CHAPTER V

Role of Compactness in Normality Conditions for Operators
In this chapter we discuss the crucial role of compactness in conditions implying normality of operators. In section 1 we show that if \( T \) is an operator with compact imaginary part such that all its invariant subspaces are reducing then \( T \) is reduction isoloid and Weyl's theorem holds for \( T \). The theorem also serves as a structure theorem for such operators and describes the spectrum completely. Moreover under the additional condition \( \omega(T) \) is countable, \( T \) becomes normal also.

In section 2, we study various normality conditions based on compactness for reduction, reduction spectraloid and reduction normaloid operators and conclude the chapter with a characterisation of convexoid operators through discs containing \( \sigma(T) \).

1. Operators with a Compact Imaginary Part:

Using the fact that a compact operator has always a nontrivial invariant subspace Andô [1] showed that a compact operator is normal if and only if all its invariant subspaces are reducing. Saitô [57] extended Andô's theorem to polynomially compact operators by using the Bernstein-Robinson theorem on invariant subspaces and further raised the question whether Andô's theorem would hold for operators with a compact imaginary part. The answer is not conclusive in this case since the existence of an invariant subspace for such an operator is as yet unknown. However our first theorem in this section answers
Theorem 5.1.1 [52]: If $T \in \mathcal{B}(\mathcal{H})$ is an operator with compact imaginary part such that all its invariant subspaces are reducing, then

1. $T$ is reduction isoloid.
2. Weyl's theorem holds for $T$.

Moreover one can write $T = T_1 \oplus T_2$, where $T_1$ is normal with pure point spectrum and $\sigma(T_2) = \omega(T_2)$ is real. If in addition $\omega(T)$ is countable then $T$ is normal.

Proof:- Let $\lambda_0$ be an isolated point in the spectrum of $T = A + iB$. Since every direct summand $T_i$ of $T$ has the property that $T_i$ has compact imaginary part and all its invariant subspaces are reducing it is sufficient to prove that $\lambda_0$ is an eigenvalue of $T$.

Let $B_0$ be the spectral projection associated with $\lambda_0$. Then $R(B_0)$ is invariant under $T$ and hence reduces $T$. Further $\sigma(T_{R(B_0)}) = \{ \lambda_0 \}$ and $R(B_0)$ reduces $A = T + T^*$ and $B = T - T^*$ also. Set $T_0 = T_{R(B_0)}$, $A_0 = A_{R(B_0)}$ and $B_0 = B_{R(B_0)}$.

Case 1:- $R(B_0)$ is finite dimensional. Then $T_0$ is a finite dimensional operator with $\sigma(T_0) = \{ \lambda_0 \}$.

Hence $\lambda_0 \in \pi_0(T_0)$ or $\lambda_0 \in \pi_0(T)$.

Case 2:- Suppose $R(B_0)$ is infinite dimensional. Then since $\sigma(T_0) = \{ \lambda_0 \}$ and $T_0$ has compact imaginary part.
Thus $A_0$ is a normal operator whose Weyl spectrum is finite. Hence $A_0$ is polynomially compact [6, Theorem 6.4].

We claim that $T_0$ is polynomially compact. Let $p(\lambda) = a_0 + a_1 \lambda + a_2 \lambda^2 + \cdots + a_n \lambda^n$ be the polynomial such that $p(A_0)$ is compact. Then

$$p(T_0) = a_0 I + a_1 (A_0 + B_0) + \cdots + a_n (A_0 + B_0)^n = p(A_0) + \text{terms involving } B_0.$$ 

Since $p(A_0)$ and $B_0$ are compact operators, clearly $p(T_0)$ is compact. Thus $T_0$ is a polynomially compact operator all of whose invariant subspaces are reducing. Hence $T_0$ is normal by Saito's theorem on polynomially compact operators.

Since $\sigma(T_0) = \{ \lambda_0 \}$ now yields

$$\lambda_0 \in \mathcal{W}_T(T_0) \quad \text{or} \quad \lambda_0 \in \mathcal{W}_T(T),$$

in fact in either case $\lambda_0$ is a pole of $R_{\lambda}(T_0)$ of order 1 and consequently is a pole of $R_{\lambda}(T)$ of order 1. [17, Theorem 20, Chapter VII]. Hence $T$ is reduction isoloid.

