Chapter 3

Optimal Policies When Supplier Credits are Linked to Order Quantity

3.0. Introduction

This chapter deals with three mathematical models which are formulated, under the effect of inflation when supplier credits linked to order quantity or delay in payments is permissible.

Model (3.1): “Optimal Ordering policy in demand declining market under inflation when supplier credits linked to order quantity”

Model (3.2): “Deteriorating inventory model in demand declining market under inflation when Supplier credits linked to order quantity”

Model (3.3): “Deteriorating inventory model for two – level credit–linked demand under permissible delay in payments”

3.1. Optimal Ordering policy in demand declining market under inflation when supplier credits linked to order quantity
Here, a lot–size model is proposed when supplier offers the retailer a credit period to settle the account if the retailer orders a large quantity. The proposed study is meant for demand declining market. Here, the retailer needs to arrive at a static decision when demand of a product is decreasing and on the other side the supplier offer the credit period if the retailer orders for more than pre – specified quantity. Shortages are not allowed and the effect of inflation is incorporated. The objective is to minimize the total cost in demand declining market under inflation when the supplier offers a credit period to the retailer if the ordered quantity is greater than or equal to pre – specified quantity. An easy – to – use flow chart is given to find the optimal replenishment time and the order quantity. The mathematical formulation is supported by a numerical example. The sensitivity analysis of parameters on the optimal solution is carried out.

3.1.1 Assumptions and Notations

The mathematical model is based on the following assumptions and notations.

3.1.1.1 Assumptions

1. The inventory system deals with single item.
2. The demand $R(t) = (a - bt)$ is decreasing function of time ‘$t$’, where ‘$a$’ denotes the constant demand and ‘$b$’ denotes the rate of change of demand with respect to time ‘$t$’. $a > 0, b > 0$, $a >> b$, $0 < b < 1$.

3. The inflation rate is constant.

4. Shortages are not allowed and the lead – time is zero or negligible.

5. If the order quantity is less than $Q_d$, then the payments for the goods received must be done immediately.

6. If the order quantity is greater than or equal to $Q_d$, then the delay in payments up to ‘$M$’ is allowed. During the permissible delay period, the account is not settled; the generated sales revenue is deposited in an interest bearing account. At the end of the delay period, the retailer pays off for all units ordered and starts paying the interest charges on the items in stock.

3.1.1.2 Notations

$H$ Length of planning horizon

$R(t) = (a - bt)$ is demand rate which is decreasing function of time ‘$t$’, where ‘$a$’ denotes the constant demand and ‘$b$’ denotes the rate of change of demand with respect to time ‘$t$’. $a > 0, b > 0$, $a >> b$, $0 < b < 1$

$h$ Holding cost rate per unit time excluding interest charges

$r$ Constant rate of inflation per unit time where $0 < r < 1$
\( P(t) \) \( P e^{Rt} \); The selling price per unit at time \( t \), where \( P \) is the unit selling price at \( t = 0 \)

\( C(t) \) \( C e^{Rt} \); The purchase cost per unit at time \( t \), where \( C \) is the unit purchase cost at \( t = 0 \)

\( A(t) \) \( A e^{Rt} \); The ordering cost per unit at time \( t \), where \( A \) is the unit ordering cost at \( t = 0 \)

\( M \) Permissible trade credit in settling the account

\( I_c \) Interest charged per $ in stocks per annum by the supplier

\( I_e \) Interest earned per $ per year Note: \( I_e < I_c \)

\( Q \) Order quantity (decision variable)

\( Q_d \) Pre – specified quantity at which the delay in payments is allowed

\( n \) Number of replenishments to be made during period \( H \)

\( T_d \) Time at which \( Q_d \) – units are depleted to zero due to declining demand

\( I(t) \) Inventory level at any instant of time \( t \), \( 0 \leq t \leq T \)

\( T \) Replenishment time (decision variable)

\( K(T) \) Total cost over finite planning horizon of length \( H \)
3.1.2 Mathematical Model

We assumed that the length of planning horizon is \( H = nT \); where \( n \) denotes the number of orders to be made during period \( H \), and \( T \) is the replenishment time. The inventory level; \( I(t) \) depletes to meet the demand. The rate of change of inventory level at any instant of time is governed by the following differential equation.

\[
\frac{dl(t)}{dt} = -R(t), 0 \leq t \leq T
\]  

(3.3.1.1)

With the boundary condition \( I(0) = Q \) and \( I(T) = 0 \). Consequently, the solution of differential equation (3.3.1.1) is given by

\[
I(t) = a \left[ T - t + \frac{b}{2}(t^2 - T^2) \right], 0 \leq t \leq T
\]  

(3.3.1.2)

and the order quantity is

\[
Q = I(0) = a \left[ T - \frac{b}{2}T^2 \right]
\]  

(3.3.1.3)

From (3.3.1.3), we can obtain the cycle time at which \( Q_d \) units are reduced to zero. In fact, \( T_d \) is the solution of

\[
T_d = \frac{a \pm \sqrt{a^2 - 2abQ_d}}{ab}
\]  

(3.3.1.4)

It is clear \( Q < Q_d \) holds if and only if \( T < T_d \).

Since the time intervals are of equal length, using (3.3.1.2) we have
The total cost for the planning horizon consists of the following cost components.

1. Ordering cost per cycle;

\[
OC = A(0) + A(T) + A(2T) + \ldots + A((n-1)T) = A\left(\frac{e^{\frac{rH}{T}} - 1}{e^{\frac{rT}{T}} - 1}\right) \tag{3.3.1.6}
\]

2. Purchase cost;

\[
PC = Q\left[C(0) + C(T) + C(2T) + \ldots + C((n-1)T)\right]
\]

\[
= a\left(T - \frac{1}{2}bT^2\right)\left(\frac{e^{\frac{rH}{T}} - 1}{e^{\frac{rT}{T}} - 1}\right) \tag{3.3.1.7}
\]

3. Inventory holding cost per cycle;

\[
IHC = \frac{h}{k}\sum_{k=0}^{n-1} C(kT)\int_0^T l(kT + t)dt = \frac{1}{6}hCaT^2(3 - 2bT)\left(\frac{e^{\frac{rH}{T}} - 1}{e^{\frac{rT}{T}} - 1}\right) \tag{3.3.1.8}
\]

For the interest charged and the interest earned, we have four possible cases based on the lengths of \(T\), \(M\) and \(T_d\).

