- k_1'' \mathbf{D}_{\perp,11} + k_{11} k_1'' \mathbf{D}_{11,\perp}
+ k_{11} k_1'' \mathbf{D}_{11,11}
\]  

(12)

The expression for different matrix element can be derived as follows:

Substituting the orbits in equation (30), we can split the integral with three terms so that

\[
\mathbf{D}_{k, k''} = \mathbf{D}_i + \mathbf{D}_{ii} + \mathbf{D}_{iii}
\]  

(13)

where

\[
\mathbf{D}_1 = \int d^3 \mathbf{v} \int d^3 \mathbf{c} \mathbf{D}_a k_2'' \mathbf{D}_b \frac{\partial \mathbf{D}_b}{\partial \mathbf{u}_1} 
\]  

(14)

\[
\mathbf{D}_{ii} = \int d^3 \mathbf{v} \int d^3 \mathbf{c} \mathbf{D}_a k_2'' \cos \theta \mathbf{u}_1 \mathbf{u}_1^{-1} \frac{\partial \mathbf{D}_b}{\partial \theta_1} 
\]  

(15)

\[
\mathbf{D}_{iii} = \int d^3 \mathbf{v} \int d^3 \mathbf{c} \mathbf{D}_a k_2'' \frac{\partial \mathbf{D}_b}{\partial \mathbf{u}_1} 
\]  

(16)
in equation (14) to equation (16), we have

\[ D_0 = i e \chi \left[ i k_1 u_{11} c (\cos \theta' - \cos \theta) + i (\omega - k_{11} u_{11}) \right] \tag{17} \]

and after integrating over \( \mathcal{T}'' \) in equation (30) we obtain

\[
D_h = \frac{i}{2 \pi} \sum_c \sum d \sum m \sum n \left\{ \int_{\mathcal{C}} \mathcal{C} \left( \frac{k_1 u_1}{\alpha} \right) \mathcal{C} \left( \frac{k_1 u_1}{\alpha} \right) \times \right.
\]
\[
\left. \times e^{i \chi} \left[ i (c - d) \left( ' \frac{\pi}{2} - \theta' \right) + i (\omega' - c \omega - k_{11} u_{11}) \right] \right\} \tag{18}
\]
where the expression for \( Z_m(k) \) is given before by equation (3). We can now integrate over \( t' \), substitute back into equation (14) to equation (16) expand the exponential in equation (17) in a double-Bessel function series and integrate over \( \mathcal{C} \). Finally we integrate over the velocity space angle \( \Theta \) integrate by parts over \( U_1 \) repeatedly and obtain

\[
\mathcal{D}_i = - \sum_{j=0}^{\infty} \sum_{b} \sum_{c} \sum_{d} \sum_{m} \sum_{n} \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{k''/2}{(w-b \cdot \alpha - k_{ii} U_{ii})} \times \mathcal{J}_m(k_1 U_{i1}) \mathcal{J}_n(k_2 U_{i2}) \times \left[- Z_m(k) \right] \frac{1}{2} k_{12}'' \times \left[- \frac{2}{U_{i1}} + \frac{(m-n) U_{i2}}{U_{i1}} \right] \sigma_{a} \frac{2}{u_{i2} k_{ii} U_{ii}} \times (s_{p-2} + s_{p}) \right)

\[
+ \frac{k_{ii} \sigma_{a} (s_{p-2} + s_{p})}{u_{i2} k_{ii} U_{ii}} \left[ - Z_m(k) \right] \frac{1}{2} k_{2}'' \times \left[- \frac{2}{U_{i2}} + \frac{(m-n) U_{i1}}{U_{i2}} \right] \sigma_{a} (s_{p} + s_{p+2})

\[
+ \frac{k_{ii} \sigma_{a} (s_{p-1} + s_{p+1})}{u_{i1} k_{ii} U_{ii}} \left[ - Z_m(k) \right] \frac{1}{2} k_{1}'' \times \left[- \frac{2}{U_{i1}} + \frac{(m-n) U_{i1}}{U_{i2}} \right] \sigma_{a} (s_{p} + s_{p+2})

\[
+ \frac{k_{ii} \sigma_{a} (s_{p-1} + s_{p+1})}{u_{i1} k_{ii} U_{ii}} \left[ - Z_m(k) \right] \frac{1}{2} k_{1}'' \times \left[- \frac{2}{U_{i1}} + \frac{(m-n) U_{i1}}{U_{i2}} \right] \sigma_{a} (s_{p} + s_{p+2})

\]