Since $T$ is reduction isoloid and the eigenspaces of $T$ reduce $T$, Weyl's theorem holds for $T$ [6, Theorem 5.1], and if in addition $\omega(T)$ is countable then $T$ is a diagonal normal operator [9, Theorem 2]. If $\mathcal{M}$ is the closed linear span of the reducing eigenspaces of $T$ then $T_1 = T_{/\mathcal{M}}$ is a normal operator with pure point spectrum. $\mathcal{M}^\perp$ also reduces $T$ and hence reduces $A$ and $B$ also. Set $T_2 = T_{/\mathcal{M}^\perp}$ and $A_2 = A_{/\mathcal{M}^\perp}$.
Then since \( \mathcal{T}_2 \) has no eigenvalues, \( \sigma(\mathcal{T}_2) = \omega(\mathcal{T}_2) = \omega(A_2) \) and hence \( \sigma(\mathcal{T}_2) \) is real.

2. Reduction \( G_1 \), Reduction normaloid and Reduction spectraloid operators:

Throughout this section we shall assume that the underlying Hilbert space \( \mathcal{H} \) is separable.

In the rest of the chapter, we denote by \( \mathcal{N}_1 \) the closed linear span of the reducing eigenspaces of \( \mathcal{T} \). \( \mathcal{T}_1 = \mathcal{N}_1 \) is normal and \( \mathcal{N}_1 \) reduces \( \mathcal{T} \).

Set \( \mathcal{T}_2 = \mathcal{N}_2 \). To determine the normality of \( \mathcal{T} \), it is sufficient to consider the normality of \( \mathcal{T}_2 = \mathcal{N}_2 \). \( \mathcal{N}_1 \) reduces \( A_1 \) and \( B_1 \) also and we denote \( \mathcal{A}/\mathcal{N}_1 \) and \( \mathcal{B}/\mathcal{N}_1 \) by \( A_2 \) and \( B_2 \) respectively.

Saitô [57] showed that a seminormal operator \( \mathcal{T} \) such that \( \langle \mathbf{a} \mathcal{T} + \mathbf{b} \mathcal{T}^* \rangle \) is polynomially compact is normal. In [59] the result was extended to quasihyponormal operators. We show below that the result is true even for reduction \( G_1 \) operators.

Theorem 5.2.1 [52]: A reduction \( G_1 \) operator such that \( \alpha \mathcal{T} + \beta \mathcal{T}^* \) is polynomially compact is normal.

Proof: We shall show that \( \mathcal{T}_2 \) is normal. Let \( \lambda \neq 0 \in \sigma(\mathcal{T}_2) \). Since \( \lambda \in \partial \sigma(\mathcal{T}_2) \), \( \lambda \) is a semibare point of \( \sigma(\mathcal{T}_2) \). Hence by a
We claim that \( \lambda \notin \sigma_{\infty}(T_2) \). If to the contrary \( \lambda \in \sigma_{\infty}(T_2) \) then \( \lambda \) is a normal eigenvalue of \( T_2 \) or \( T \) has a misplaced eigenvector which is a contradiction. Thus \( \lambda \in \sigma_{\infty}(T_2) \) and hence there exists a sequence \( \{ x_n \} \) of unit vectors such that \( x_n \to 0 \) weakly in \( T^* \).

\[ \| (T_2 - \lambda) x_n \| \to 0 \quad \text{and} \quad \| (T_2^* - \lambda^*) x_n \| \to 0. \]

Since \( \| (T_2 + \beta T_2^*) \| \) is finite, it follows that \( \alpha x + \beta T \in \sigma_{\infty}(\alpha T_2 + \beta T_2^*) \). Since \( \alpha T_2 + \beta T_2^* \) is polynomially compact, \( \sigma_{\infty}(\alpha T_2 + \beta T_2^*) \) is finite. Hence

\[ Z = \{ \alpha \lambda + \beta \lambda : \lambda \notin \sigma_{\infty}(T_2) \} \]

is finite. Since \( |\alpha| + |\beta| > 0 \), \( \alpha \lambda_1 + \beta \lambda_1 = \alpha \lambda_2 + \beta \lambda_2 \) whenever \( \lambda_1 = \lambda_2 \).

Hence the finiteness of \( Z \) implies that \( \sigma_{\infty}(T_2) \) and consequently \( \sigma(T_2) \) is finite. Thus \( T_2 \) is a reduction \( G_1 \) operator with finite spectrum and hence is normal.

[9, Theorem 1].

In the case of reduction convexoid or reduction normaloid operators it is not known whether the polynomial compactness of \( \alpha T + \beta T^* \) would imply normality.

However the stronger assertion namely \( \alpha T + \beta T^* \subset \sigma(T) \) is compact guarantees normality in these cases.

