CHAPTER III

TOPOLOGICAL PROPERTIES OF OPERATORS
In this chapter we study some topological properties of different classes of operators. Using the continuity of the numerical range and that of the centre of mass w.r. to the norm topology we show that the classes of centroid and centraloid operators are norm closed. Further we show by using a technique of Luecke and the faithful representation of Berberian that if dim $H > 2$, these two classes are nowhere dense in $B(H)$. This modification enables us to give a simpler proof of a similar result of Luecke on convexoid operators and to improve a result of Patel on a topological property of spectraloid operators.

Section 2 deals with the Halmos class $\mathcal{R}_1$. Denoting the classes of essentially convexoid and essentially spectraloid operators by $\mathcal{E}(\mathcal{B})$ and $\mathcal{E}(\mathcal{F})$ respectively we show that $\mathcal{E}(\mathcal{B}) + \mathcal{K}$ and $\mathcal{E}(\mathcal{F}) + \mathcal{K}$ are contained in $\mathcal{R}_1$ where $\mathcal{K}$ is the ideal of compact operators on $H$. Similar results are proved for essentially centroid and essentially centraloid operators also.

1. Topological Properties:

Throughout this section we shall assume that $B(H)$ has the norm topology. Since $\sigma(\tau)$ and $\overline{w(\tau)}$ are functions with domain $B(H)$ and their range consists of compact subsets of the complex plane it is natural to ask whether they are continuous. To study the continuity of $\sigma(\tau)$ and $\overline{w(\tau)}$, Halmos [26] invoked the Hausdorff metric
topology for compact subsets of the plane. He showed that while $\omega(T)$ and $\omega^{(T)}$ are continuous functions, $\sigma(T)$ and $\lambda(T)$ need only be upper semicontinuous functions w.r. to the norm topology. The following lemma puts in a nutshell the behaviour of $\sigma(T)$, $\omega(T)$, $\omega^{(T)}$ and $\lambda(T)$ under a small perturbation.

Lemma F: [17, 26] - Let $T \in \mathcal{B}(H)$ and let $\epsilon > 0$.

Then there is a $\delta > 0$ such that if $\|T - T_0\| < \delta$ then $\sigma(T) \subset \sigma(T_0) + \epsilon$, $\omega(T) \subset \omega(T_0) + \epsilon$, and

$$\|\omega(T) - \omega(T_0)\| < \epsilon$$

wherein for any subset $S$ of the plane, $S + (\epsilon)$ stands for the set $\{z : d(z, S) < \epsilon\}$.

Our first theorem in this section is:

Theorem 3.1.1: The class $\mathcal{J}$ of centroid operators is a norm closed subset of $\mathcal{B}(H)$.

Proof: - Let $\{T_n\}$ be a sequence of centroid operators such that $\|T_n - T\| \to 0$. We shall show that $M = W_T$ so that $T$ is centroid by theorem 2.2.2.

Since each $T_n$ is centroid, $Z_{T_n} = m_{T_n} = W_{T_n}$ and $R_{T_n} = W_{T_n} = M_{T_n}$. Stampfli [63] showed that the mapping $T \mapsto m_T$ is continuous w.r. to the norm topology.

Thus $\lim_{n \to \infty} m_{T_n} = m_T$ and $\lim_{n \to \infty} \|T_n - m_{T_n}\| = \lim_{n \to \infty} \|T - m_T\| = \lim_{n \to \infty} M_{T_n} = \lim_{n \to \infty} W_{T_n} = \lim_{n \to \infty} W_{T_n}$.