(19)
where \( f = (a-b+c-d+n-m) \) and

\[
\mathcal{F}_n = \sum_{a}^\infty \sum_{b}^\infty \sum_{c}^\infty \sum_{\alpha=1}^\infty \sum_{\beta=1}^\infty \left[ \int_{0}^{\infty} \int_{0}^{\infty} \frac{k_{n}}{\omega_i u_{ii}} \right] \frac{1}{\omega_i} \left( \sum_{\beta=1}^{\infty} \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \sum_{\alpha=1}^{\infty} \frac{k_{n}}{\omega_i u_{ii}} \right)
\]

\[
D_{ij} = \sum_{a}^\infty \sum_{b}^\infty \sum_{c}^\infty \sum_{\alpha=1}^\infty \sum_{\beta=1}^\infty \left[ \int_{0}^{\infty} \int_{0}^{\infty} \frac{k_{n}}{\omega_i u_{ii}} \right] \frac{1}{\omega_i} \left( \sum_{\beta=1}^{\infty} \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \sum_{\alpha=1}^{\infty} \frac{k_{n}}{\omega_i u_{ii}} \right)
\]

\[
\left\{ J_{m} \left( k_{n} u_{ii} \right) J_{m} \left( k_{n} u_{ii} \right) \right\}
\]

\[
\times \left[ \frac{1}{2} Z_m(k) \frac{k''}{\omega_i} \left[ \frac{\omega_i}{\omega_i} + (m-n) \right] \right] \frac{\psi_{b}}{\beta} \left( c-d+n-m+1 \right) \left( - \psi_{p+2} + \psi_{b} \right)
\]

\[
\frac{1}{\omega_i} \left( (c-m+n-1) \omega_i - k_{n} u_{ii} \right)
\]

\[
+ \frac{1}{2} Z_m(k) \frac{k''}{\omega_i} \left[ \frac{\omega_i}{\omega_i} + (m-n) \right] \frac{\psi_{b}}{\beta} \left( c-d+n-m+1 \right) \left( + \psi_{p} - \psi_{p+2} \right)
\]

\[
\frac{1}{\omega_i} \left( (c-m+n-1) \omega_i - k_{n} u_{ii} \right)
\]

\[
+ \frac{1}{2} Z_m(k) \frac{k''}{\omega_i} \left[ \frac{\omega_i}{\omega_i} + (m-n) \right] \frac{\psi_{b}}{\beta} \left( c-d+n-m+1 \right) \left( + \psi_{p} - \psi_{p+2} \right)
\]

\[
\frac{1}{\omega_i} \left( (c-m+n-1) \omega_i - k_{n} u_{ii} \right)
\]

\[
+ \frac{1}{2} Z_m(k) \frac{k''}{\omega_i} \left[ \frac{\omega_i}{\omega_i} + (m-n) \right] \frac{\psi_{b}}{\beta} \left( c-d+n-m+1 \right) \left( + \psi_{p} - \psi_{p+2} \right)
\]

\[
\frac{1}{\omega_i} \left( (c-m+n-1) \omega_i - k_{n} u_{ii} \right)
\]

\[
+ \frac{1}{2} Z_m(k) \frac{k''}{\omega_i} \left[ \frac{\omega_i}{\omega_i} + (m-n) \right] \frac{\psi_{b}}{\beta} \left( c-d+n-m+1 \right) \left( + \psi_{p} - \psi_{p+2} \right)
\]

\[
\frac{1}{\omega_i} \left( (c-m+n-1) \omega_i - k_{n} u_{ii} \right)
\]

\[
+ \frac{1}{2} Z_m(k) \frac{k''}{\omega_i} \left[ \frac{\omega_i}{\omega_i} + (m-n) \right] \frac{\psi_{b}}{\beta} \left( c-d+n-m+1 \right) \left( + \psi_{p} - \psi_{p+2} \right)
\]

\[
\frac{1}{\omega_i} \left( (c-m+n-1) \omega_i - k_{n} u_{ii} \right)
\]

\[
+ \frac{1}{2} Z_m(k) \frac{k''}{\omega_i} \left[ \frac{\omega_i}{\omega_i} + (m-n) \right] \frac{\psi_{b}}{\beta} \left( c-d+n-m+1 \right) \left( + \psi_{p} - \psi_{p+2} \right)
\]

\[
\frac{1}{\omega_i} \left( (c-m+n-1) \omega_i - k_{n} u_{ii} \right)
\]

\[
+ \frac{1}{2} Z_m(k) \frac{k''}{\omega_i} \left[ \frac{\omega_i}{\omega_i} + (m-n) \right] \frac{\psi_{b}}{\beta} \left( c-d+n-m+1 \right) \left( + \psi_{p} - \psi_{p+2} \right)
\]

\[
\frac{1}{\omega_i} \left( (c-m+n-1) \omega_i - k_{n} u_{ii} \right)
\]
Where

\[ D_{\text{iii}} = \sum_{a} \sum_{b} \sum_{c} \sum_{d} \sum_{m} \sum_{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dV_{\text{ll}} dV_{l} k''_{\text{ll}} \]

