CHAPTER V

CRITICAL LENGTHS IN ELECTRON BEAM INJECTED MIRROR MACHINES

1. Introduction:

In the preceding chapter we had studied the interaction between HFCLC waves and electron beam induced Langmuir waves in the regime 
\[ \omega'' \left( \approx \omega_{pe} \frac{k''/k''}{} \right) \approx \omega \left( \approx \omega_{pe} \frac{k''/k''}{} \right), \quad k'' = k'' \text{ etc.} \]

It was shown there that scattering of beam electrons from Langmuir waves produces enough anomalous resistivity to stabilize the HFCLC modes. The coupling was envisaged according to the equation

\[
\frac{\omega - \omega''}{k_{||} - k_{||}''} = \gamma_1 \tag{a}
\]

In this chapter, we will study the interaction of HFCLC waves and Langmuir waves (electron beam induced) in the regime \( \omega = \omega_{pe} \frac{k_{||}/k_{||}'}{\omega_{pe}} \gg \omega'' \), \( \omega_{be} \frac{k''_{||}/k_{||}'}{} \approx \omega_{be} \left[ k_{||} - k_{||}' \right] \) and \( -k_{||} \approx k_{||}' \). This regime is...
appropriate to those Langmuir waves which travel almost along the field lines \( \text{so that } - k_{\parallel} - k \). Clearly then equation (a) for this regime becomes

\[
\frac{\omega}{k_{\parallel}} = \nu_{\parallel}
\]

which simply expresses the resonance of Langmuir waves with electrons i.e. the HFCLC get decoupled from the Langmuir waves. Thus we see that in this regime the coupling between HFCLC waves and Langmuir waves by non-linear Landau damping becomes weak. However, as we show in this Chapter the Langmuir waves and HFCLC waves can still interact in this regime. The appropriate technique to study the interaction in this regime is the 'adiabatic approximation' which considers interaction between waves of widely different properties i.e. \( \omega \gg \omega'' \) etc.

In Chapter IV it was shown that in the interactions considered in the regime \( \omega \approx \omega'' \), the spectral features of Langmuir waves do not play any role. However, in the regime considered here the spectral features of Langmuir waves play an important role. The magnitude of the effect of Langmuir waves on HFCLC waves varies in its course of non-linear evolution. Thus, while studying the interaction it becomes important to take into consideration the non-linear evolution of Langmuir waves.

The non-linear evolution of Langmuir turbulence has already been discussed in general in Chapter III. However, for the sake of continuity we briefly discuss here some of its features relevant to the present problem. Accordingly, the general scenario is as follows: If the injected beam is warm enough i.e. \( \Delta V_b / V_b \gtrsim (\gamma_b / \gamma_{hb})^{1/3} \)
(\(V_b, n_b\) are the beam velocity and density, \(n_p\) is the plasma density), a narrow spectrum of wave-number spread
\[ k \Delta k = \frac{\Delta V_b}{V_b} \frac{U_{th e}}{V_b} \]
(where \(U_{th e}\) is thermal velocity of electrons) centred around
\[ k_0 = \frac{\omega_{te}}{V_b} \]
is generated. The quasi-linear theory fails to explain the saturation of this unstable spectrum (1,2). It is explained when strong turbulence effects are taken into account. Accordingly it has been shown by a number of authors (1-5) that, when amplitude becomes large enough, various non-linear processes like parametric decay, oscillating two stream instability etc. can cause scattering in K-space. Computer simulation (6,7) have shown that these processes lead to the formation of 'spiky turbulence', where there are localised structures of intense electric fields (\(W = \langle E^2 \rangle / \varepsilon_0 k T_e \gtrsim 1\), \(\langle E^2 / \varepsilon_0 \rangle\) is the wave energy density). It has also been shown numerically by solving appropriate kinetic-wave equations, that these processes lead to a significant broadening of the plasmon spectrum (\(\Delta \sim k_0\), where \(\Delta\) is the width of plasmon spectrum), which can be approximated by a Gaussian centred around \(k_0\) (8). The non-linear evolution of Langmuir turbulence proceeds on the time scale of a few tens of \(1000 \omega_{pe}^{-1}\), i.e. a few \(\mu s\) for typical experimental situations. Now, in experiments employing parallel injection of electron beam, the electron gun and the plasma gun are fired simultaneously. It takes a few \(\mu s\) (\(\sim\) a few ion transit time) for the ion distribution to shape itself to develop the hole, so that the loss cone instabilities are triggered a few \(\mu s\) after the firing of the plasma gun. Hence in such experiments we expect a broad and energetic (\(\Delta \sim k_0\), \(W \ll 1\)) spectrum of Langmuir waves to come into existence by the time HFCLC waves are excited. We wish
to investigate the effect of this spectrum on HFCLC modes. Later, we will investigate the experimental situation in which narrow spectrum of Langmuir turbulence interacts with HFCLC waves.