**Case 1: \(0 < T < T_d\)**

Here, replenishment cycle time \(T\) is less than \(T_d\) and so the delay in payments is not allowed. The retailer must pay immediately for the units received. This is the scenario of the classical EOQ model. The retailer has to pay interest
charges for all unsold items. Hence, interest charged payable during planning horizon $H$ is

$$IC_1 = I_c \sum_{k=0}^{n-1} C(kT) \left[ \int_0^T (kT + t)dt \right] = \frac{1}{6} Cl_c a T^2 (3 - 2bT) \left( \frac{e^{rt}}{e^{rt} - 1} \right)$$ (3.3.1.9)

Therefore, the total cost during planning horizon is

$$K_1(T) = OC + PC + IHC + IC_1$$ (3.3.1.10)

**Case 2: $T_d \leq T < M$**

Since, $T < M$, there is no interest charges and interest earned during planning horizon is

$$IE_2 = I_c \sum_{k=0}^{n-1} P(kT) \left[ \int_0^T R(t)dt + R(T)T(M - T) \right]$$

$$= Pr_c \left[ \frac{T^2 (3 - 2bT)}{6} + (1 - bT)T(M - T) \right] \left( \frac{e^{rt}}{e^{rt} - 1} \right)$$ (3.3.1.11)

As a result, total cost in this case during the planning horizon is

$$K_2(T) = OC + PC + IHC - IE_2$$ (3.3.1.12)

**Case 3: $T_d \leq M \leq T$**

In this case, cycle time is greater than both $T_d$ and $M$, hence, the trade credit is allowed. Here, total cost of an inventory system will have two additional components viz. interest charged and interest earned. The interest charged during planning horizon is
\[ IC_1 = I_c \sum_{k=0}^{n-1} C(kT) \left[ \int_M^T I(kT + t)dt \right] \]

\[ = CI_c \left[ \frac{1}{2} T^2 - \frac{1}{3} bT^3 - TM + \frac{1}{2} bMT^2 - \frac{1}{6} bM^2 + \frac{1}{2} M^2 \right] \left( \frac{e^{rT} - 1}{e^{rT} - 1} \right) \] (3.3.1.13)

and interest earned is

\[ IE_2 = I_e \sum_{k=0}^{n-1} P(kT) \int_0^M R(t)dt = \frac{1}{6} Pl_e aM^2 (3 - 2bM) \left( \frac{e^{rT} - 1}{e^{rT} - 1} \right) \] (3.3.1.14)

Hence, the total cost is

\[ K_3(T) = OC + PC + IHC + IC_3 - IE_3 \] (3.3.1.15)

**Case 4: \( M \leq T_d \leq T \)**

In this case, the replenishment cycle time \( T \) is greater than or equal to both \( T_d \) and \( M \). Thus, case 4 is similar to case 3, therefore, the total cost during planning horizon is

\[ K_4(T) = OC + PC + IHC + IC_3 - IE_3 \] (3.3.1.16)

### 3.1.3 Theoretical discussion

Since, the value of inflation \( r \) is very small, using a truncated Taylor series expansion for the exponential term, we have

\[ e^{rt} \approx 1 + rt + \frac{r^2 t^2}{2} \] (3.3.1.17)
\[ e^T \approx 1 + rT + \frac{r^2T^2}{2} \quad (3.3.1.18) \]

Using the above approximations, the total cost \( K_i(T), i=1,2,3,4 \) can be written as

\[
K_1(T) = \left( A + Ca \left( T - \frac{1}{2}bT^2 \right) \right) \left( \frac{rH + \frac{1}{2}r^2H^2}{rT + \frac{1}{2}r^2T^2} \right) \\
+ \left( (h + I_c)CaT^2 \frac{(3-2bT)}{6} \right) \left( \frac{rH + \frac{1}{2}r^2H^2}{rT + \frac{1}{2}r^2T^2} \right) \\
(3.3.1.19)
\]

\[
K_2(T) = \left( A + Ca \left( T - \frac{1}{2}bT^2 \right) \right) \left( \frac{rH + \frac{1}{2}r^2H^2}{rT + \frac{1}{2}r^2T^2} \right) \\
+ hCaT^2 \frac{(3-2bT)}{6} \left( \frac{rH + \frac{1}{2}r^2H^2}{rT + \frac{1}{2}r^2T^2} \right) \\
+ P_l a(T^2 \frac{(3-2bT)}{6}) \left( \frac{rH + \frac{1}{2}r^2H^2}{rT + \frac{1}{2}r^2T^2} \right) \\
+ (1-bT)T(M - T) \left( \frac{rH + \frac{1}{2}r^2H^2}{rT + \frac{1}{2}r^2T^2} \right) \\
(3.3.1.20)
\]

\[
K_3(T) = \left( A + Ca \left( T - \frac{1}{2}bT^2 \right) \right) \left( \frac{rH + \frac{1}{2}r^2H^2}{rT + \frac{1}{2}r^2T^2} \right) \\
+ hCaT^2 \frac{(3-2bT)}{6} \left( \frac{rH + \frac{1}{2}r^2H^2}{rT + \frac{1}{2}r^2T^2} \right) \\
+ cL_a \left( \frac{1}{2}T^2 - \frac{1}{3}bT^3 - TM \right) \left( \frac{rH + \frac{1}{2}r^2H^2}{rT + \frac{1}{2}r^2T^2} \right) \\
+ \left( I_c \frac{1}{2}bMT^2 - \frac{1}{3}bM^3 + \frac{1}{2}M^2 \right) \left( \frac{rH + \frac{1}{2}r^2H^2}{rT + \frac{1}{2}r^2T^2} \right) \\
- P_l aM^2 \frac{(3-2bM)}{6} \left( \frac{rH + \frac{1}{2}r^2H^2}{rT + \frac{1}{2}r^2T^2} \right) \\
(3.3.1.21)
\]
\[
K_4(T) = \left\{ \begin{array}{c}
Ca \left( T - \frac{1}{2} b T^2 \right) + h Ca T^2 \left( \frac{3 - 2b T}{6} \right) \\
- P_l \left( \frac{3 - 2b M}{6} \right) + A \\
+ C_l a \left[ \frac{1}{2} T^2 - \frac{1}{3} b T^3 - TM \right] \\
+ \frac{1}{2} b M T^2 - \frac{1}{6} b M^3 + \frac{1}{2} M^2 \\
\end{array} \right\} \left( r H + \frac{1}{2} r^2 H^2 \right) \\
\left( r T + \frac{1}{2} r^2 T^2 \right) \right) \quad (3.3.1.22)
\]