\[
\left\{ 2n(k) J_{n}(k_{1}l_{1}) J_{n}(k_{1}l_{1}) v \left[ \frac{1}{2} k_{1}'' \left[ \frac{2}{v_{l_{1}}} + \frac{(m-n)}{v_{l_{1}}} \right] \right] \frac{\varphi_{c} S_{b+1}}{\omega' - (c - n + m - 1) \omega - k''_{\text{ll}} V_{\text{ll}}} \right. \\
- \left. \frac{1}{2} k_{1}'' \left[ \frac{2}{v_{l_{1}}} + \frac{(m-n)}{v_{l_{1}}} \right] \right] \frac{\varphi_{c} S_{b+1}}{\omega' - (c - n + m - 1) \omega - k''_{\text{ll}} V_{\text{ll}}} \right] \left[ \frac{2}{v_{l_{1}}} \left( (w - b \omega - k''_{\text{ll}} V_{\text{ll}})^{-1} \right) \right] \\
- \left[ \frac{1}{2} k_{1}'' \left[ \frac{2}{v_{l_{1}}} + \frac{(m-n)}{v_{l_{1}}} \right] \right] \frac{\varphi_{c} S_{b+1}}{\omega' - (c - n + m - 1) \omega - k''_{\text{ll}} V_{\text{ll}}} \right] \left[ \frac{2}{v_{l_{1}}} \left( (w - b \omega - k''_{\text{ll}} V_{\text{ll}})^{-1} \right) \right] \\
+ k''_{\text{ll}} \frac{2}{v_{l_{1}}} \frac{\varphi_{c} S_{b}}{\omega - (c - n + m - 1) \omega - k''_{\text{ll}} V_{\text{ll}}} \left[ \frac{2}{v_{l_{1}}} \left( (w - b \omega - k''_{\text{ll}} V_{\text{ll}})^{-1} \right) \right] \\
\right \}
\]

where \( \varphi_{c} = u_{l} \varphi_{b} \). We further note that in the foregoing expressions the derivative \( \partial / \partial v_{l_{1}} \) and \( \partial / \partial V_{l_{1}} \) operate on all quantities to their right. To proceed further, we add \( D_{i} \) and \( D_{ii} \).
and begin to sum over indices with the help of the kronecker deltas. Then a considerable amount of manipulation with the Bessel function identities is necessary to combine the large numbers of terms. Furthermore integration by parts over $U_{ij}$ is performed wherever convenient. Thus after a considerable amount of tedious algebra, we obtain following expressions for the matrix elements:

$$D_{11} = \left\{ \sum_{\gamma=\pm} \frac{1}{2 \Omega} \gamma k_1 \left[ k_1'' Z_\eta(k) - k_1 Z_{\eta,c}(k'') \right] \right\}$$

$$\left[ (\omega - (b+\gamma) \Omega - k_{11} U_{11})^2 - \Omega^2 \right]\left[ (\omega' - (c+\gamma) \Omega - k_{11} U_{11})^2 - \Omega^2 \right]$$

$$- \left\{ \sum_{\gamma=\pm} \frac{1}{2 \Omega} \gamma k_{11} \left[ k_1'' Z_\eta(k) - k_1 Z_{\eta,c}(k'') \right] \right\}$$

$$\left[ (\omega - b - \Omega - k_{11} U_{11})^2 - \Omega^2 \right]\left[ (\omega' - c - \Omega - k_{11} U_{11})^2 - \Omega^2 \right]$$

$$D_{1,11} = \left\{ \sum_{\gamma=\pm} \frac{1}{2 \Omega} \gamma k_{11} \left[ k_1'' Z_\eta(k) - k_1 Z_{\eta,c}(k'') \right] \right\}$$

$$\left[ (\omega - (b+\gamma) \Omega - k_{11} U_{11})^2 - \Omega^2 \right]\left[ (\omega' - (c+\gamma) \Omega - k_{11} U_{11})^2 - \Omega^2 \right]$$

$$- \frac{k_1''}{k_1} \left\{ \frac{k_1'' Z_\eta(k) - k_1 Z_{\eta,c}(k'')} \left[ (\omega - b - \Omega - k_{11} U_{11})^2 - \Omega^2 \right]\left[ (\omega' - c - \Omega - k_{11} U_{11})^2 - \Omega^2 \right] \right\}$$

$$- - - - (23)$$
Here \( Z_n(k) \) has been defined before. Equation (4) is also called 'Compton scattering term' or scattering from bare particle.

We may now write equation (34) in the following form

\[
\mathcal{D}_{II,11} = -\frac{k_1}{k_{II}} \left[ (w - b \Omega - k_{II} \Omega_{II})^2 - \Omega^2 \right]^{-1} \left[ (w' - c \Omega - k_{II} \Omega_{II})^{-1} \right] \frac{\partial}{\partial \Omega_{II}} \left[ k_{II}^\prime Z_n(k) - k_{II} Z_n-c(k'') \right] 
\]

\[
\mathcal{D}_{II,II} = -1 \left( w - b \Omega - k_{II} \Omega_{II} \right)^{-2} \left( w' - c \Omega - k_{II} \Omega_{II} \right)^{-1} \frac{\partial}{\partial \Omega_{II}} \left[ k_{II}^\prime Z_n(k) - k_{II} Z_n-c(k'') \right] 
\]

\[
L_{k,k''} = \frac{4 \omega_p^4}{|\partial \epsilon / \partial \omega_0| |\partial \epsilon'' / \partial \omega''| k_{II}^\prime k_1^2 k_2^2 m^2 \eta_p^2} \text{Im} \left[ T_{k,k''} + \omega_p^2 \frac{k_{II}^2}{k_{II}^2 - \epsilon'(\omega')} \frac{H_{k,k',k} H_{k',k,-k''}}{\epsilon(\omega', k')} \right] 
\]

where \( \omega' = (\omega - \omega'') \), \( k' = (k - k'') \). From
equation (26) we see that only Im part of the matrix element is required. In particular, we shall be interested mainly in the imaginary terms associated with the resonance condition (16) of the text and hence the poles associated with \( \omega \) and \( \omega'' \) will be neglected. Thus in the foregoing expressions we split the integrals with \( \omega' \) poles in the form

\[
\left( \omega' - \eta \Omega - k' u_{11} \right)^{-1} = P \left( \omega' - \eta \Omega - k'' u_{11} \right)^{-1} - i \pi \delta \left( \omega' - \eta \Omega - k'' u_{11} \right).
\]