2. The effect of Langmuir Spectrum on HFCLC Waves:

In our theoretical model we consider a low-β mirror plasma with the Z-axis along the mirror axis. In this machine we consider a HFCLC mode \((\omega'', \vec{k}'')\) which is mostly in Y-direction i.e. \(k_{y}' > k_{z}''\) with frequency in the vicinity of \(\omega'' \approx \omega_{pe} \approx k_{y}'' \approx \omega_{pi} \approx k_{z}'' U_{th}\) (where \(\omega_{pe}\) is the ion plasma frequency and \(U_{th}\) is the ion thermal velocity). In the same geometry, we consider a packet of Langmuir waves \((\omega, \vec{k})\) in Y-Z plane, mostly along Z-direction i.e. \(k_{y} = k_{z}\). Let the packet be inclined at a small angle \(\theta\) from the field lines and centred around \(k_{0} = \omega_{pe}/V_{b}\).

The evolution of the Langmuir plasmon distribution function will be studied by a wave-kinetic equation developed by Vedenov et al (9).

\[
\frac{\partial N_{k}}{\partial t} = \frac{\partial N_{k}}{\partial \vec{y}} \cdot \vec{V}_{g} - \frac{\partial (\omega_{k} N_{k})}{\partial \vec{y}} + \frac{\partial N_{k}}{\partial \vec{k}} = 0
\]

(1)

where \(\vec{V}_{g} = \omega/\partial \vec{k}\) is the group velocity of the Langmuir waves.

In brief, the effect of Langmuir turbulence arises as follows i.e. the low frequency perturbation creates a perturbation in the plasmon density. The gradient of this plasmon density gives rise to a ponderomotive force (P.F.) which reacts back on the low frequency waves (HFCLC) to modify its characteristics. The P.F. on ions is \(m_{i}/m_{e}\) times smaller than that on electrons, hence dropped.
The plasma distribution function is perturbed as follows:

\[ \tilde{N}_k = N_k + \eta_k \]  \hspace{1cm} (2)

where \( \tilde{N}_k \) is the equilibrium distribution function normalised as

\[ \int \tilde{N}_{k0} dk^\prime = 1 \]  \hspace{1cm} (3)

The space and time dependence of \( \eta_k \) is given by

\[ \eta_k = \tilde{\eta}_k e^{i(\vec{k} \cdot \vec{\xi} - \omega'' t)} \]  \hspace{1cm} (4)

where \( (\omega'', \vec{k}'') \) is the low frequency mode. From equations (1), (2), and (3), we have

\[ \frac{\partial \omega_k}{\partial y} \cdot \frac{\partial N_{k0}}{\partial k} \]

\[ \frac{1}{i(\vec{k}'', \vec{\xi}' - \omega'')} \]

\[ \tilde{\eta}_k = \frac{2 \omega_k \cdot \omega N_{k0}}{\omega'' - \omega'''} \]

(5)

The dependence of \( \tilde{\eta}_k \) on the low frequency density perturbation comes from the term \( \partial \omega_k / \partial y \). For plasma waves in the regime \( \omega_{pe} / \omega_{ce} > 1 \)

\[ \omega_k^2 = \omega_{pe}^2 + k^2 \frac{v_{he}^2}{\omega_{pe}} \]  \hspace{1cm} (6)

which gives

\[ \frac{\partial \omega_k}{\partial y} = \frac{i}{2} k'' \omega_{pe} \tilde{\eta}_{c1} \tilde{\eta}_{c1} \frac{\omega_{ce}}{\omega_{ce}} \]  \hspace{1cm} (7)

where \( \tilde{\eta}_{c1} \) is the low frequency density perturbation. Following Post and Rosenbluth we shall use the straight line orbit approximation (10) i.e. on the time scale of the growth of this instability \( \sim \omega_{pe}^{-1} \) the ion motion is taken to be rectilinear and mainly perpendicular to the
field lines while electron motion is mainly along the field lines. Hence we shall include the effect of ponderomotive force only in the electron equation for the motion parallel to the magnetic field, which thus becomes

$$m_e \frac{\partial U_z}{\partial t} = -eE_z - m_e \frac{\partial U_z}{\partial z}$$