The necessary condition for \( K_1(T) \) in (3.3.1.19) is (3.3.1.23). Similarly for all \( K_i(T) \), \( i = 2,3,4 \)

\[
\frac{dK_1}{dT} = \left\{ \begin{array}{c}
A(r + r^2 T) + Ca(1 - b T) - Ca \left( T - \frac{1}{2} b T^2 \right)(r + r^2 T) \\
\left( r T + \frac{1}{2} r^2 T^2 \right) \\
+ \frac{h Ca T(3 - 2b T)}{3} - \frac{h Ca T^2 b}{3} - \frac{h Ca T^2(3 - 2b T)(r + r^2 T)}{6} \\
\left( r T + \frac{1}{2} r^2 T^2 \right) \\
+ \frac{l_c Ca T(3 - 2b T)}{3} - \frac{l_c Ca T^2 b}{3} - \frac{l_c Ca T^2(3 - 2b T)(r + r^2 T)}{6} \\
\left( r T + \frac{1}{2} r^2 T^2 \right) \\
\end{array} \right\} \left( \frac{r H + \frac{1}{2} r^2 H^2}{r T + \frac{1}{2} r^2 T^2} \right) = 0 \quad (3.3.1.23)
\]
\[
\frac{dK_2}{dT} = \begin{pmatrix}
-A(r+r^2T) & \frac{Ca}{rT + \frac{1}{2}r^2T^2} & \frac{Ca}{rT + \frac{1}{2}r^2T^2} \\
\frac{hCaT(3-2bT)}{3} & \frac{hCaT^2b}{3} & \frac{hCaT^2(3-2bT)(r+r^2T)}{6(rT + \frac{1}{2}r^2T^2)} \\
+P_{le}a\left(\frac{T(3-2bT)}{3} - bT^2 - bT(M - T) + (1-bT)(M - 2T)\right) & +\frac{P_{le}aT^2(3-2bT)}{6} + (1-bT)T(M - T)(r+r^2T) & \left(rT + \frac{1}{2}r^2T^2\right)
\end{pmatrix}
\]

\[
\frac{dK_3}{dT} = \begin{pmatrix}
-A(r+r^2T) & \frac{Ca}{rT + \frac{1}{2}r^2T^2} & \frac{Ca}{rT + \frac{1}{2}r^2T^2} \\
\frac{hCaT(3-2bT)}{3} & \frac{hCaT^2b}{3} & \frac{hCaT^2(3-2bT)(r+r^2T)}{6(rT + \frac{1}{2}r^2T^2)} \\
+Cl_a\left(T - bT^2 - M + bMT\right) & +\frac{P_{le}aM^2(3-2bM)(r+r^2T)}{6} & \left(rT + \frac{1}{2}r^2T^2\right)
\end{pmatrix}
\]

(3.3.1.24)
\[
\frac{dK_4}{dT} = \left\{ \begin{array}{l}
\frac{A\left(r+r^2T\right)}{rT+\frac{1}{2}r^2T^2} + \frac{Ca\left(1-bT\right)}{rT+\frac{1}{2}r^2T^2} - \frac{Ca\left(T-\frac{1}{2}bT^2\right)}{rT+\frac{1}{2}r^2T^2} \\
+ \frac{hCaT(3-2bT)}{3} - \frac{hCaT^2b}{3} - \frac{hCaT^2(3-2bT)}{6} \left(\frac{rT+\frac{1}{2}r^2T^2}{rT+\frac{1}{2}r^2T^2}\right) \\
+ Cl_c\left(\frac{1}{2}T^2 - \frac{1}{3}bT^3 - TM + \frac{1}{2}bMT^2 - \frac{1}{6}bM^3 + \frac{1}{2}M^2\right) \left(\frac{rT+\frac{1}{2}r^2T^2}{rT+\frac{1}{2}r^2T^2}\right)
\end{array} \right.
\]

\[\frac{\left(rT+\frac{1}{2}r^2T^2\right)^2}{\left(rT+\frac{1}{2}r^2T^2\right)} = 0 \]

(3.3.1.26)

The sufficient conditions are

\[
\frac{d^2K_1}{dT^2} = \frac{-2H(2+rT)}{3T^3(2+rT)^3} \left\{ \begin{array}{c}
-12A - 9Ar^2T^2 - 18ArT \\
-3CaT^3r^2 - 3CaT^3br \\
+3hCaT^3r + 4hCaT^3b \\
+3l_cCaT^3r + 4l_cCaT^3b
\end{array} \right\} > 0 \quad (3.3.1.27)
\]

\[
\frac{d^2K_2}{dT^2} = \frac{2H(2+rT)}{3T^3(2+rT)^3} \left\{ \begin{array}{c}
12A + 9Ar^2T^2 + 18ArT \\
+3CaT^3r^2 + 3CaT^3br \\
-3hCaT^3r - 4hCaT^3b \\
+3l_ePaT^3r + 8l_ePaT^3b \\
+6l_ePaT^3bMr + 3l_ePaT^3Mr^2
\end{array} \right\} > 0 \quad (3.3.1.28)
\]
\[
\frac{d^2 K_a}{dT^2} = \frac{H(2 + rH)}{3T^3(2 + rT)^3} \begin{pmatrix}
-24A - 18Ar^2T^2 - 36ArT \\
-6CaT^3r^2 - 6CaT^3br \\
+6hCaT^3r + 8hCaT^3b \\
+6l_cCaT^3r + 8l_cCaT^3b \\
-12Cl_c aM^2 + 6Cl_c aT^3bMr \\
+4l_c CabM^3 + 6l_c CabM^3rT \\
-18l_c CaM^2rT + 12Pl_e aM^2 \\
+18Pl_e aM^2rT - 8Pl_e aM^3b \\
-12Pl_e aM^3brT + 6Cl_c aT^3Mr^2 \\
+3Cl_c abM^3r^2T^2 - 9l_c CaM^2r^2T^2 \\
+9Pl_e aM^2r^2T^2 - 6Pl_e aM^3br^2T^2
\end{pmatrix} > 0 \quad (3.3.1.29)
\]
3.1.4 Computational flow – chart

To obtain optimal solution, decision maker can use following flow – chart.

START

Take parametric values of parameters

Compute $T_1$ using equation (3.3.1.23)

Is $0 < T_1 < T_d$ ?

YES

Compute $K_1(T_1)$

NO

Compute $T_2$ using equation (3.3.1.24)

Is $T_d \leq T < M$ ?