(27)

where \( P \) denotes the principle value part. Utilizing this equation, we will now determine \( L_{k',k''} \) for the case under consideration. To do this we make following assumptions:

\[
k_{11} < k'_1, \quad k''_1 < k''_1, \quad \left( k''_1 \left< \frac{\partial F_{11}}{\partial u_{11}} \right> \right) \ll \eta \Omega \left< u_{11} \right> \frac{\partial F_{0}}{\partial u_{11}}.
\]

(28)

Under these approximations we retain \( k_{11} \) only at poles so that we have following relation for the matrix elements H.

\[
H_{k',k'',k'} = \sum_{s} \sum_{l} \int_{0}^{\beta} du_{11} \int_{-\infty}^{\infty} du_{1} \tilde{Y}_{l,s} \frac{\partial F_{0}}{\partial s} \frac{x_{1} k'_{1} k''_{1}}{(w - s - \Omega - k_{11} u_{11})^{2} - g_{2}^{2}}
\]

\[
x \left[ \frac{(s - \Omega) k''_{1}}{w'' - (s - \Omega - \Omega - k_{11} u_{11})} + \frac{\Omega k''_{1}}{w' - \left( \Omega - k_{11} u_{11} \right)} \right] - (29)
\]
After this we write down the following symmetry relations:

\[ H_{k', k''} = H_{k', k'} \]

These relations are proved in Appendix D. Furthermore in Appendix B, we have proved the relation

\[ \text{Im} H_{k, k', k} = \text{Im} H_{k', k, -k'} \]  \hspace{1cm} (31)

Equation (31) and equation (32) allows us to write

\[ H_{k, k', k'} H_{k', k''} = H_{k, k'', k'} \]  \hspace{1cm} (33)

\[ \text{Im} T_{k, k''} \] is obtained in Appendix C. To proceed further, we can easily establish the relationship
\[
\text{Im} \left[ \frac{\omega_p^2}{k_{11}^2} \frac{H_{k,k',k''}}{\varepsilon'(\omega,k')} \right] = \frac{\omega_p^2}{k_{11}^2} \left[ - \frac{H_{k,k',k''}}{\varepsilon'(\omega,k')} \right]^2
\]

\[\times \text{Im} \varepsilon'(\omega,k') + 2 \text{Re} \left[ \frac{H_{k,k',k''}}{\varepsilon'(\omega,k')} \right]^2 \times \text{Im} H_{k,k'',k'} \]

--- (34)

From equation (29), we obtain the imaginary part of \( H_{k,k'',k'}^{(n)} \) as follows:

\[
\text{Im} H_{k,k'',k'}^{(n)} = - \sum_{s} \int_{-\infty}^{0} d\psi_{11} \int_{0}^{\infty} d\psi_{1} \int_{0}^{\infty} \gamma_{m,s}^{(1)} \left( \psi_{1} \right)
\]

\[\times K_{11} K_{k''}, m = 1 \pi \delta \left( \psi_{1} - m \omega_{1} - k_{11} \psi_{11} \right) \]

\[\left( \omega_{1} - \omega_{2} - k_{11} \psi_{11} \right)^2 - \omega_{1}^2 \]

--- (35)

And from equation (6), we obtain \( \text{Im} \varepsilon'(\omega,k') \) as:

\[
\text{Im} \varepsilon'(\omega,k') = - \pi \sum_{j} \frac{\omega_{p j}^2}{k_{11}^2} \int_{-\infty}^{0} d\psi_{11} \int_{0}^{\infty} d\psi_{1} \frac{\delta_{11}}{\psi_{1}}
\]

\[m = 1 \int_{0}^{\infty} \left( \frac{k_{11} \psi_{11}}{\omega_{1}} \right) \delta \left( \omega_{1} - m \omega_{1} - k_{11} \psi_{11} \right) \]

--- (36)
Then substituting equation (34) and the result of Appendix B in equation (26) and further using equation (35) and equation (36) for \( \text{Im} \mathcal{E}(\omega',k') \) and \( \text{Im} H_{k,k',k''} \) and after some algebra we have the following simple and useful form of \( L_{k,k''}^{\text{(m)}} \)

\[
L_{k,k''}^{\text{(m)}} = \sum_{j \neq 0} \frac{4 \pi \omega_{p}^{4} m_{e}^{2}}{|\partial \mathcal{E}/\partial \omega| \partial \mathcal{E}''/\partial \omega'} k_{2} k_{2}^{''} m_{e} n_{0}
\]

\[
\times \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial F_{0}}{\partial u_{1}} \delta \left( \omega' - \omega - \mathcal{E}(\omega',k_{1}) \right) \right)
\]

\[
\times \left| \sum_{k} \frac{K_{k} K_{k}'' J_{p}(x') J_{p-\mathcal{E}}(x)}{(\omega - \mathcal{E}/k_{1} u_{1})^2 - \omega^2} \right|^{-\frac{2}{k_{1}^{2}}}
\]

(37)

where \( H_{k,k'',k'} \) and \( \mathcal{E}(\omega',k') \) in equation (37) are given by equation (29) and equation (6) respectively.
To set \( X'V \), we may proceed as follows.