(8)

In equation (8), we write

$$U_z = \tilde{U}_{zS} + \tilde{U}_{zf}, \quad E_z = \tilde{E}_{zS} + \tilde{E}_{zf}$$

(9)

where $\tilde{U}_{zS}$ and $\tilde{E}_{zS}$ are parallel velocity and electric field sustained by HFCLC (identified as the low frequency perturbation as compared to the Langmuir field frequency) while $\tilde{U}_{zf}$ and $\tilde{E}_{zf}$ are those sustained by Langmuir waves identified as the high frequency field. Averaging equation (8) over an ensemble of random set of Langmuir waves

$$m_e \frac{\partial \tilde{U}_{zs}}{\partial t} = -e \tilde{E}_{zs} - m_e \frac{\partial \tilde{U}_{zs}}{\partial z} \frac{\partial |\tilde{V}_{zf}|^2}{\partial z}$$

(10)

where $\tilde{U}_{zf}$ may be evaluated from the following equation

$$m_e \frac{\partial \tilde{U}_{zf}}{\partial t} = -e \tilde{E}_{zf}$$

(11)

From equation (11) $\tilde{U}_{zf}$ may be evaluated and substituted in equation (10) which becomes

$$m_e \frac{\partial \tilde{U}_{zs}}{\partial t} = -e \tilde{E}_{zs} - \frac{e^2}{\kappa^2} \frac{\partial |\tilde{E}_{zf}|^2}{\partial z}$$

(12)

where $\tilde{E}_{zf}$, the $z$-component of the Langmuir field is assumed to oscillate with frequencies $\omega_R$, $\tilde{E}_{zf} = e^{-i\omega_R t} E_{zf \kappa}$.
If $E_t$ is the Langmuir field amplitude in a direction $\theta$ with respect to Z-axis, then

$$E_{t+}^2 = E_{t-}^2 \cos^2 \theta, \quad \langle E_{t+} E_{t-} \rangle / 8\pi = N_0^2 N_o \omega$$

where $N_o$ is the average plasmon density. Using these relations in equation (12), we have:

$$m_e \frac{d \vec{U}_{Z_{\perp}}}{dt} = -e \vec{E}_{Z_{\perp}} - \sum \frac{4\pi e^2}{k^2} \frac{1}{\omega} \cos^2 \theta N_o \frac{\partial \vec{\eta}_k}{\partial z} \tag{13}$$

Now from equation (5), we have $\vec{\eta}_k$ as

$$\vec{\eta}_k = \frac{i}{2} \kappa'' \eta_{\perp} \frac{\partial N_k}{\partial k} \frac{1}{i (k'' \cdot v_g - \omega')} \tag{14}$$

Using this expression for $\vec{\eta}_k$ in equation (13), we obtain:

$$m_e \frac{d \vec{U}_{Z_{\perp}}}{dt} = -e \vec{E}_{Z_{\perp}} - \sum \frac{4\pi e^2}{k^2} \frac{1}{\omega} \cos^2 \theta N_o \frac{1}{i (k'' \cdot v_g - \omega')} \frac{\partial \eta_{\perp}}{\partial z} \tag{15}$$

which gives

$$m_e \frac{d \vec{U}_{Z_{\perp}}}{dt} = -e \vec{E}_{Z_{\perp}} - \sum \frac{4\pi e^2}{k^2} \frac{1}{\omega} \cos^2 \theta N_o \frac{k'' \cdot \partial N_k}{\partial \kappa} \frac{1}{i (k'' \cdot v_g - \omega')} \frac{\partial \eta_{\perp}}{\partial \kappa} \tag{16}$$