Also by lemma F, there exists a positive integer $N$ such that for all $n \geq N$, ...
\[
\overline{W(T_n)} \subset \overline{W(T)} + (\varepsilon) \subset D_T + (\varepsilon)
\]

and
\[
\overline{W(T)} \subset \overline{W(T_n)} + (\varepsilon) \subset D_{T_n} + (\varepsilon)
\]

Hence for all \( n \geq N \)
\[
W_{T_n} - \varepsilon \leq W_T \leq W_{T_n} + \varepsilon \quad \text{or}
\]
\[
\lim_{n \to \infty} W_{T_n} = W_T . \quad \text{Thus } M_{T_n} W_T \text{ and } T \text{ is centroid.}
\]

The faithful \( * \) representation \( T \to \mathfrak{T}^* \) defined by Berberian [6] plays a vital role in the ensuing theorems. This representation is isometric and \( * \) isomorphic with
\[
\pi(T) = \pi(T^*) = \pi_0(C^0) \quad \text{and} \quad W(T) = W(T^*) .
\]

Using the Berberian representation and a technique of Luecke [36] we show,

Theorem 3.1.2 [54] :- If \( \dim H > 1 \), the class \( Z \) of centroid operators is nowhere dense.

Proof:- In view of theorem 3.1.1, it is sufficient to show that \( \text{Int}(Z) = \emptyset \).

We shall first show that if \( T \in \text{Int}(Z) \) then \( \sigma(T) \) must have at least two points. Suppose \( \sigma(T) = \{ \alpha \} \).

Since \( T \) is centroid, \( T = \alpha T \) by theorem 1.4.1. Write
\[
H = M \oplus M^\perp
\]
where \( M \) is a two dimensional subspace of \( H \). Given \( \varepsilon > 0 \), we define an operator \( A \) on \( H \) as follows:
\[
A = \begin{bmatrix} \varepsilon & 0 \\ 0 & 0 \end{bmatrix} \quad \text{on } M \quad \text{and} \quad A = 0 \quad \text{on } M^\perp .
\]
Then
\[
\sigma(T + A) = \{ \alpha \} . \quad \text{Suppose } T + A \text{ is centroid.}
\]
Then by theorem 1.4.1, \( T + A = aI \) or \( A \) is the zero operator which is a contradiction. Thus \( T + A \) is not centroid. However \( \| T + A - T \| = \| A \| = \epsilon \). Since \( \epsilon \) is arbitrary it follows that \( T \notin \text{Int}(Z) \).

Suppose \( \sigma(T) \) has two or more points. We shall show that \( T \notin \text{Int}(Z) \). Since the class of centroid operators is translation invariant (Theorem 1.1.1) we may assume without loss of generality that \( z_\gamma = 0 \). Since \( z_\gamma \in \text{convexhull} [\sigma(T) \cap C_T] \) [Lemma 1.1.3], there exists two or three distinct points in \( \sigma(T) \cap C_T \) such that \( z_\gamma = 0 \) is their convex hull. Let \( \lambda \) and \( \mu \) be the two such points. Since \( |\lambda| = |\mu| = R_T \| T \| \), \( \lambda \) and \( \mu \) are normal approximate eigenvalues of \( T \). Let \( T^0 \) be the image of \( T \) under the Berberian representation. Since \( T \) is centroid if and only if \( T^0 \) is centroid, it is sufficient to show that \( T^0 \) is not in the interior of the class of centroid operators on \( \mathcal{K} \) where \( \mathcal{K} \) is the extension of \( H \) in the Berberian representation. Thus we can assume that \( \lambda \) and \( \mu \) are normal eigenvalues of \( T \).

If \( x \) and \( y \) are the corresponding eigenvectors then clearly \( x \bot y \). Let \( M \) be the closed linear span of \( x \) and \( y \). Define an operator \( S \) on \( H \) as follows:

\[
Sx = \epsilon y, \quad Sy = \lambda x
\]
and \( S \varphi = \varphi \) for all \( \varphi \in M^\perp \) where \( \epsilon > 0 \) is arbitrary. Then \( T + S = A \otimes B \) where

\[
A = \begin{bmatrix}
\lambda & \epsilon \\
0 & \mu
\end{bmatrix}
\]
on \( M \) and \( B = \frac{T}{\| T \|} \) on \( M^\perp \).
Thus \( \sigma(T+S) = \sigma(T) \) and therefore \( R_{T+S} = R_T \).