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**Theorem of Saito** [5, 3] on the boundary spectra of \( G_1 \) operators, if \( \lambda \) is an eigenvalue (resp. approximate eigenvalue) then \( \lambda \) is a normal eigenvalue (resp. normal approximate eigenvalue). Since \( \sigma(T_2) \subset T_{\infty}(T_2) \cup \sigma_{\infty}(T_2) \), either \( \lambda \in T_{\infty}(T_2) \) or \( \lambda \in \sigma_{\infty}(T_2) \).
Theorem 5.2.2: A reduction spectraloid operator $\tau$ such that $\alpha \tau + \beta \tau^*$ (where $\alpha \neq \beta$) is compact is normal.

Proof: We shall prove that $\tau$ is the zero operator.

Suppose to the contrary $\omega(\tau) \neq 0$. Then there exists a $\lambda \in \sigma(\tau)$ such that $|\lambda| = \lambda(\tau) = \omega(\tau)$.

Clearly $\lambda \in \sigma(\tau)$, Thus either $\lambda \in \Pi_{00}(\tau)$ or $\lambda \in \sigma_e(\tau)$. Suppose $\lambda \in \Pi_{00}(\tau)$.

Since $\lambda \in \sigma_e(\tau)$, it follows that $\lambda$ is a normal eigenvalue of $\tau$ or $\tau$ has a misplaced eigenvector, a contradiction. Thus $\lambda \in \sigma_e(\tau)$ and there exists a sequence $\{\chi_n\}$ of unit vectors such that $\chi_n \to 0$ weakly and $\|\tau^* - \lambda\|\chi_n\| \to 0$ as well as $\|\tau - \lambda\|\chi_n\| \to 0$.

Clearly $\alpha \lambda + \beta \lambda \in \sigma_e(\omega(\tau^2 + \beta \tau^2))$.

Since $\alpha \tau + \beta \tau^*$ is compact, $\sigma_e(\omega(\tau^2 + \beta \tau^2)) = \{0\}$ and thus $\alpha \lambda + \beta \lambda = 0$. Since $\lambda \neq 0$ we have $|\alpha| = |\beta|$.

which is a contradiction. Hence $\omega(\tau) = 0$ or $\tau$ is the zero operator. Thus $\tau = 0$ and $\tau$ is normal.

The technique of the above theorem will not work if $|\alpha| = |\beta|$. However if $\tau$ is a reduction convexoid operator and $\alpha = 1$ and $\beta \neq -1$ i.e., if $\tau$ has compact imaginary part then $\tau$ is normal.

Theorem 5.2.3: A reduction convexoid operator with compact imaginary part is normal.

Proof: Let $\lambda \in E(\omega(\tau))$. Since $\tau$ is convexoid, $\lambda \in \sigma_e(\tau)$, Thus either $\lambda \in \Pi_{00}(\tau)$,
or \( \lambda \in \sigma_c(T_2) \). As in the previous theorems one can show that \( \lambda \in \sigma_c(T_2) = \sigma_c(T_2) \). Thus 
\[
E\left( \bar{W}(T_2) \right) \text{ is real or } T_2 \text{ is self-adjoint. Hence } \]
\( T = T_1 \oplus T_2 \) is normal.

I.H. Sheth [56] showed that a transloid operator \( T \) such that \( AT = T^*A \) for an arbitrary operator \( A \) for which \( 0 \notin \bar{W}(A) \) is self-adjoint. In view of theorem 5.2.3 we extend this result as follows:

Theorem 5.2.4 [52]: If \( T \) is a reduction transloid operator which is also essentially transloid and if \( AT = T^*A + k \) where \( k \) is compact and \( A \) is an arbitrary operator such that \( 0 \notin \bar{W}(A) \) then \( T \) is normal.

Proof: \( AT = T^*A + k \Rightarrow A \uparrow = \uparrow^*A \)

Since \( \uparrow \) is transloid and \( 0 \notin \bar{W}(A) \) it follows that \( \uparrow \) is self-adjoint [56], i.e. \( T - T^* = 0 \) or \( T - T^* \) is compact. That \( T \) is normal follows from theorem 5.2.3.

The importance of this theorem lies in the fact that it offers a simple proof to the following well-known result [46] relating to hyponormal operators.

Corollary 5.2.1: A hyponormal operator \( T \) such that \( AT = T^*A + k \) where \( k \) is compact and \( A \) is an operator such that \( 0 \notin \bar{W}(A) \) is normal.
Our next theorem gives a normality condition for reduction normaloid operators based only on the behaviour of the numerical range.

**Theorem 5.2.5** [2]: If \( T \) is reduction normaloid and \( W(T\pi) = \overline{W(T\pi)} \) for every reducing subspace \( \pi \) of \( T \) then \( T \) is normal.