From this equation we can calculate $\vec{V}_{Z_{\perp}}$ which is given as (after using $\vec{E}_{Z_{\perp}} = -i k_{Z_{\perp}} \phi$)

$$\vec{V}_{Z_{\perp}} = -\frac{e}{m_e} \frac{k_{Z_{\perp}}^2}{\omega''} \phi + \sum \frac{4\pi e^2}{k^2} \frac{1}{\omega} \cos^2 \theta N_o \frac{k''}{\omega''} \frac{1}{\omega''} \frac{\partial \eta_{\perp}}{\partial \kappa} \tag{17}$$
Next, the equations for the perpendicular motion of the electrons give
(taking the perturbation in the perpendicular motion to be on the slow time
scale of the HFCLC modes)

\[-i \omega'' m_e \frac{\partial \tilde{u}_{xs}}{\partial t} = -i \frac{\tilde{U}_{ys} B_0}{\omega''} \]  \hspace{1cm} (18)

\[-i \omega'' m_e \frac{\partial \tilde{U}_{ys}}{\partial t} = i e k'' \frac{\Phi}{\sqrt{y} \gamma} + e \frac{\tilde{U}_{xs} B_0}{\omega''} \]  \hspace{1cm} (19)

From these equations we have

\[\tilde{U}_{ys} = \frac{-e k'' \Phi / m_e \omega''}{1 - \omega c^2 / \omega''^2} \]  \hspace{1cm} (20)

Also from the equation of continuity on the slow time scale of HFCLC mode
we have

\[\frac{\partial \tilde{\rho} \tilde{\epsilon}_1}{\partial t} = -i \tilde{\eta}_c k'' \sqrt{y} \tilde{U}_{ys} - i \tilde{\eta}_0 k'' \tilde{U}_{zs} \]  \hspace{1cm} (21)

Substituting for \(\tilde{U}_{ys}\) and \(\tilde{U}_{zs}\) from equation (20) and equation
(17), we have the electron density perturbation as
\[
\tilde{\eta}_{ei} = \left[ - \frac{n_0 k_z^2 \varrho_1}{m_e \omega_0^2} \left\{ 1 - \frac{e^2}{m_e \omega_0^2} \right\} - \frac{e}{m_e} \frac{k_z^2}{\omega_0^2} \left( \varrho_1, \eta_0 \right) \right] \\
\times \left[ 1 - \frac{k_z^2}{\omega_0^2} \eta_{ei} \right]^{-1} \left\{ \frac{c_0 \varrho_2 \theta \kappa_0}{\omega_0^2} \frac{\partial N_k}{\partial \kappa_0} \alpha_i^2 \right\}
\]

(22)

where \( \varpi = \frac{4 \pi e^2}{\gamma \rho_0 \omega_0} \). The summation over \( K \)-plane has been replaced by integration which has to be done according to Landau's prescription. The ion density perturbation is unaffected by Langmuir waves and hence can be calculated in the manner shown by Post et al (10). Using electron density perturbation and ion density perturbation in the Poisson's equation we get the modified dispersion relation for the HFCLC modes as

\[
1 = \frac{\omega_{pi}^2}{\omega_0^2} - \frac{\omega_{pe}^2}{\omega_0^2} - \frac{\omega_{pe}^2}{\omega_{ce}^2} \left[ 1 + \frac{\varpi \omega_0^2}{\omega_{ce}^2} \left( \frac{\kappa_0}{\Delta} \right)^2 \{ 1 + \varphi_i^2 (\varpi_i) \} \right] \\
+ \frac{\omega_{pi}^2}{\kappa_0^2} \int \left( \overrightarrow{V}_0 \right)^2 \frac{\partial F_i}{\partial \overrightarrow{V}} d\overrightarrow{V} \left( \omega_i - \frac{\varpi_0}{\kappa_0} \left( \omega_0 \right) + i \omega_i \right)
\]