YES

Compute $K_2(T_2)$

NO

Compute $T_3$ and $T_4$ using equations (3.3.1.25) and (3.3.1.26)

STOP

Or $K_4(T_4)$
3.1.5 Numerical Examples

Consider following parametric values in proper units.

\[
\begin{bmatrix}
H, a, b, h, l_c, l_e, r, C, P, M, A
\end{bmatrix} = \\
[1, 50, 0.10, 2, 0.10, 0.06, 0.05, 20, 50, 30 / 365, 120]
\]

then if \( Q_d = 6 \) and \( 15 \), solving (3.3.1.26) for \( T_4 \) gives \( 0.3636 \) years and corresponding minimum total cost is \$ 1702.84 and corresponding

\[
\frac{d^2K_4(T_4)}{dT_4^2} = 4930 > 0 \text{ for all } T_4.
\]

Example 3.3.1.1 (for first case)

\[
\begin{bmatrix}
H, a, b, h, l_c, l_e, r, C, P, M, A
\end{bmatrix} = \\
[1, 50, 0.10, 2, 0.10, 0.06, 0.05, 20, 50, 30 / 365, 120]
\]

and \( Q_d = 35 \) gives \( T_d = 0.7264 \). Solving (3.3.1.23) for \( T_1 \) gives \( 0.3637 \) years and corresponding minimum total cost is \$ 1710.94 and corresponding

\[
\frac{d^2K_1(T_1)}{dT_1^2} = 4926 > 0 \text{ for all } T_1.
\]

Example 3.3.1.2 (for second case)

\[
\begin{bmatrix}
H, a, b, h, l_c, l_e, r, C, P, M, A
\end{bmatrix} = \\
[1, 50, 0.10, 2, 0.10, 0.06, 0.05, 20, 50, 150 / 365, 120]
\]
then $Q_d=15$ gives $T_d=0.3046$. Solving (3.3.1.26) for $T_2$ gives $0.3842$ years and corresponding minimum total cost is $1712.37$ and corresponding

$$\frac{d^2K_2(T_2)}{dT_2^2} = 4973 > 0$$

for all $T_2$.

**Example 3.3.1.3** (for third case)

\[
\begin{bmatrix}
  H, a, b, h, l_c, l_e, r, C, P, M, A
\end{bmatrix} =
\begin{bmatrix}
  1, 50, 0.10, 2, 0.10, 0.06, 0.05, 20, 50, 120 / 365, 120
\end{bmatrix}
\]

and $Q_d=15$ gives $T_d=0.3046$. Solving (3.3.1.26) for $T_3$ gives $0.3641$ years and corresponding minimum total cost is $1679.79$ and corresponding

$$\frac{d^2K_3(T_3)}{dT_3^2} = 4934 > 0$$

for all $T_3$.

**Example 3.3.1.4** (for fourth case)

\[
\begin{bmatrix}
  H, a, b, h, l_c, l_e, r, C, P, M, A
\end{bmatrix} =
\begin{bmatrix}
  1, 50, 0.10, 2, 0.10, 0.06, 0.05, 20, 50, 30 / 365, 120
\end{bmatrix}
\]

and $Q_d = 15$ gives $T_d = 0.3046$. Solving (3.26) for $T_4$ gives $0.3636$ years and corresponding minimum total cost is $1702.84$ and corresponding

$$\frac{d^2K_4(T_4)}{dT_4^2} = 4930 > 0$$

for all $T_4$.

Next we study effect of changes in critical parameter on decision variables and objective function with data same as given in example 3.3.1.4.
Table 3.3.1.1 Variation in $Q_d$

<table>
<thead>
<tr>
<th>$Q_d$</th>
<th>$T$</th>
<th>$Q$</th>
<th>$K$</th>
<th>Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>0.36359345</td>
<td>17.84917222</td>
<td>1694.935738</td>
<td>4</td>
</tr>
<tr>
<td>15</td>
<td>0.36359345</td>
<td>17.84917222</td>
<td>1694.935738</td>
<td>4</td>
</tr>
<tr>
<td>20</td>
<td>0.36366507</td>
<td>17.85262265</td>
<td>1710.935707</td>
<td>1</td>
</tr>
</tbody>
</table>

The number of pre-specified units to be procured is varied in Table 3.3.1.1. When $Q_d$ increases it increases the total cost of an inventory system.

Table 3.3.1.2 Variation in $r$

<table>
<thead>
<tr>
<th>$r$</th>
<th>$T$</th>
<th>$Q$</th>
<th>$K$</th>
<th>Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.3636</td>
<td>17.85</td>
<td>1702.84</td>
<td>3</td>
</tr>
<tr>
<td>0.07</td>
<td>0.3670</td>
<td>18.01</td>
<td>1713.25</td>
<td>3</td>
</tr>
<tr>
<td>0.09</td>
<td>0.3705</td>
<td>18.18</td>
<td>1723.52</td>
<td>3</td>
</tr>
</tbody>
</table>

The inflation rate is increased in Table 3.3.1.2. Increase in inflation rate increase cycle time, optimum procurement quantity and total cost of an inventory system.

Table 3.3.1.3 Variation in $a$

<table>
<thead>
<tr>
<th>$a$</th>
<th>$T$</th>
<th>$Q$</th>
<th>$K$</th>
<th>Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>0.3636</td>
<td>17.85</td>
<td>1702.84</td>
<td>3</td>
</tr>
<tr>
<td>150</td>
<td>0.2067</td>
<td>30.69</td>
<td>4250.01</td>
<td>3</td>
</tr>
<tr>
<td>200</td>
<td>0.1785</td>
<td>35.39</td>
<td>5454.83</td>
<td>3</td>
</tr>
</tbody>
</table>

The increase in constant demand decreases cycle time, increases optimum purchase quantities and total cost of an inventory system. The decision variables and objective function is very sensitive to changes in ‘$a$’.
The rate of change of demand; ‘\(b\)’ increase cycle time and decreases purchase units and total cost of an inventory system marginally.

**Table 3.3.1.4 Variation in \(b\)**

<table>
<thead>
<tr>
<th>(b)</th>
<th>(T)</th>
<th>(Q)</th>
<th>(K)</th>
<th>Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>0.3636</td>
<td>17.85</td>
<td>1702.84</td>
<td>3</td>
</tr>
<tr>
<td>0.12</td>
<td>0.3678</td>
<td>17.98</td>
<td>1697.25</td>
<td>3</td>
</tr>
<tr>
<td>0.15</td>
<td>0.3745</td>
<td>18.19</td>
<td>1688.71</td>
<td>3</td>
</tr>
</tbody>
</table>

The effect of variations in ordering cost is very significant. Increase in ordering cost increases cycle time, optimum procurement quantities and total cost of an inventory system.