We write

\[
\left[ \left( \omega' - \eta \cdot \Omega - k''_{ii} \Omega_i \right)^2 - \Omega^2 \right]^{-1}
\]

\[
= - (2 \Omega)^{-1} \left\{ \left[ \omega' - \left( \eta_{i-1} \right) \cdot \Omega - k''_{ii} \Omega_i \right]^{-1}
- \left[ \omega' - \left( \eta_{i+1} \right) \cdot \Omega - k''_{ii} \Omega_i \right]^{-1} \right\}
\]

(1)

Then we let \( n \rightarrow n+1 \) and \( n \rightarrow n-1 \) and obtain

\[
H_{k', k_i, -k''} = \frac{1}{2} \left( k_1 k'_1 k''_1 \right)
\]

\[
x \sum_{n_{k_1}} \sum_{k_{p_{n-1}}} \int_0^\infty \int_0^\infty \frac{dF_{\theta_{l} u_{l}}}{(\omega' - \eta \cdot \Omega - k''_{ii} \Omega_i)}
\]

\[
= \frac{\partial F_{\theta_{l} u_{l}}}{\partial F_{\theta_{l} u_{l}}} \frac{J_{D_{p-n-1}}(x)}{(p_{n-1}) \cdot \Omega - k''_{ii} \Omega_i)
\]

\[
- \left\{ \left( \frac{p_{n-1}}{k''_{ii}} \right) \int \frac{J_{n_{i-1}}(x') \cdot J_{p-n-1}(x'')}{\omega'' - (p_{n-1}) \cdot \Omega - k''_{ii} \Omega_i}
\right\}
\]

\[
+ \left\{ \left( \frac{p_{n+1}}{k''_{ii}} \right) \int \frac{J_{n_{i-1}}(x') \cdot J_{p-n+1}(x'')}{\omega'' - (p_{n+1}) \cdot \Omega - k''_{ii} \Omega_i}
\right\}
\]
where we have used the notation
\[ x = \frac{k_1 u_1}{\omega_2} \]

Now using the resonance condition we pick up the pole \( n = m \), write
\[ \omega'' - (\beta - n) \omega - k''_1 u_{11} = (\omega - \beta \omega - k_{11} u_{11}) \]

in the last two terms let \( \beta \neq 1 \rightarrow \beta \) and obtain
\[ \text{Im} \, H_{k', k', k''} = \frac{1}{2} \left( -1 + k'_1 \right) \]
\[ \times \left\{ \begin{array}{l}
\frac{1}{\beta} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt_1 \, dt_2 \, \frac{2F_0}{t_1} \left( \frac{\omega'}{\omega} \frac{m - \omega - k_{11} u_{11}}{\omega - \beta \omega - k_{11} u_{11}} \right) \\
\end{array} \right. \]
\[ \begin{array}{l}
k''_1 \left( \beta - m \right) J_{\beta - m}(x') \left[ J_{m+1}(x') J_{p-m+1}(x'') - J_{m+1}(x') J_{p-m+1}(x'') \right] \\
- \left. k_1 \right|_{n = m} \left( \beta - m \right) J_{p-m}(x') \left[ J_{m+1}(x') J_{p+1}(x) - J_{m+1}(x') J_{p+1}(x) \right] \\
- J_{m+1}(x') J_{p-1}(x) \end{array} \]
Now we employ the identity

\[ 2 \, \Omega \, J_{\Omega}(x) = x \left[ J_{\Omega-1}(x) + J_{\Omega+1}(x) \right] \]  

(2)

to split the \( \Omega \, J_{\Omega}(x) \) and \((\Omega-m) \, J_{\Omega-m}(x)\), then collect terms and finally obtain

\[ \text{Im} \, H_{m, k_1', k_2', -k_1''} = \frac{1}{4 \pi} \left( -i \pi \, k_1' \, k_1'' \right) \]

\[ \times \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{\partial F_{0}}{\partial \nu_1} \frac{m \, \delta \left( \omega' - m \omega - k_1' \nu_1 \right)}{(\omega - \Omega^2 - k_1'' \nu_1)} \]

\[ \times \left[ J_{\Omega-1}(x') \, J_{\Omega-m-1}(x'') - J_{\Omega+1}(x') \, J_{\Omega-m+1}(x'') \right] \]

Now let \( \Omega \to \Omega \) in the first and second terms respectively and get

\[ \text{Im} \, H_{m, k_1', k_2', -k_1''} = \frac{1}{4 \pi} \, k_1' \, k_1'' \sum_{\Omega} \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{\partial F_{0}}{\partial \nu_1} \frac{m \, \delta \left( \omega' - m \omega - k_1' \nu_1 \right)}{(\omega - \Omega^2 - k_1'' \nu_1)} \]

\[ \times \left( \frac{m \omega \, J_{m}(x') \, J_{\Omega-m}(x'') \, \bar{J}_{\Omega}(x) \, \delta \left( \omega' - m \omega - k_1' \nu_1 \right)}{(\omega - \Omega^2 - k_1'' \nu_1)^2 - \Omega^2} \right) \]