Also \( z_{T+S} = z_T = 0 \). But \(|T+S| \geq \sqrt{\|T\|^2 + \|S\|^2} > |\lambda| = R_T = R_{T+S} \). Thus \( T+S \) fails to be centroid.

But \( \|T+S-T\| = \|S\| = \epsilon \). Since \( \epsilon \) is arbitrary it follows that \( T \notin \text{Int}(\mathcal{Z}) \).

The technique of switching over to the Berberian representation helps us to give a simpler proof of a theorem of Luecke [36] on convexoid operators.

Theorem 3.1.3 [34] : If \( \dim H > 2 \), the class \( \mathcal{G} \) of convexoid operators on \( H \) is nowhere dense.

Proof : Since \( \mathcal{G} \) is norm closed [36] it is sufficient to show that \( \text{Int}(\mathcal{G}) = \emptyset \). Luecke [36] showed that if \( T \in \text{Int}(\mathcal{G}) \) then \( \sigma(T) \) must have at least two points.

We shall show that if \( \sigma(T) \) has two or more points then \( T \notin \text{Int}(\mathcal{G}) \).

Since \( \sigma(T) \) has two or more points, \( E(\mathcal{R}(T)) \) contains at least two points say \( \lambda \) and \( \mu \). By Krein-Milman theorem, \( \lambda \) and \( \mu \) are approximate eigenvalues of \( T \) and since \( \lambda, \mu \in \sigma(T) \), they are normal approximate eigenvalues of \( T \) also [28]. Since \( T \) is convexoid if and only if \( T^* \) is convexoid, as in theorem 3.1.2, we may assume that \( \lambda \) and \( \mu \) are normal eigenvalues of \( T \). Let \( S \) be as in theorem 3.1.2. Then \( \|T+S-T\| = \epsilon \) but \( T+S \) is not a convexoid operator [36]. Hence \( T \notin \text{Int}(\mathcal{G}) \) or \( \text{Int}(\mathcal{G}) = \emptyset \).

Let \( \mathcal{Y} \) denote the class of spectraloid operators.
on $H$. Patel [36] showed that if $T \in \mathcal{G}$ and if there are two points $a$ and $b$ such that

1. $a, b \in \partial \mathcal{H}(T) \cap \Pi_0(T)$
2. $|a| = \omega(T)$

then $T \notin \text{Int} (\mathcal{G})$.

We modify this result in the next theorem. For this we need the notion of the peripheral spectrum of $T$.

Definition 3.1.1: The peripheral spectrum $\sigma_{p(T)}$ of $T$ is the set

$\{ z : z \in \sigma(T) \text{ and } |z| = \lambda(T) \}$.

Theorem 3.1.4: [54] If $T \in \mathcal{G}$ and $\sigma_{p(T)}$ has two or more points then $T \notin \text{Int} (\mathcal{G})$.

Proofs: Let $\lambda, \mu \in \sigma_{p(T)}$. Since $|\lambda| = |\mu| = \lambda(T) = \omega(T)$, $\lambda$ and $\mu$ are normal approximate eigenvalues of $T$. Since $T$ is spectraloid if and only if $T^*$ is spectraloid as in the previous theorems we may assume that $\lambda$ and $\mu$ are normal eigenvalues of $T$. Let $S$ be as in the above theorems. Then

$\sigma(T + S) = \sigma(T)$ and hence $\lambda(T + S) = \lambda(T)$.

Now $\mathcal{W}(A)$ is the closed elliptical disc with foci at $a$ and $b$ [26]. Therefore $|a| < \omega(A)$ and hence

$\omega(T) = |a| < \omega(A)$. Since $\omega(T + S) = \max \{ \omega(A), \omega(B) \}$,

$\omega(T + S) = \omega(A) > \omega(T) = \lambda(T) = \lambda(T + S)$. Thus though $\|T + S - T\| = \varepsilon$, $T + S$ fails to be spectraloid or $T \notin \text{Int} (\mathcal{G})$.