**Proof:** Since \( T \) is normaloid, there exists a \( \lambda \in \sigma(T) \) such that \( |\lambda| = \|T\pi\| \). Now \( \lambda \in \overline{W(T\pi)} = W(T\pi) \) implies that \( \lambda \) is a normal eigenvalue of \( T \) in other words \( T \) has a misplaced eigenvector, a contradiction. Thus \( \overline{W(T\pi)} = \{0\} \) or \( T \) is normal.

We shall say that an operator \( T \) is of class-\( ? \) if \( E(\overline{W(T)}) \cap W(T) \subset T_0(T) \). The following theorem gives a normality condition for reduction-\( ? \) operators.

**Theorem 5.2.6** [2]: A reduction-\( ? \) operator with compact imaginary part is normal.

**Proof:** We first observe that \( \overline{W(T_2)} = W_e(T_2) \). Suppose \( \lambda \in E(\overline{W(T_2)}) \). Since \( E(\overline{W(T_2)}) \subset W(T) \cup W_e(T_2) \) either \( \lambda \in W(T) \) or \( \lambda \in W_e(T_2) \). If \( \lambda \in W(T) \) then clearly \( \lambda \) is a normal eigenvalue of \( T_2 \) which leads to a contradiction via a misplaced eigenvector. Hence \( E(\overline{W(T_2)}) \subset W_e(T_2) \) or \( \overline{W(T_2)} = W_e(T_2) \).

Since \( T \) has compact imaginary part, \( B_2 \) is compact. Hence \( \overline{W(T_2)} = W_e(T_2) = W_e(A_2) \) which is real. Thus
is real or \( T \) is self-adjoint and hence
\[ T = T_i \oplus T_b \]
is normal.

It is known that an essentially convexoid operator need not be convexoid [30]. However an essentially convexoid operator of class- \( P \) is necessarily convexoid.

**Theorem 5.2.7** [30]: An essentially convexoid operator of class- \( P \) is convexoid.

**Proof:** Let \( \lambda \in E(\overline{W(T)}) \). Then either \( \lambda \in W(T) \) or \( \lambda \in W_\infty(T) \). If \( \lambda \in W(T) \) then \( \lambda \in \pi_\eta(T) \).

If \( \lambda \in W_\infty(T) \) then \( \lambda \in E(W_\infty(T)) \) and since \( \overset{\circ}{T} \) is convexoid, \( \lambda \in \sigma(T) \subset \sigma(T) \). In either case
\[ E(\overline{W(T)}) \subset \sigma(T) \text{ or } \overline{W(T)} \subset \xi(T). \]
Thus \( E(T) = \overline{W(T)} \) and \( \overset{\circ}{T} \) is convexoid.

3. Convexoid Operators:

We conclude this chapter with a remark on convexoid operators. Fujii and Nakamoto [30] characterised transloid and normaloid operators in terms of spectral sets and showed that an operator \( T \) is transloid if and only if every disc containing \( \sigma(T) \) is a spectral set for \( T \).

They further showed that \( T \) is transloid if and only if every disc containing \( W(T) \) is a spectral set for \( T \).

While they gave an independent proof of the latter result they claimed that it is a consequence of the former in view of the fact that a disc contains \( \sigma(T) \) if and only if
it contains \( W(\tau) \). The following example shows that a disc containing \( \sigma(\tau) \) need not always contain \( W(\tau) \).

**Example 5.3.1**: If \( \tau = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} \otimes \frac{1}{4} I \) then the circle with centre \((\frac{1}{4}, 0)\) and radius \(\frac{1}{4}\) contains \(\sigma(\tau)\) but not \(W(\tau)\).

However if \( \tau \) is convexoid, every disc containing \(\sigma(\tau)\) also contains \(\mathcal{Z}(\tau) = \overline{W(\tau)}\). The following theorem shows that this property characterises convexoid operators.

**Theorem 5.3.1** [55]: An operator \( \tau \) is convexoid if and only if every disc containing \(\sigma(\tau)\) also contains \(W(\tau)\).

**Proof**: Since the necessity is obvious we need only prove the sufficiency. For any complex \( \lambda \), \( \lambda (\tau - \lambda) D + \lambda \) contains \(\sigma(\tau)\) and hence \(W(\tau)\). Thus \(W(\tau - \lambda) \subseteq \mathcal{Z}(\tau - \lambda)\) or \(\omega(\tau - \lambda) \subseteq \mathcal{Z}(\tau - \lambda)\). In other words \(\tau - \lambda\) is spectraloid for all complex \( \lambda \) and hence by a theorem of Furuta and Nakamoto [23] \( \tau \) is convexoid.