**Table 3.3.1.5 Variation in \(A\)**

<table>
<thead>
<tr>
<th>(A)</th>
<th>(T)</th>
<th>(Q)</th>
<th>(K)</th>
<th>Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.3308</td>
<td>16.27</td>
<td>1644.29</td>
<td>3</td>
</tr>
<tr>
<td>120</td>
<td>0.3636</td>
<td>17.85</td>
<td>1702.84</td>
<td>3</td>
</tr>
<tr>
<td>150</td>
<td>0.4083</td>
<td>20.00</td>
<td>1781.75</td>
<td>3</td>
</tr>
<tr>
<td>170</td>
<td>0.4359</td>
<td>21.32</td>
<td>1829.81</td>
<td>3</td>
</tr>
</tbody>
</table>

The increase in delay period decreases total cost because retailer can earn more interest and having some cost savings.

The following managerial insights are observed:
(1) A higher value of minimum order quantity for avail of a permissible trade credit lowers the optimal order quantity and increases the total cost.

(2) Increase in ordering cost increases optimal cycle time and total cost of an inventory system.

(3) Increase in delay period lowers the total cost of an inventory system;

(4) Increase in inflation rate increases order quantity and total cost of an inventory system.

(5) Increase in demand rate increases total cost of an inventory system.

---

3.2. Deteriorating inventory model in demand declining market under inflation when Supplier credits linked to order quantity

In this section, an inventory model is developed when supplier offers the retailer a credit period to settle the account if the retailer orders a large quantity. The proposed study is meant for demand declining market. Shortages are not allowed and the effect of inflation is incorporated. The units in inventory are subject to constant rate of deterioration. The total cost is minimized for deteriorating items in demand declining market under inflation when the supplier offers a credit period to the retailer if the order quantity is greater than or equal to a pre – specified quantity. An easy – to – use algorithm is exhibited to find the optimal order quantity and the replenishment
time. The mathematical formulation is explored by a numerical example. The sensitivity analysis of parameters on the optimal solution is carried out.

### 3.2.1 Assumptions and Notations

The assumptions and notations for the proposed model are:

#### 3.2.1.1 Assumptions

1. The inventory system under consideration deals with a single item.
2. The demand is partially constant and partially decreases with time.
3. The inflation rate is constant.
4. Shortages are not allowed and the lead – time is zero.
5. The planning horizon is finite.
6. The dues for the items procured must be made immediately if the order quantity is less than $Q_d$ (pre-specified quantity offered by the supplier).

However, if the order quantity is greater than or equal to $Q_d$, then the delay in payment up to $M$ is allowed. During this credit period, the generated sales revenue is deposited in an interest bearing account. At the end of the delay period, the retailer can settle the account and after that supplier charges interest on the unsold stock in the inventory system.

#### 3.2.1.2 Notations

$H$ Length of finite planning horizon
\[ R(t) = a(1 - bt), \] where \( a > 0 \) denotes the constant demand and \( b > 0 \) denotes the rate of change of demand with respect to time \( t \). \( a \gg b, 0 < b < 1 \)

\[ h \] Holding cost rate per unit time excluding interest charges

\[ r \] Constant rate of inflation per unit time where \( 0 < r < 1 \)

\[ P(t) = Pe^{rt}; \] The selling price per unit at time \( t \), where \( P \) is the unit selling price at \( t = 0 \)

\[ C(t) = Ce^{rt}; \] The purchase cost per unit at time \( t \), where \( C \) is the unit purchase cost at \( t = 0 \)

\[ A(t) = Ae^{rt}; \] The ordering cost per unit at time \( t \), where \( A \) is the unit ordering cost at \( t = 0 \)

\[ M \] Allowable credit period in settling the account

\[ I_c \] Interest charged per $ for un-sold stock per annum by the supplier

\[ I_e \] Interest earned per $ per annum

\[ Q \] Order quantity (a decision variable)

\[ Q_d \] Minimum order quantity pre–specified by the supplier at which the delay in payment is permissible

\[ n \] Number of replenishments to be made during period \( H \)
\( T_d \)  
Time length at which \( Q_d \) – units are depleted to zero

\( \theta \)  
Constant deterioration rate, where \( 0 \leq \theta \leq 1 \)

\( I(t) \)  
Inventory level at any instant of time \( t \), \( 0 \leq t \leq T \)

\( T \)  
Total cycle time (a decision variable)

\( K(T) \)  
Total cost of an inventory system during the planning horizon

### 3.2.2 Mathematical Model

The retailer can make \( n \) – replenishments after every \( T \) – time units during the planning horizon \( H \). Thus, \( H = nT \), where \( n \) is an integer. The inventory level depletes due to demand and the deterioration of units. This rate of change of inventory level is governed by the differential equation.

\[
\frac{dI(t)}{dt} = -R(t) - \theta I(t), \quad 0 \leq t \leq T
\]  
(3.3.2.1)

with the boundary condition \( I(0) = Q \) and \( I(T) = 0 \). Hence the solution of (3.3.2.1) is given by

\[
I(t) = \frac{a}{\theta^2} \left[ \theta(1 - bT)e^{\theta(T-t)} + be^{\theta(T-t)} - \theta(1 - bt) - b \right], \quad 0 \leq t \leq T
\]  
(3.3.2.2)

and the order quantity is

\[
Q = I(0) = \frac{a}{\theta^2} \left[ \theta e^{\theta T} (1 - bT) + b(e^{\theta T} - 1) - \theta \right]
\]  
(3.3.2.3)

Using (3.3.2.3), the pre – specified \( Q_d \) units are given by
\[ Q_d = \frac{a}{\theta^r} \left[ \theta e^{\theta T_d} (1 - b T_d) + b(e^{\theta T_d} - 1) - \theta \right] \quad (3.3.2.4) \]

where \( T_d \) is the time at which \( Q_d \) units depletes to zero. The value of \( T_d \) is given by

\[ T_d = \frac{a \theta \pm \sqrt{a^2 \theta^2 - 4b \theta (\theta Q_d + 1)}}{2 \theta b} \quad (3.3.2.5) \]