(23)

where in the electron term, the integration in the \( K \)-plane has been performed by modelling \( N_k \varphi_0 = \frac{1}{\sqrt{2 \pi} \Delta} \exp \left\{ - \left( \frac{\overrightarrow{k} - \overrightarrow{k_0}}{2} \right)^2 / 2 \Delta^2 \right\} \), where \( \Delta \) is the width of the spectrum. Thus \( \varphi_i(\varpi_i) \) is the plasma dispersion function with \( \overrightarrow{V}_0 = \overrightarrow{k_0} \lambda \varphi_0 \), \( \varphi_i = \frac{\overrightarrow{k_0} \cdot \kappa_0}{\Delta \kappa_0} \left( \frac{\omega_0}{\kappa_0} - 1 \right) \).
\[ W = N_0 \omega_{pe} / n_0 k T_e \], \( \omega^2 = (\omega_{pe}^2 / k^2, \omega_{ce}^2 / k^2) \). In writing down the electron we have also used the approximation

\[ \omega_{ce}^2 \gg \omega''^2 \quad \text{which gives} \quad \omega_{pe}^2 / \omega''^2, \omega_{ce}^2 \alpha \omega_{ce}^2 / \omega_{ce}^2 \].

In the ion term under small growth rate \([ \omega_2''']\) approximation, the principle value and the pole part can be separated and the \( \phi \) integration \((\phi\) is the angle between \( \vec{k''} \) and \( \vec{V}_1 \)) in the pole part can be performed to give equation (23) as (11)

\[ I = \frac{\omega_{pe}^2 k_x^2}{\omega''^2 k_y^2} - \frac{\omega_{pe}^2}{\omega_{ce}^2} \left[ 1 + W \cdot \left( \frac{\omega''^2}{\omega''^2} \right)^2 \left\{ 1 + \xi(y) \right\} \right] \]

\[ + \frac{\omega_{pi}^2}{k''^2 U_{hi}^2} Z_1(y) \]

where \( y = \omega''/k'' U_{hi} \), \( x = v_L^2 / U_{hi}^2 \),

\[ Z_1(y) = \int_{-\infty}^{\infty} \frac{g_0(x)}{\sqrt{x/y^2 - 1}} \]  

\[ Z_2(y) = \int_0^y \frac{g(x)}{1 - x/y^2} \]  

\[ g(x) = 2 \pi U_{hi}^2 \int_0^x \frac{f_i(U_{hi}, U_{li})}{d\xi} \]  

\[ d\xi g(x) = 2 \]
is the loss cone ion distribution (i.e. $F_1(\mu_{\perp}=0)=0$). Here the resonance condition is $x > y^2$ or $V_\perp > \frac{\omega''/k''_j}{k''_i}$, in which case only $Z_1$ term is important and $Z_2$ may be neglected. $\cos^2 \Theta$ has been put $\sim 1$. In the absence of Langmuir turbulence $W = 0$, equation (24) reduces that obtained by Post and Rosenbluth. At the ends $\omega_{pe} \rightarrow \infty$ and hence $|k_{||}'' \rightarrow \infty$ but before this, the condition for electron Landau damping i.e. $\omega'' |k_{||}'' U_{\perp_{\mathrm{ke}}}| < 1$ is satisfied. Hence we assume in this analysis that waves are effectively absorbed at the $\langle y \rangle$, $|F(y)|$ for collisional equilibrium distribution function is $\leq 1$ (10). In such cases the critical length can be evaluated by assuming real $\omega''$, $\frac{\omega''}{k''} U_{\perp_{\mathrm{ke}}} \leq 1$ and solving for $\Im k''_i$ by making binomial expansion in equation (24) which gives $\frac{\omega''}{U_{\perp_{\mathrm{ke}}}} \left( \frac{\omega_{pe}}{\omega'_{ki}} \right)^{-1}$.

For growth $Z_1(y) > 0$. We assume that the critical length is about 10 times the growth length calculated from equation (25). Hence the critical length comes out to be

$$L = 20 \left( \frac{m_i}{m_e} \right)^{1/2} \frac{|y Z_1(y) \omega_{pe}}{\omega_{pe}} \left( \frac{\omega_{pe}}{\omega'_{ki}} \right)$$

where $\omega_{pe}^2/\omega'_{ki} > 1$ has been used which is the typical density regime of present day machines. It should be noted that $Z_1(y)$ is a smooth function of $y$ and acquires positive and negative value with maximum value of order unity on both sides. We will now evaluate the critical length in the presence of the electron beam.
3. **Electron Gun and Plasma Gun Fired Simultaneously:**

In this case as stated earlier the HFCLC waves encounter a broad spectrum of Langmuir waves, hence we may use \( \beta Q_t \approx \beta Q_{Q \theta} \frac{\lambda_{0}}{\lambda_{0}} \)

\[ \left( \frac{\omega''}{k_{n}^{'}, V_{o}} \right) \ll 1 \] so that \( \chi(\epsilon_{g}) \) can be approximated as \( \chi = i \sqrt{\pi} \)