The next theorem illustrates the topological properties of the class $V$ of centraloid operators.
Theorem 3.1.5: The class $V$ of centraloid operators is norm closed and if $d(x, H) \geq \delta$, then $V$ is nowhere dense.

Proof: Let $\{T_n\}$ be a sequence of centraloid operators such that $\|T_n - T\| \to 0$. We shall show that $R_T T_n \to R_T$.

Suppose $R_T T_n \to T = W_T$. Set $\epsilon = d/4$ where $d = W_T - R_T$.

By lemma 2, there exists a positive integer $N_1$ such that for all $n > N_1$,

$$\sigma(T_n) \subset \sigma(T) + (\epsilon) \subset d + (\epsilon)$$

Thus for all $n > N_1$, $R_T T_n = W_{T_n} \leq R_T + \epsilon$ --- I

Also $\lim_{n \to \infty} W_{T_n} = W_T$ [Theorem 3.1.1]. Hence there exists a positive integer $N_2$ such that for all $n > N_1$,

$$W_T - \epsilon \leq W_{T_n} \leq W_T + \epsilon$$

--- II

If $N_0 \geq \max \{ N_1, N_2 \}$, then we have

$$R_{T_{N_0}} = W_{T_{N_0}} \leq R_T + d/4$$

from I and

$$R_T + 3d/4 \leq R_{T_{N_0}} \leq W_{T_{N_0}} \leq R_T + 5d/4$$

from II

which is absurd. Hence $R_T = W_T$ or $T$ is centraloid.

Suppose $\dim H \geq 2$. We shall show that

$\text{Ind}(V) = \phi$. Since a quasinilpotent centraloid operator is necessarily the zero operator it is sufficient to consider the case when $\sigma(T)$ has two or more points. Since the class of centraloid operators is translation invariant we may assume $x_T = W_T = 0$. Further by lemma 1.1.3, there exist two or three distinct points in $\sigma(T) \cap C_T$ such that
...is in their convex hull. If \( \lambda \) and \( \mu \) are two such points, since \( |\lambda| = |\mu| = \omega(\tau) = \omega(\tau) \), they are normal approximate eigenvalues of \( T \). As in theorem 3.1.2, we may assume that \( \lambda \) and \( \mu \) are normal eigenvalues of \( T \).

If \( S \) is the operator defined in theorem 3.1.2, then
\[
\sigma(T+S) = \sigma(T) \quad \text{and hence} \quad x_{T+S} = 0.
\]
Thus
\[
T+S - x_{T+S} = T+S,
\]
which as shown in theorem 3.1.4 is not spectraloid. Hence though \( \|T+S-T\| = \epsilon \), \( T+S \) fails to be centraloid or \( T \notin \text{Int}(V) \Rightarrow \text{Int}(V) \neq \emptyset \).

Using the same technique as in theorem 3.1.5, we show below that the class \( \mathcal{Y} \) of spectraloid operators on \( H \) is norm closed.

Theorem 3.1.6 [54]: The class \( \mathcal{Y} \) of spectraloid operators on \( H \) is norm closed.

Proof: Let \( \{T_n\} \) be a sequence of spectraloid operators such that \( \|T_n - T\| \to 0 \). We shall show that \( \lambda(T) = \omega(T) \).

Suppose \( \lambda(T) \neq \omega(T) \). Set \( \epsilon = \frac{d}{4} \) where
\[
d = \omega(T) - \lambda(T)\]
By lemma F, there exists a positive integer \( N \) such that for all \( \tau, N \),
\[
\lambda(T_n) \leq \omega(T_n) \leq \lambda(T) + \epsilon \quad (I)
\]
and \( \omega(T) - \epsilon \leq \omega(T_n) \leq \omega(T) + \epsilon \quad (II) \)

From \( I \) and \( II \) we have,
\[
\lambda(T_n) \leq \lambda(T) + \frac{d}{4} \quad \text{and} \quad \lambda(T) + \frac{3d}{4} \leq \lambda(T_n) = \omega(T_n) \leq \lambda(T) + \frac{5d}{4}.
\]
which is absurd.