Obviously, \( Q < Q_d \) holds if and only if \( T < T_d \). Under the assumption that the lengths of time interval are equal, using (3.3.2.2), we have

\[ l(kT + t) = a \left[ T - t + \frac{b}{2} (t^2 - T^2) \right], \quad 0 \leq k \leq n - 1, \quad 0 \leq t \leq T \quad (3.3.2.6) \]

The cost components of the total cost of an inventory system during planning horizon of length are as follows:

1. Ordering cost per cycle;

\[ OC = A(0) + A(T) + A(2T) + \ldots + A((n-1)T) = A\left(\frac{e^{rT} - 1}{e^{rT} - 1}\right) \quad (3.3.2.7) \]

2. Purchase cost;

\[ PC = [C(0) + C(T) + C(2T) + \ldots + C((n-1)T)]Q = CQ\left(\frac{e^{rT} - 1}{e^{rT} - 1}\right) \quad (3.3.2.8) \]

3. Inventory holding cost;

\[ IHC = \sum_{k=0}^{n-1} \frac{C(kT) l(kT + t)dt}{h} = C \left[ \frac{-2\theta - 2b - 2\theta^2 T + \theta^2 T^2}{2\theta^3} + 2\theta e^{rT} - 2\theta b e^{rT} + 2be^{rT} \right] \left(\frac{e^{rT} - 1}{e^{rT} - 1}\right) \quad (3.3.2.9) \]
Regarding interest charges and earned, the following four cases are possible depending on the lengths of \(T, T_d\) and \(M\).

**Case 1** \(0 < T < T_d\) (Figure 3.3.2.1)

Since the cycle time \(T\) is less than \(T_d\), supplier will not facilitate the retailer for the trade credit to settle the account. The retailer will have to pay immediately for the units procured. This is the case of classical economic order quantity (EOQ). The interest charged for unsold items during finite planning horizon is

\[
IC_c = I_c \sum_{k=0}^{n-1} C(kT) \left[ \int_0^T I(kT + t)dt \right]
\]

\[
= \frac{Cl_c a \left( -2\theta - 2b - 2\theta^2 T + \theta^2 T^2 b \right)}{2\theta^3 \left( +2\theta e^{\theta T} - 2\theta b T e^{\theta T} + 2b e^{\theta T} \right)} \left( \frac{e^{\theta T} - 1}{e^{\theta T} - 1} \right) \quad (3.3.2.10)
\]

Therefore, the total cost in \([0,H]\) is
The necessary condition for $K_1(T)$ to be minimum, is to set derivative of $K_1(T)$ with respect to $T$ be zero;

$$
\begin{align}
&\left(-\frac{A r^T e^{\theta t}}{(e^{\theta t} - 1)^2} + \frac{C a e^{2^T t}(1-bT)(e^{\theta t} - 1)}{(e^{\theta t} - 1)^2} - \frac{(h+ I_c) C a (1-bT)(1-e^{\theta t})(e^{\theta t} - 1)}{\theta e^{\theta t} - 1}\right) \\
+ &\frac{C a e^{2^T t}(\theta + b + \theta b T e^{\theta t} - \theta e^{\theta t} - b e^{\theta t})(e^{\theta t} - 1)}{\theta^2 e^{\theta t} - 1^2} \\
+ &\frac{(h+ I_c) C a e^{2^T t}(-2\theta - 2b - 2\theta^2 T + \theta^2 T^2 b + 2\theta e^{\theta t} - 2\theta e^{\theta t} b T + 2 b e^{\theta t})(e^{\theta t} - 1)}{2\theta^3 e^{\theta t} - 1^2} \\
= &0
\end{align}
$$

(3.3.2.12)

Solve equation (3.3.2.12) for $T = T_1$ by mathematical software. The obtained $T = T_1$ will minimize total cost provided

$$
\frac{d^2 K_1}{dT^2} = \frac{A r^2 e^{\theta t}(e^{\theta t} - 1)(e^{\theta t} + 1)}{(e^{\theta t} - 1)^3} + \frac{C a r^2 e^{2^T t}(-\theta e^{\theta t} + \theta e^{\theta t} b T)}{\theta^2 e^{\theta t} - 1^2} \\
- \frac{2 C a r^2 e^{2^T t}(-\theta e^{\theta t} + \theta e^{\theta t} b T)}{\theta^2 e^{\theta t} - 1^3} + \frac{2 C a r^2 e^{2^T t}(b T - 1)(e^{\theta t} - 1)}{(e^{\theta t} - 1)^2} \\
- \frac{C a e^{2^T t}(\theta(b T - 1) + b)(e^{\theta t} - 1)}{(e^{\theta t} - 1)} + \frac{h C a (b + \theta e^{\theta t} - \theta e^{\theta t} b T - b e^{\theta t})(e^{\theta t} - 1)}{\theta e^{\theta t} - 1} \\
+ \frac{2 h C a e^{2^T t}(1-b T)(1-e^{\theta t})(e^{\theta t} - 1)}{\theta e^{\theta t} - 1^2} - \frac{I_c C a (b + \theta e^{\theta t} - \theta e^{\theta t} b T - e^{\theta t} b)(e^{\theta t} - 1)}{\theta e^{\theta t} - 1}
$$
\[ + \frac{h\text{Car}^2e^{\theta T}(-2\theta - 2b - 2\theta^2T + \theta^2T^2b + 2\theta e^{\theta T} - 2\theta e^{\theta T} b T + 2 e^{\theta T} b)(e^{\theta H} - 1)(e^{\theta T} + 1)}{2\theta^3 \left(e^{\theta T} - 1\right)^3} \]

\[ - \frac{I_{c}\text{Car}^{e^{\theta T}}(-2\theta^2 + 2\theta^2Tb + 2\theta^2e^{\theta T} - 2\theta^2e^{\theta T} b T)(e^{\theta H} - 1)}{\theta^3 \left(e^{\theta T} - 1\right)^2} \]

\[ = \frac{I_{c}\text{Car}^2e^{\theta T}\left(-2\theta - 2b - 2\theta^2T + \theta^2T^2b + 2\theta e^{\theta T} - 2\theta e^{\theta T} b T + 2 e^{\theta T} b\right)(e^{\theta H} - 1)(e^{\theta T} + 1)}{2\theta^3 \left(e^{\theta T} - 1\right)^3} > 0, \text{ for all } T (3.3.2.13) \]