In this limit the Landau resonance between the wave and the Langmuir plasmons becomes significant which brings about substantial changes in the temporal and spatial growth of HFCLC modes. We further assume \( \omega'' = \omega_{\phi} \approx \frac{\omega_{be} k_{n}''}{k_{n}''} \)

and \( \omega(\frac{k_{n}''}{\lambda})^2 >> 1 \) (typically as will be shown later). Using this we may write \( |k_{n}''|^2 \) from equation (24) as

\[
|k_{n}''|^2 = \frac{(\omega'')^2 k_{n}''^2}{\omega_{be}^2} \left[ 1 + \alpha + T + i \sqrt{\pi} T \epsilon_{g} - i \frac{\omega_{bi}^2}{k_{n}''^2 U_{hi}^2} z_{i} \right. 
\]

\[ - i \frac{T}{k_{n}''^2 U_{hi}^2} z_{i} + \sqrt{\pi} T \epsilon_{g} \frac{\omega_{bi}^2}{k_{n}''^2 U_{hi}^2} z_{i} \epsilon_{g} \]

and evaluate the square root to give \( k_{n}'' \) as

\[
k_{n}'' = \frac{(\omega'')^2 k_{n}''}{\omega_{be}^2} \left( \alpha + T \right)^{1/2} \left[ 1 + \frac{T}{\alpha + T} \epsilon_{g} - i \frac{\omega_{bi}^2}{k_{n}''^2 U_{hi}^2} z_{i} \left( 1 + T \right) \right.
\]

\[ + \frac{T}{\alpha + T} \frac{\omega_{bi}^2}{k_{n}''^2 U_{hi}^2} z_{i} \epsilon_{g} \left. \right] \frac{1}{2}
\]

(28)
Since $a, T > 1$, $\xi < 1 \quad \frac{\omega_{pi}^2}{k''^2 U_{hi}^2} \ll 1 \quad \xi, \gamma < 1$

etc., the three terms other than unity in bracket are less than unity in which case we may make a binomial expansion to get the $\text{Im} K_{ii}''$ as

$$\text{Im} K_{ii}'' = \frac{\omega'' k'' (\alpha + T)^{1/2}}{\alpha^{1/2} U_{hi}^2} \left[ \frac{\sqrt{\pi T} \xi}{(\alpha + T)^{1/2}} \frac{\omega_{pi}^2}{k''^2 U_{hi}^2} \frac{Z_i (1 + T)}{(\alpha + T)^{1/2}} \right]$$

(29)

The first term in equation (29) is smaller than the second on account of $\xi$, hence we neglect it to get $\text{Im} K_{ii}''$ as

$$\text{Im} K_{ii}'' = -\frac{1}{2} \frac{\omega'' k'' (\alpha + T)^{1/2}}{\omega_{pe} \omega_{pi}} \frac{\omega_{pi}^2}{k''^2 U_{hi}^2} \frac{Z_i (1 + T)}{(\alpha + T)^{1/2}}$$

(30)

which can be rearranged as

$$\text{Im} K_{ii}'' = \left[ \frac{m_e}{m_i} \right]^{1/2} - \frac{1}{2} \frac{\sqrt{\alpha \gamma}}{\omega_{pe} \omega_{pi}} \frac{U_{hi}}{U_{hi}} \frac{\omega_{pi}}{\omega_{pe}} (\alpha + T)^{1/2} (31)$$

Thus all those HFCLC modes which have their perpendicular phase velocities

$$\left[ \gamma = \frac{\omega''}{k'' U_{hi}} \right]$$

such that $\gamma Z_i (\gamma)$ is $> 0$ will grow spatially with a growth length given by equation (31). The modified critical length for a mirror machine where these modes are excited is accordingly given by

$$L_m = 2 \left[ \frac{m_i}{m_i} \right]^{1/2} \frac{1}{\gamma Z_i (\gamma)} \frac{U_{hi}}{U_{pi} \omega_{pe}} \left( \frac{\omega_{pe}}{\omega_{pi}} \right) \sqrt{\alpha \gamma} / (\alpha + T)^{1/2}$$

(32)
Using equation 26, we may write equation (32) as

\[ L_{\infty} = \frac{L}{\sqrt{\frac{\alpha T}{(1+\alpha)}}} \]  \hspace{1cm} (33)

where \( L \) is the critical length in the absence of the beam. Since

\[ \sqrt{\frac{\alpha T}{(1+\alpha)}} > 1 \]

we see that the critical length is reduced thereby endangering the open-ended confinement.