Hence \( \lambda(\tau) = \omega(\tau) \) and \( \tau \) is spectraloid.

**Topological Properties of Operators in the Calkin Algebra:**

We conclude the section with some observations on the topological properties of operators in the Calkin Algebra. Since \( \mathcal{B}(H)/\mathcal{K}(H) \) is a \( \mathcal{C}^* \)-algebra, by the Gelfand-Naimark representation theorem there exists a Hilbert space \( H_0 \) and a unital *-representation \( \gamma : \mathcal{B}(H)/\mathcal{K}(H) \rightarrow \mathcal{B}(H_0) \). Further the mapping \( \alpha \rightarrow \gamma(\alpha) \) is an algebra homomorphism of \( \mathcal{B}(H)/\mathcal{K}(H) \) into \( \mathcal{B}(H_0) \). It is known that \( \sigma(\hat{\tau}) = \sigma(\gamma(\hat{\tau})) \) for \( \hat{\tau} \in \mathcal{B}(H)/\mathcal{K}(H) \) \[16\]

In fact a similar result holds for \( \omega_e(\tau) \) also.

**Lemma 3.1.1** [54]: \( \omega_e(\tau) = \sigma(\gamma(\hat{\tau})) \)

**Proof:** Williams [57] showed that if \( x \) is an element of a Banach algebra with unit then

\[
\text{W} (\gamma(\hat{\tau})) = \bigcap_{\mu} \{ \lambda : |\lambda - \mu| \leq \| \gamma(\hat{\tau}) - \mu \| \}
\]

Thus we have,

\[
\text{W} (\gamma(\hat{\tau})) = \bigcap_{\mu} \{ \lambda : |\lambda - \mu| \leq \| \gamma(\hat{\tau}) - \mu \| \}
\]

But since \( \gamma \) is unital,

\[
\| \gamma(\hat{\tau}) - \mu \| = \| \gamma(\hat{\tau}) - \mu \| = \| \hat{\tau} - \mu \|
\]

Hence

\[
\text{W} (\gamma(\hat{\tau})) = \bigcap_{\mu} \{ \lambda : |\lambda - \mu| \leq \| \hat{\tau} - \mu \| \} = \omega_e(\tau)
\]

Lueckë [36] showed that the class \( e(\gamma) \) of essentially convexoid operators on \( H \) is norm closed. Using lemma 3.1.1 and theorem 3.1.6, we extend this result to the class \( e(\gamma) \) of essentially spectraloid operators on \( H \).
Theorem 3.1.7 [54]: The class \( \mathcal{C}(\mathcal{R}) \) of essentially spectraloid operators on \( H \) is norm closed.

Proof:- Let \( \| \tau_n - \tau \| \to 0 \) where each \( \tau_n \) is essentially spectraloid. We need to show that \( \tau \) is essentially spectraloid. In view of lemma 3.1.1 and the fact that \( \sigma(\gamma \cdot \hat{\tau}) = \sigma(\hat{\tau}) \) it follows that \( \tau \in \mathcal{C}(\mathcal{R}) \) if and only if \( \gamma \cdot \hat{\tau} \) is spectraloid in \( \mathcal{B}(H^\omega) \). Since
\[
\| \gamma \cdot \hat{\tau}_n - \gamma \cdot \hat{\tau} \| \to 0
\]
and each \( \gamma \cdot \hat{\tau}_n \) is spectraloid, by theorem 3.1.6, \( \gamma \cdot \hat{\tau} \) is spectraloid. Thus \( \hat{\tau} \) is spectraloid and hence \( \tau \in \mathcal{C}(\mathcal{R}) \).

Luecke [37] introduced the class \( \mathcal{R} \) of operators which satisfy the equality \( \| (\tau - \lambda I)^{-1} \| = \frac{1}{d(\lambda, \mathcal{W}(\tau))} \) for all \( \lambda \notin \mathcal{W}(\tau) \) and showed that this class is norm closed.