**Case 2** \( T_d \leq T < M \) (Figure 3.3.2.2)

Since, \( T_d \leq T < M \) there are no interest charges. Interest earned during \([0, H]\) is

\[ I_{E_2} = I_{c} \sum_{k=0}^{n-1} P(kT) \left[ \int_0^T R(t)dt + a(1 - b T)T(M - T) \right] \]

\[ = P I_{c} \left[ \frac{T^2}{2} - \frac{bT^3}{3} + (1 - b T)T(M - T) \right] \frac{e^{\theta H} - 1}{e^{\theta T} - 1} (3.3.2.14) \]

**Figure 3.3.2.2:** For \( T_d \leq T < M \)

Therefore, the total cost during \([0, H]\) is

\[ K_2(T) = OC + PC + IHC - I_{E_2} \] (3.3.2.15)
The necessary and sufficient conditions for $K_2(T)$ to be minimum at $T = T_2$ are

$$
\frac{dK_2}{dT} = -\frac{Ar^T(e^{\theta_1} - 1)}{(e^{\theta_1} - 1)^2} + \frac{Car^T(1 - bT)(e^{\theta_1} - 1)}{(e^{\theta_1} - 1)} + \frac{I_0 Pa(M - T)(1 - 2bT)(e^{\theta_1} - 1)}{(e^{\theta_1} - 1)}
$$

$$
\frac{Care^T(-\theta e^{\theta_1} + \theta e^{\theta_1} bT - be^{\theta_1} + \theta + b)(e^{\theta_1} - 1)}{\theta^2 (e^{\theta_1} - 1)^2} - \frac{hCa(1 - bT)(1 - e^{\theta_1})(e^{\theta_1} - 1)}{\theta (e^{\theta_1} - 1)}
$$

$$
- \frac{hCare^T(-2\theta - 2b - 2\theta^2 T + \theta^2 T^2 b + 2\theta e^{\theta_1} + 2\theta e^{\theta_1} bT + 2be^{\theta_1})(e^{\theta_1} - 1)}{\theta^3 (e^{\theta_1} - 1)^2}
$$

$$
- \frac{I_0 Pare^T(T^2(3 - 2bT) + 6(1 - bT)T(M - T))(e^{\theta_1} - 1)}{6(e^{\theta_1} - 1)^2} = 0 \quad (3.3.2.16)
$$

and

$$
\frac{d^2K_2}{dT^2} = \frac{Ar^T(e^{\theta_1} - 1)(e^{\theta_1} + 1)}{(e^{\theta_1} - 1)^3} + \frac{Car^T(-\theta e^{\theta_1} + \theta e^{\theta_1} bT)}{\theta^2 (e^{\theta_1} - 1)^2} + \frac{2Car^T(-\theta e^{\theta_1} + \theta e^{\theta_1} bT)(e^{\theta_1} - 1)}{\theta (e^{\theta_1} - 1)}
$$

$$
- \frac{Care^T(\theta(bT - 1) + b)(e^{\theta_1} - 1)}{(e^{\theta_1} - 1)} - \frac{2Care^T(-\theta e^{\theta_1} + \theta e^{\theta_1} bT)}{(e^{\theta_1} - 1)^3} + \frac{hCa(b + \theta e^{\theta_1} - \theta e^{\theta_1} bT - be^{\theta_1})(e^{\theta_1} - 1)}{\theta (e^{\theta_1} - 1)}
$$

$$
+ \frac{2hCare^T(1 - bT)(1 - e^{\theta_1})(e^{\theta_1} - 1)}{\theta (e^{\theta_1} - 1)^2} + \frac{I_0 Pare^T(2b(T - M) - T)(e^{\theta_1} - 1)}{(e^{\theta_1} - 1)^2}
$$

$$
+ \frac{hCar^T(-2\theta - 2b - 2\theta^2 T + \theta^2 T^2 b + 2\theta e^{\theta_1} + 2\theta e^{\theta_1} bT + 2be^{\theta_1})(e^{\theta_1} - 1)(e^{\theta_1} + 1)}{2\theta^3 (e^{\theta_1} - 1)^3}
$$
\[ \frac{I_p Par^2 e^{rT} (3T^2 - 2bT^3 + 6(1 - bT)T(M - T))(e^{rt} - 1)(e^{rt} + 1)}{6(e^{rt} - 1)^3} > 0 \text{ for all } T \] (3.3.2.17)

**Case 3** \[ T_d \leq M \leq T \] (Figure 3.3.2.3)

![Inventory Level Graph](image)

**Figure 3.3.2.3** \[ T_d \leq M \leq T \]

Here, cycle time; \( T \) is greater than \( T_d \) and \( M \) both. Therefore, delay in payment is allowed. The interest earned in \([0, H]\) is

\[ lE_3 = l_c \sum_{k=0}^{n-1} P(kT) \left[ \int_0^M R(t)tdt \right] = PI_o a \left( \frac{M^2}{2} - \frac{bM^3}{3} \right) \left( \frac{e^{rt} - 1}{e^{rt} - 1} \right) \] (3.3.2.18)

And interest charged during \([0, H]\) is

\[ lC_3 = l_c \sum_{k=0}^{n-1} C(kT) \left[ \int_M^T l(kT + t)dt \right] = \frac{Cl_o a}{\theta^3} \left[ -2\theta - 2\theta^2 T + \theta^2 T^2 b + 2\theta e^{\theta(T-M)}(1 - bT) + \frac{e^{rt} - 1}{e^{rt} - 1} \right] \] (3.3.2.19)
Consequently, the total cost in \([0, H]\) is

\[
K_3(T) = OC + PC + IHC + IC_3 - IE_3
\]  

(3.3.2.20)

The total cost can be minimized for \(T = T_3\) (say) by setting

\[
\begin{align*}
\frac{dK_3}{dT} &= -\frac{Ae^{rT}(e^{rT} - 1)}{(e^{rT} - 1)^2} + \frac{Ca e^{rT}(1 - bT)(e^{rT} - 1)}{(e^{rT} - 1)} - \frac{hCa(1 - bT)(1 - e^{rT})(e^{rT} - 1)}{\theta(e^{rT} - 1)} \\
&\quad - \frac{Care^{rT}(-\theta e^{rT} + \theta e^{rT} bT - be^{rT} + \theta + b)(e^{rT} - 1)}{\theta^2(e^{rT} - 1)^2} + \frac{l_Ca(1 - bT)(1 - e^{rT(T-M)})(e^{rT} - 1)}{\theta(e^{rT} - 1)} \\
&\quad - \frac{hCare^{rT}(-2\theta - 2b - 2\theta^2T + \theta^2T^2b + 2\theta e^{rT} - 2\theta e^{rT} bT + 2be^{rT})(e^{rT} - 1)}{\theta^3(e^{rT} - 1)^2} \\
&\quad + \frac{l_Ca e^{rT} M^2(2bM - 3)(e^{rT} - 1)}{6(e^{rT} - 1)^2} = 0
\end{align*}
\]  