(3)
APPENDIX C

Derivation of $\mathcal{I}_{\eta\lambda} \mathcal{D}_{1,1}$

Splitting terms into partial fractions we can write the equation (22) of Appendix A as

$$\sum_{\eta} k_{\eta} k_{\eta}'' \mathcal{D}_{\omega} = \sum_{\eta} \sum_{b} \sum_{c} \frac{1}{2} \sum_{\gamma=\pm 1}$$

$$\left[ k_{\eta} J_{\eta}(x) J_{b-\gamma}(x) J_{b-\gamma}(x'') J_{\eta-c}(x'') - k_{\eta}'' J_{\eta}(x) J_{b}(x) J_{b-c}(x'') J_{\eta-c}(x'') \right] \Bigg/ \left( (\omega - b - c - k_{\eta} u_{11})^{2} - \omega_{e}^{2} \right)$$

$$\times \left[ \frac{\gamma k_{\eta} k_{\eta}''}{(\omega - c - \gamma) \omega_{e} - k_{\eta} u_{11}} \right]$$

$$\times \left[ \frac{\gamma k_{\eta}''}{(\omega - \eta u_{11} - k_{\eta} u_{11})} - \frac{(\eta - c) k_{\eta}}{(\omega - k_{\eta} u_{11})} \right] \frac{\partial \Phi}{\partial v_{1}}$$

(1)

Now we let $c \rightarrow (c - \gamma)$. and in the last term $(\eta - c) \rightarrow \eta$

Then, we write out all the terms perform a series of manipulation with the Bessel function identity i.e.

$$\partial_{p} \partial_{p} J_{p}(x) = x \left[ J_{p-1}(x) + J_{p+1}(x) \right]$$

Collect terms and obtain
\[
\sum \frac{\prod k_i' k_i'' u_i D_{i,i}}{m_{n,b} c} \left\{ \frac{\frac{\prod k_i' k_i''^2 \Gamma_{m}(x)}{m_2 - \Omega^2 - \Omega_i u_i}}{(\omega - \Omega^2 - \Omega_i u_i)_{n,b}^2} \right\} \\
\times \left[ c^2 J_{m-c+1}(x') J_b(x) J_{b-c}(x'') + (n-c) J_{m-c}(x') \frac{\xi_y}{\xi''} \right] + \left[ c J_{m-1}(x') J_b(x) J_{b-c}(x'') + n J_m(x) \frac{\xi_y}{\xi''} \right]^2 \\
\times \frac{1/u_i \cdot \frac{\Re u_i}{\Re u_i}}{(\omega^2 - \Omega^2 - \Omega_i u_i) \left[ (\omega - \Omega^2 - \Omega_i u_i)^2 - \Omega_i^2 \right]}
\]

where \( \xi_y = -x J_{b+c}(x) J_{b-c}(x') + x'' J_b(x) J_{b-c-1}(x'') \)

Now we use the resonance condition equation (16) of the text to obtain the imaginary part and hence pick \( \Omega = \omega \) from the summation and obtain

\[
\Im \sum \frac{\prod k_i' k_i'' u_i D_{i,i}}{m_{n,b} c} \\
= -\sum \frac{\prod (u_i \cdot \frac{\Re u_i}{\Re u_i}) k_i' k_i'' \Omega}{m_2 - \Omega^2} J_{b-m}(x') J_b(x) \\
\times \left[ k_i'' n m J_m(x') J_{m-m+1}(x') - k_i' m(n-m) J_{m-1}(x) J_{m-m}(x'') \right] \\
\frac{x \left[ \frac{\Omega_1}{\Omega_1^2 - \Omega_i u_i^2} \right]}{(\omega - \Omega^2 - \Omega_i u_i)_{n,b}^2 \left[ (\omega - \Omega^2 - \Omega_i u_i)^2 - \Omega_i^2 \right]}
where \( \delta(\cdot) = \delta(\omega' - \omega - k_{11}v_{11}) \) and the terms with \( e_j \) cancel out.

Splitting \( \gamma J_\eta(x) \) and \( (\eta-a) J_{\eta-1}(x''') \) according to the Bessel function identity collecting terms gives us, for the square bracket

\[
\left(-\Omega \cdot x \cdot x'''/2u_4\right) \left[ J_{\eta+1}(x) J_{\eta-m+1}(x''') - J_{\eta-m}(x''') \right]
\]