Thus similar to theorem 3.1.7 we have,

Theorem 3.1.8: The class \( \mathcal{C}(\mathcal{R}) \) of essentially- \( \mathcal{R} \) operators on \( H \) is norm closed.

2. Halmos Class \( \overline{\mathcal{R}_1} \).

Throughout this section we assume that \( H \) is separable.

While studying irreducible operators Halmos [27] introduced the class \( \overline{\mathcal{R}_1} \) where the bar denotes the closure in the norm topology. The class \( \mathcal{R}_1 \) consists of all operators in \( \mathcal{B}(H^\omega) \) which have a one dimensional reducing subspace. Halmos showed that every isometry and every normal operator is in \( \overline{\mathcal{R}_1} \).
Stampfli [62] showed that the class $\overline{R}$ is much larger. He observed that $T \in \overline{R}$ if and only if $T$ has a normal approximate eigenvalue and showed that if $T$ is of any one of the following forms (1) Hyponormal + compact (2) Toeplitz + compact and (3) $T-\lambda$ is normaloid for some $\lambda$ then $T \in \overline{R}$. Istrătescu [31] showed that Norm closure $\ell_2 G + \mathcal{K} \subset \overline{R}$ where $\ell_2 G$ is the class of all $G$ operators on $H$ and $\mathcal{K}$ is the ideal of compact operators on $H$. We extend this result to essentially convexoid, essentially spectraloid, essentially centroid and essentially centroidoid operators.

The following lemma of Joel Anderson [4] will be used repeatedly in this section.

Lemma 6: If $\lambda \in \sigma(T)$ and if there exists a sequence $\xi_n$ of unit vectors such that $\xi_n \to 0$ weakly and $\| (T-\lambda) \xi_n \| \to 0$ then $\| (T^* - \overline{\lambda}) \xi_n \| \to 0$.

Our first result in this section is

Theorem 3.2.1 [55]: Let $G_h$ be the class of restriction convexoid operators on $H$. Then Norm closure $\ell_2 (G_h + \mathcal{K})$ is contained in $\overline{R}$.

Proof: Let $A = T + k$ where $T \in G_h$ and $k \in \mathcal{K}$. In view of Stampfli's observation it is sufficient to show that $A$ has a normal approximate eigenvalue.

If $\mathcal{M}$ is the closed linear span of the reducing eigenspaces of $T$ then $\mathcal{M}$ is normal. If $\mathcal{M}$ is infinite
dimensional then we are through. Otherwise set \( T_0 = \tau \nabla T \)
and consider \( \lambda \in E(\widetilde{\mathcal{M}}(T_0)) \). Since \( T_0 \) is convexoid,
\( \lambda \in \pi(T_0) \) by Krein-Milman theorem. Clearly
\( \lambda \in \partial \sigma(T_0) \subset \pi_0(T_0) \cup \sigma_c(\hat{T}_0) \). Suppose \( \lambda \in \pi_0(T_0) \).
Then since \( \lambda \in \pi_0(T_0) \cap \partial W(T_0) \) it follows that \( \lambda \) is a
normal eigenvalue of \( T_0 \) \(^{[28]} \) or \( T \) has a misplaced eigen-
vector in \( \mathcal{H}_0 \) which is a contradiction. Thus
\( \lambda \in \sigma_c(\hat{T}_0) \cap \partial W(T_0) \). Hence there exists a sequence
\( x_n \) of unit vectors such that \( x_n \to 0 \) weakly in \( \mathcal{H}_0 \)
and \( \| (T_0 - \lambda) x_n \| \to 0 \) as well as \( || (T_0 - \lambda) x_n || \to 0. \)
Since \( x_n \to 0 \) weakly in \( \mathcal{H} \) also we have \( || (T + k - \lambda) x_n || \to 0 \)
and \( || (T + k - \lambda) x_n || \to 0 \). In other words \( T + k \) has a
normal approximate eigenvalue and the proof is complete.