We now make a numerical estimate of this reduction of critical length in actual experimental situations. In Constance II mirror machine (12), the electron beam was injected to suppress DCLC fluctuations. Hollow beam (1 cm diameter, 0.1 cm thick) of 8 KV, 7A was injected in a plasma produced by the TІ-washer gun with following parameter: \( n_p = 2 \times 10^{13} \text{ cm}^{-3} \), \( T_I = 400 \text{ eV} \), \( T_e = 10 \text{ eV} \). In the experiment \( \beta \) of the plasma was \( \lesssim 4 \times 10^{-3} \), a low-\( \beta \) plasma. The critical length in the absence of wave reflection at the ends was about 200 cm, while with the wave reflection (due to different mechanism mentioned earlier), the critical length was a few cms. In the experiment, without the beam, no oscillations at \( \omega \) were observed which implies that there was no significant wave-reflection at the ends and that the waves were damped by the electrons at the throats before they could grow to a significant level. We can apply our results as we have assumed a low-\( \beta \) plasma and the absence of wave-reflections at the ends. During the injection strong signals at 30 GHz were observed which confirmed the existence of Langmuir turbulence in the machine \( (\omega_{\text{pl}} \lesssim 30 \text{ GHz for } n_p \lesssim 10^{13} \text{ cm}^{-3}) \). In the experiment electron gun and plasma gun were fired simultaneously hence as stated earlier HFCLC modes will encounter a broad, energetic spectrum with \( \Delta \lesssim K_0, W \lesssim 0.2 \).
From the electron beam parameter, the beam density comes out to be
\[ n_b = 10^{10} \text{ cm}^{-3}, \quad v_b = 5 \times 10^9 \text{ cm/sec}. \]
As typically \( k'' = \frac{\omega_b}{\Omega_{tb}} \),
\[ \Delta = \frac{k_o}{\omega_{pr}/v_o}, \quad \nu \geq 0.2, \quad \nu (q/\Delta)^2 = 20 \text{ and } \]
\[ a = \frac{\omega_b c^2}{\omega_c e} = 10 \text{ so that } a^2 = 3. \]
Thus \( \sqrt{a} \tau/\varepsilon = 12 \), which gives \( L_m \approx L/12. \) From this it is clear that in these experiments
the critical length is expected to be reduced by at least an order of magnitude i.e. from a few hundred to a few tens of cm. This can seriously jeopardize the mirror confinement.

4. **Electron Gun Fired After the Plasma Gun:**

Let us now see whether this situation can be salvaged by delaying the injection of the beam. Such situations are generally encountered in mirror experiments employing relativistic electron beam for heating purposes (13). If the injection of electron beam is delayed by a few \( \mu s \) after the plasma gun turn off, then the HFCLC modes will encounter the Langmuir wave-spectrum in its initial stage. In this stage the spectrum is narrow, the width is typically given by \[ \frac{\omega_{pr}}{v_o} \left( \frac{v_o}{\Omega_p} \right)^{1/3} \] and contains roughly \( 1/3 \) energy of the beam (11). Hence \( \nu = \frac{1}{3} \frac{n_b m_e v_b^2}{n_b m_e \Omega_{tb}^2} \).

In equation (24) this situation is characterized by
\[ \epsilon_Q = \frac{k_o l_b m_0}{\lambda} \left( \frac{\omega''}{k''}, \frac{1}{V_{\theta}} \right) \gg 1 . \]
In this limit \( Z(\epsilon_Q) \) is given as
\[ Z(\epsilon_Q) = \left[ 1 - \frac{1}{\epsilon_Q} - \frac{1}{2} \epsilon_Q^3 \right] \]
\[ (34) \]