Now we let \( n \rightarrow n+1 \) in the first and second terms respectively, combine the poles \( [\omega - (\eta + 1) - \Omega - k_{11}v_{11}]^{-1/2} \) and obtain

\[
\Im \sum_{n} \sum_{b} \frac{k_1 k_1^n u_1 w_1 D_{1,1}}{\Pi m = 2 k_1^2 k_1^{2n} W(u_1) \frac{F_0}{\sqrt{\pi}} \frac{\delta(\omega' - m - \omega - k_{11}v_{11})}{\left[ (\omega' - m - \omega - k_{11}v_{11})^2 - \omega^2 \right]^2}}
\]

where \( W(u_1) \) is defined in the text.
We wish to show the following relations:

\[ H_{k', k'', k'} = H_{k, k', k''} \]  \hspace{1cm} (1)

\[ H_{k', k, -k''} = H_{k', k'', k'} \]  \hspace{1cm} (2)

where \( H_{k', k'', k'} \) is as defined in the text and \( H_{k', k, -k''} \) is written in the following form

\[
H_{k', k, -k''} = - \sum_{\ell} \sum_{\ell'} \sum_{\ell''} k_\ell k_{\ell'} k_{\ell''}
\]

\[
\chi \int_{-\infty}^{\infty} dx_{\ell} \int_{-\infty}^{\infty} dx_{\ell'} \int_{-\infty}^{\infty} dx_{\ell''} \frac{J_{\ell'}(x') J_{\ell''}(x'') J_{\ell}(x)}{(\omega' - \ell - 2 - k_{\ell'} u_{\ell'})^2 - \omega^2}
\]

\[
\chi \left[ \frac{S-\Omega/k_{\ell'}}{(\omega - S-\Omega - k_{\ell'} u_{\ell''})} - \frac{(S-\ell) S-\Omega/k_{\ell''}}{(\omega - (S-\ell) S-\Omega - k_{\ell''} u_{\ell''})} \right]
\]

Equation (3) follows from equation (30) of Appendix A by letting \( p \rightarrow s \) and \( n \rightarrow 2 \). All other symmetry relations follow from equation (1) and equation (2). We see that equation (1) follows immediately from the form \( H_{k', k'', k'} \). In order to show equation (2), we proceed as follows: In equation (3) we write...
\[
\left[ (\omega' - \Omega - k''_{i''}u_{i''}) \right]^{-1} \left( x_{2}^{2} \right) \left[ (\omega' - l\Omega + \Omega) \right]^{-1} = (\mathbf{R} - \Omega)^{-1} \\
\left[ (\omega' - l\Omega - \Omega) \right]^{-1}
\]

where we have used the notation \( \omega' = (\omega' - k'_{i'}u_{i'}) \). Similarly, in the following we shall abbreviate \( \omega = (\omega - k_{i}u_{i}) \) and \( \omega'' = (\omega'' - k''_{i''}u_{i''}) \) Now we let \( \ell' \rightarrow \ell + 1, s \rightarrow s + 1 \) and obtain

\[
\frac{\partial F_{\ell}}{\partial u_{1}} \left( \frac{J_{s-\ell}(x'')}{(\omega' - \Omega u_{2})} \right) \times
\begin{align*}
\sum_{\ell} \left\{ \left[ \omega - (s + 1)\Omega \right] (s - 1) J_{-\ell-1}(x') J_{\ell+1}(x) - \left[ \omega' - (s - 1)\Omega \right] (s + 1) \\
\times J_{s-\ell-1}(x') J_{s+1}(x) \right\} \\
\left[ \frac{1}{k_{1}''} \left[ \omega'' - (s - \ell)\Omega \right] \right]
\end{align*}
\]

where in the first term we cross multiplicative by the denominators. Now in the first we can apply the Bessel's function density mentioned in Appendix B to rewrite the Bessel function \( J_{\ell+1}(x') \) in terms of \( J_{\ell}(x') \), \( J_{\ell+2}(x') \) and \( J_{s+1}(x) \). After regrouping terms we obtain
\[ H_{k', k, -k''} = H_{k', k, -k''}^{(1)} + H_{k', k, -k''}^{(2)} \]  
\[ \text{(5)} \]

where
\[ H_{k', k, -k''}^{(1)} = \sum_{l} \sum_{s} k'_l k_s l'' \int_{-\infty}^{\infty} d\upsilon_{1l} \int_{0}^{\infty} d\upsilon_{1l} \]
\[ \frac{\partial F_{0}}{\partial \upsilon_{1l}} \frac{J_{s}(x)}{J_{s-l}(x')} \frac{J_{l}(x)}{l_{-2}/k_{l}} \]
\[ \left[ (\tilde{w} - s\omega)^{2} - \omega^{2} \right] \left[ \tilde{w}' - l \omega \right] \]
\[ \text{(6)} \]
\[ H_{k', k, -k''}^{(2)} = - \sum_{l} \sum_{s} \frac{1}{2} (k'_l k'_s l''') \int_{-\infty}^{\infty} d\upsilon_{1l} \int_{0}^{\infty} d\upsilon_{1l} \]
\[ \frac{\partial F_{0}}{\partial \upsilon_{1l}} \frac{J_{s-l}(x)}{(\tilde{w}' - l \omega)} \left\{ a - b - c \right\} \]
\[ \text{(7)} \]

where
\[ a = (\tilde{w} - s\omega) \left[ (s-1) J_{s+1}(x) J_{s-1}(x') - (s+1) J_{s+1}(x) J_{s+1}(x') \right] \]
\[ \frac{k_{l}}{k_{l} \left[ (\tilde{w} - s\omega)^{2} - \omega^{2} \right]} \]
Applying the previous mentioned Bessel's identity in (A) and (B) we may rewrite the terms \( (s+1) \frac{J_{s+1}(x)}{k_1} \) and \( \ell \frac{J_{\ell}(x')}{k_1'} \) and obtain

\[
\left( \frac{\mu}{\omega_1} \right) \left\{ \begin{array}{l}
\left[ (\bar{\omega} - (s+1) \omega) \right] \left[ J_s(x) J_{\ell-1}(x') - J_{s+2}(x) J_{\ell+1}(x') \right] \\
+ \left[ (\bar{\omega} - (s+1) \omega) \right] \left[ J_{s-2}(x) J_{\ell-1}(x') - J_{s}(x) J_{\ell+1}(x') \right] \end{array} \right\}
\]