While discussing the topological properties of ope-
rators in the Calkin algebra Luecke \([38]\) conjectured that
if \( T \) is an essentially \( \mathcal{S}_1 \) or an essentially convexoid
operator then there exists a compact operator \( k \) such that
\( T + k \) is \( \mathcal{S}_1 \) or convexoid respectively. Fujii and Nakamoto
\([21]\) answered the question positively in the case of essen-
tially convexoid operators by using the following theorem
of Chui, Smith, Smith and Ward.

Theorem H \([44]\) : For a given operator \( T \), there exists a
compact operator \( k \) such that
\[ \mathcal{W}(T+k) = \mathcal{W}(T) \quad \text{and} \quad \| T+k \| = \| T+k \| \]

Fujii and Nakamoto \([21]\) showed that the compact
operator \( k \) can be chosen such that \( T+k \) is convexoid

\( \mathcal{H}(T) \)
(resp. normaloid, transloid, of class $\mathcal{R}$) if $T$ is essentially convexoid (resp. essentially normaloid, essentially transloid, essentially $\mathcal{R}$). We shall refer to this result as the Fujii-Nakamoto theorem in the following.

Note: The lemma $G$ has a simple proof via Theorem H as shown below:

Proof of Lemma $G$: Let $\lambda \in \mathcal{W}(T)$ and suppose $T_k$ is a sequence of unit vectors such that $x_n \to 0$ weakly in $H$ and $\| (T - \lambda)x_n \| \to 0$. By Theorem H, there exists a compact operator $K$ such that $W(T + K) = W(T)$. Clearly $\lambda \in \mathcal{W}(T + K)$ and $\| (T + K - \lambda)x_n \| \to 0$. Hence $\| (T^* + K^* - \lambda)x_n \| \to 0$ or $\| (T^* - \lambda)x_n \| \to 0$.

The Fujii-Nakamoto theorem plays a fundamental role in the ensuing theorems.

Theorem 3.2.2 [15]: Let $e(G)$ be the class of essentially convexoid operators on $H$. Then

1. $e(G) + K$ is norm closed and
2. $e(G) + K \subseteq \overline{e(G)}$

Proof: Let $T_{\gamma} = \frac{1}{\gamma} \sum \gamma T_n + k_n \to T$ where $T_n \in e(G)$ and $k_n \in K$. Since $\| T_{\gamma} + k_n - T \| = \| T_{\gamma} - T \| + \| k_n - T \|$, $\| T_{\gamma} - T \| \to 0$ and hence $\| \gamma T_{\gamma} - \gamma T \| \to 0$. Thus the sequence $\{ \gamma T_{\gamma} \}$ of convexoid operators converges to $\gamma \hat{T}$ in $B(H_0)$. Since the class of convexoid operators is norm closed [16] $\gamma \hat{T}$ is convexoid and consequently $\hat{T}$ is convexoid. Thus $T$ is essentially convexoid and by the
Fujii-Nakamato theorem there exists a compact operator $k$ such that $T_1 = T + k$ is convexoid. Clearly $T_1 \in e(\mathbb{G})$, thus $T = T_1 - k$ where $T_1 \in e(\mathbb{G})$ and $k \in K$. Hence $e(\mathbb{G}) + K$ is norm closed.

Let $A \in e(\mathbb{G}) + K$. Then $A = T + k$ where $T \in e(\mathbb{G})$ and $K \subseteq K$. Consider $\lambda \in E(\mathbb{G})$. Since $\mathbb{T}$ is convexoid, $\lambda \in \sigma(\mathbb{T})$. Moreover $\lambda \in \sigma_0(\mathbb{T}) \subseteq \sigma_0(\mathbb{T}) \cup \sigma_0(\mathbb{T})$. Hence there exists a sequence $\{x_n\}$ of unit vectors such that $x_n \to 0$ weakly in $H$ and $\| (T - \lambda) x_n \| \to 0$. By lemma G, $\| (T^* - \lambda) x_n \| \to 0$. Since $x_n \to 0$ weakly we have,

$$ \| (T + k - \lambda) x_n \| \to 0 \text{ and } \| (T^* + k^* - \lambda) x_n \| \to 0 $$

Thus $T + k$ has a normal approximate eigenvalue and $e(\mathbb{G}) + K \subseteq K_1$.