Using this approximation in equation (24) we have \( k_{n}''^2 \) as
\[ k_{n}''^2 = \frac{\omega_{pr}^2}{\omega_{pe}^2} \left[ X \left\{ 1 + \frac{\omega_{pe}^2}{\omega_c e^2 x} \right\} - \frac{\Omega_{tb}^2}{k_{H''}^2 \varepsilon} X \right] \]
\[ (35) \]
where \[ X = \left[ 1 - W \left( \frac{\omega^*}{\omega} \right)^2 \left( \frac{k''}{\Delta} \right)^{\frac{1}{2}} \right] \]

Typically as we will show later \[ \frac{\omega_{pc}^2}{\omega_{cc}^2} \left| x \right| \] is \( \geq 1 \) in which case we may write \( k''_{II} \) (after evaluating the square root as

\[ k''_{II} = \frac{\omega''}{\omega_{pc}} \frac{\omega_{pc}}{\omega_{cc}} \left[ 1 - i \frac{\omega_{pc}^2}{k''_{II} \omega_{cc}^2} \right] \]

which gives the modified critical length \( L_m \) as

\[ L_m = \frac{L}{\left| X \right|} \]

Let us now evaluate \( |x| \) for the parameters of Constance II. Using the expressions for \( W \) and \( a \) given before we have for Constance II parameters \( W = 1/10, a = 10 \left( \frac{k''/\omega'}{\Delta} \right)^{\frac{1}{2}} \) so that \( W \left( \frac{k''}{\omega'} \right)^{\frac{1}{2}} \frac{a}{\sqrt{a}} \geq 5 \) so that

\[ |x| \leq 4 \]

and as \( \frac{\omega_{pc}^2}{\omega_{cc}^2} \geq 10 \) we have \( \frac{\omega_{pc}^2}{\omega_{cc}^2} \left| X \right| \geq 2 \), \( 3 \leq 1 \)

thus justifying our approximation stated before. We remark that this approximation will all the more hold for future generation of high density mirror machines. From equation (38) we have that typically for present day mirror machine with electron beam injection \( L_m \geq 1/4 \), the critical length is reduced by about a factor of four. This is not as dangerous as the previous case where the two guns were fired simultaneously but reduction of the critical length by four times can seriously risk the mirror confinement.
5. **Discussion:**

From these calculations we see that there exists a definite risk of the worsening of mirror-confinement in the machines which employ parallel injection of electron beam for the purpose of controlling drift cyclotron loss cone turbulence or for heating purposes (REB). If the plasma and the beam gun are fired simultaneously the critical length is reduced by about an order of magnitude, this can seriously jeopardize the confinement. On the other hand we see that if we delay the injection of electron beam i.e. fire it a few \( \mu \) s after the plasma gun firing the critical length is still reduced by about four to five times. Thus we do not gain much by way of improving the confinement or eliminating the danger associated with the previous case. On the other hand, there is yet another strong reason against delaying the injection of the electron beam. There is a strong possibility that the very purpose of electron beam injection is defeated.

As stated earlier, after the firing of plasma gun, it takes a few ion transit time (a few \( \mu \) s) for DCLC instabilities to get triggered. These instabilities saturate very quickly (on the time scale of \( \frac{1}{\Omega_\perp} \), about 1/10 of 1 \( \mu \)s for typical real situations) by diffusion in velocity space (14). In that process they push substantial amount of plasma in the loss cone from where it is lost. By injecting electron beams one tries to inhibit this process by super thermal electrons which are generated a few \( \mu \)s after the injection of the beam. Thus if electron beam injection is delayed by a few \( \mu \)s then it may so happen that by the time the hot electrons appear DCLC turbulence has already developed and saturated resulting in the concomittant particle loss. For precisely these reasons it has been found safer to fire the two guns simultaneously. Thus we see that by delaying the injection of
electron beam we do not gain much by way of improving the confinement on the contrary we run the risk of defeating the very purpose of electron beam injection. Clearly we see that this harmful effect of electron beam injection is unavoidable hence serious. We remark that if from the reactor point of view we do a more exact calculation including all the mechanism for wave-reflection and high-$f^2$ effects the critical length will be further reduced making the confinement still worse. This calculation shows that even in the simplest case electron beam can be harmful enough.

In view of this discussion we suggest the technique of electron cyclotron resonance heating employed by Ioffe et al for creating the hot electrons to suppress the DCLC instabilities. It is much safer and free from these complications.
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