Now we break up the denominator \( (\bar{\omega} - s \omega)^2 - \omega^2 \) according to the relation (4), let \( S \rightarrow S+1 \), collect terms and regroup them as follows:

\[
\mathbf{H}^{(2)}_{k',k,-k''} = \mathbf{H}^{(2a)}_{k,k',-k''} + \mathbf{H}^{(2b)}_{k,k',-k''} \tag{8}
\]

where
\[ H^{(2n)}_{k, k', k''} = - \sum_{\ell} \sum_{s} \frac{k_{s} k_{s}'}{4 \omega} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial \Phi_{\ell}}{\partial u_{s}} \left( \frac{\omega'}{\omega - \ell \omega} \right) \times \left[ \left( \frac{1}{\omega' - \omega} \right)^{-1} \left( \omega'' - \ell \omega \right)^{-1} \right] \times \left[ J_{s-\ell-1}(\omega') + J_{s-\ell+1}(\omega') \right] \times \left[ J_{s-1}(\omega') J_{\ell-1}(\omega') - J_{s+1}(\omega') J_{\ell+1}(\omega') \right] \] 

\[ \text{(9)} \]

\[ H^{(2n)}_{k, k', k''} = - \sum_{\ell} \sum_{s} \frac{k_{s} k_{s}''}{4 \omega} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial \Phi_{\ell}}{\partial u_{s}} \left( \frac{\omega'}{\omega - \ell \omega} \right) \times \left[ \left( \omega' - \omega \right)^{-1} \left( \omega'' - \ell \omega \right)^{-1} \right] \times \left[ J_{s-2}(\omega') J_{s-\ell-1}(\omega'') J_{\ell-1}(\omega') \right. \\
- \left. S J_{s}(\omega') \times \left[ J_{s-\ell+1}(\omega'') J_{\ell-1}(\omega') + J_{s+\ell-1}(\omega'') \right] \right] \\
+ (s+2) J_{s+2}(\omega') J_{s-\ell+1}(\omega'') J_{\ell+1}(\omega') \] 

\[ \text{(10)} \]

Now using \( \omega' = \omega - \omega'' \) and \( \omega'' = \omega' - (s-\ell) \omega \) we have
\[ \omega'' - \omega (s-\ell) - (\omega' - \omega'' \ell \omega) = - (\omega' - \ell \omega) \]
and in equation (9),
we have

\[
\left[ (\tilde{\omega}^2 - \xi \omega )^{-1} \right] = \left[ (\tilde{\omega}^2 - \xi \omega )^{-1} - (\tilde{\omega}''^2 - \xi \omega )^{-1} \right]^{-1} = - \left[ (\tilde{\omega} - \xi \omega ) (\tilde{\omega}'' - \xi \omega ) \right]^{-1}
\]

Now let \( \ell, \xi \rightarrow \ell \pm 1 \) in equation (9) and using the Bessel function identity, we obtain

\[
H_{\ell, \ell', -\ell''}^{(2)}(x) = \sum_{\ell} \sum_{\ell' \neq 0} \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{dV_{\ell'}}{V_{\ell'}} \frac{dV_{\ell''}}{V_{\ell''}} \times \frac{1}{(\tilde{\omega} - \xi \omega )^2 - \zeta \omega ^2}
\]

where once more we used equation (9) for combining the \( \left[ (\tilde{\omega}^2 - \xi \omega )^{-1} \right] \) poles. It is now straightforward to show by a long series of manipulations using the Bessel function identity and resumming with the index \( S \) that the Bessel functions in equation (10) cancel identically, and we have

\[
H_{\ell, \ell', -\ell''}^{(2)} = 0
\]

Adding equations (6), (11) and (12), we obtain
\[
H_{k', k'' - k} = \sum_{l} \sum_{s} k_{l} k_{l'} k_{l''} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{cl_{s} \phi_{e}}{\psi_{l}}
\]

\[
x \frac{J_{s}(\omega) J_{s-\ell}(\omega') J_{l'}(\omega')}{(\omega - s \omega - k_{11} u_{1})^{2} - \omega^{2}}
\]

\[
x \left[ \frac{\ell \omega/k_{l}}{(\omega - \ell \omega - k_{11} u_{1})} + \frac{(s - \ell) \omega/k_{l}''}{(\omega'' - (s - \ell) \omega - k_{11}'' u_{1})} \right]
\]

which is identically equal to \( H_{k', k'' - k} \). Hence the symmetric relation (2) is proven.

Hence the
REFERENCES