The result holds for essentially spectraloid operators also.

Theorem 3.2.3 [55]: Let $e(\mathbb{G})$ be the class of essentially spectraloid operators on $H$. Then

1. $e(\mathbb{G}) + K$ is norm closed and
2. $e(\mathbb{G}) + K \subseteq K_1$

As a first step towards proving this theorem we observe that the Fujii-Nakamato theorem holds for essentially spectraloid operators also.
Lemma 3.2.1 If \( T \) is an essentially spectraloid operator then there exists a compact operator \( k \) such that \( T + k \) is spectraloid.

Proof: By theorem 3.2.2, there exists a compact operator \( k \) such that \( \lambda(T+k) = \lambda(T) \) and \( \|\lambda\| = \|T+k\| \).

Then

\[
\lambda(T+k) \leq \omega(T+k) = \omega(T) = \lambda(T) \leq \lambda(T+k)
\]

as \( \sigma(T) = \sigma(T+k) \subseteq \sigma(T+k) \).

Thus \( \lambda(T+k) = \omega(T+k) \) or \( T+k \) is spectraloid.

Proof of Theorem: The first part of the theorem is similar to theorem 3.2.2.

To prove that \( e(R) + k \subseteq R \), consider \( A = T+k \) where \( T \in e(R) \) and \( k \in R \). Since \( T \) is spectraloid, there exists a \( \lambda \in \sigma(T) \) such that \( 1\lambda = \lambda(T) = \omega(T) \).

Clearly \( \lambda \in \sigma(T) \) and consequently \( \lambda \in \sigma_e(T) \). The rest of the proof follows now as in the previous theorem.

In view of theorem 3.1.5 and the Fujii-Nakamoto theorem for essentially-\( R \) operators we have similarly,

Theorem 3.2.4: If \( e(R) \) is the class of essentially-\( R \) operators on \( H \) then,
(1) \( e(C) + \mathcal{K} \) is norm closed and

(2) \( e(C) + \mathcal{K} \subset \overline{R_1} \)

Since Luecke's conjecture is still open for essentially \( G_{11} \) operators, it is not known if \( e(C) + \mathcal{K} \) is norm closed. However, since \( e(C) \) is a subset of the class \( e(C) \), we have as a corollary to theorem 3.2.2.

**Corollary:** Norm Closure \( \sum e(C) + \mathcal{K} \subset \overline{R_1} \)

We conclude the chapter with a similar result for essentially centroid and essentially centraloid operators.

Theorem 3.2.4:- If \( e(Z) \) (resp. \( e(V) \)) is the class of essentially centroid (resp. essentially centraloid) operators on \( H \) then \( e(Z) + \mathcal{K} \) (resp. \( e(V) + \mathcal{K} \)) is norm closed and is contained in \( \overline{R_1} \).

Proof:- Since the classes \( Z \) and \( V \) are norm closed, it is sufficient to show that the Fujii-Nakamoto theorem holds for essentially centroid and essentially centraloid operators.

Without loss of generality, we can assume that \( W_q = 0 \).

By theorem H, there exists a compact operator \( k \) such that \( \overline{W(T + k)} = W_q(T) \) and \( \| T + k \| = \| T \| \).
Clearly $n_{T+K} = n_T = 0$.

If $T$ is essentially centroid then,

$$
||T+K - w_{T+K}|| = \|T+K\| = \|T\| = \omega(T) = \omega(T+K)
$$

so that $T+K$ is centroid by theorem 2.2.3.

If $T$ is essentially centraloid then

$$
\lambda(T+K - w_{T+K}) = \lambda(T+K) \leq \omega(T+K) = \omega(T) \leq \lambda(T) \leq \lambda(T+K)
$$

so that $T+K$ is centraloid by theorem 2.2.4.

The rest of the proof follows exactly in the same lines as the proof of theorem 3.2.2.