(3.3.2.21)

Provided

\[
\begin{align*}
\frac{d^2K_3}{dT^2} &= \frac{Ar^2 e^{rT}(e^{rT} - 1)(e^{rT} + 1)}{(e^{rT} - 1)^3} + \frac{Car^2 e^{rT}(-\theta e^{rT} + \theta e^{rT} bT - be^{rT} + \theta + b)}{e^{rT} - 1) \\
&\quad - \frac{Ca e^{rT} (\theta(bT - 1) + b)(e^{rT} - 1)}{(e^{rT} - 1)^2} - \frac{2Car^2 e^{rT}(-\theta e^{rT} + \theta e^{rT} bT - be^{rT} + \theta + b)}{e^{rT} - 1)^3} \\
&\quad + \frac{2Car^2 e^{rT}(bT - 1)(e^{rT} - 1)}{(e^{rT} - 1)^2} + \frac{l_Ca(b + \theta e^{rT(T-M)} - \theta e^{rT(T-M)} Tb - be^{rT(T-M)})(e^{rT} - 1)}{\theta(e^{rT} - 1)} \\
&\quad + \frac{hCa(b + \theta e^{rT} - \theta e^{rT} bT - be^{rT})(e^{rT} - 1)}{\theta(e^{rT} - 1)} + \frac{2hCare^{rT}(1 - bT)(1 - e^{rT})(e^{rT} - 1)}{\theta(e^{rT} - 1)^2}
\end{align*}
\]
\[
\frac{h Car^2 e^{rT} \left( -2\theta - 2b - 2\theta^2 T + \theta^2 T^2 b + \frac{2\theta e^{\theta T} - 2\theta e^{\theta T} b T + 2\theta e^{\theta T}}{2\theta^3} \right) (e^{rH} - 1)(e^{rT} + 1)}{2\theta^3 (e^{rT} - 1)^3} \\
+ \frac{2L_c Car^2 (1-bT)(1-e^{\theta(T-M)})(e^{rH} - 1)}{\theta (e^{rT} - 1)^2} \\
+ \frac{2L_c PaM^2(2bM - 3)r^2 e^{rT}(e^{rH} - 1)(e^{rT} + 1)}{6(e^{rT} - 1)^3}
\]

\[
I_c Car^2 e^{rT} \left( \frac{-2\theta - 2b - 2\theta^2 T + \theta^2 T^2 b + 2\theta e^{\theta(T-M)} b T + 2\theta e^{\theta(T-M)} + 2\theta^2 M - \theta^2 M^2 b + 2bM\theta}{2\theta^3} \right) (e^{rH} - 1)(e^{rT} + 1) > 0 \text{ for all } T (3.3.2.22)
\]

**Case 4** \( M \leq T_d \leq T \) (Figure 3.3.2.4)

Here, also cycle time is greater than or equal to both \( T_d \) and \( M \). And hence Case 4 is similar to Case 3, therefore, the total cost during \([0,H]\) is

\[
K_4(T) = OC + PC + IHC + IC_3 - IE_3 (3.3.2.23)
\]

![Figure 3.3.2.4: For \( M \leq T_d \leq T \)]
3.2.3 Computational Algorithm

The retailer can decide optimal policy using following steps.

**Step 1**: Initialize all parameters.

**Step 2**: Compute $T = T_1$ using equation (3.3.2.12). If $T_1 < T_d$ then $K_1(T_1)$ is minimum; otherwise go to Step 3.

**Step 3**: Compute $T = T_2$ using equation (3.3.2.16). If $T_d < T_2 < T$ then $K_2(T_2)$ is minimum; otherwise go to Step 4.

**Step 4**: Compute $T = T_3$ using equation (3.3.2.21) and corresponding $K_3(T_3)$ is minimum; (Equivalently, $K_4(T_3)$ is minimum; otherwise go to Step 4.)

**Step 5**: Stop.

3.2.4 Numerical Example

**Example 3.3.2.1** Consider following parameters in proper units.

\[
[H, a, b, h, l_1, l_2, r, C, P, M, Q_d, A, \theta] = \\
[1, 50, 0.10, 2, 0.10, 0.06, 0.05, 20, 30, 30 / 365, 20, 120, 0.05]
\]

Following, algorithm defined in section 3.2.3, $T_1 = 0.3566 < T_d = 0.4042$ years. Hence Case 1 is optimal decision. The minimum cost is $1722.78 and optimum purchase quantity is 17.67 units.

**Example 3.3.2.2** Consider following parameters in proper units.
[\(H, a, b, h, l, \theta, r, C, P, M, Q_d, A, \theta\)] =
[1, 50, 0.10, 2, 0.10, 0.06, 0.05, 20, 30, 120/365, 10, 120, 0.05]

Then \(T_d = 0.2010 < M = 0.3288\) years. Thus, Case 2 is optimal decision policy. The optimal cycle time \(T_2 = 0.3571\), minimum cost is \(K_2(T_2) = 1691.53\) and purchase quantity is 17.69 units.

**Example 3.3.2.3** Consider following parameters in proper units.

\[
[H, a, b, h, l, \theta, r, C, P, M, Q_d, A, \theta] =
[1, 50, 0.10, 2, 0.10, 0.06, 0.05, 20, 30, 30/365, 15, 120, 0.05]
\]

Then \(T_d = 0.3023 \text{ years} > M = 0.0822 \text{ years}\). From Case 4, optimal cycle time \(T_4\) is 0.3566 years, minimum cost \(K_4(T_4)\) is $1714.62. See Figure 3.3.2.5.

![Figure 3.3.2.5: Convexity of total cost](image)
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Cycle time $T$ (in years)</th>
<th>Total Cost $K(T)$ (in $)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>0.05 0.3566 1714.62</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.10 0.3498 1725.67</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.15 0.3433 1736.59</td>
<td></td>
</tr>
<tr>
<td></td>
<td>50 0.3566 1714.62</td>
<td></td>
</tr>
<tr>
<td>$a$</td>
<td>60 0.3246 1985.90</td>
<td></td>
</tr>
<tr>
<td></td>
<td>70 0.2999 2251.73</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.10 0.3566 1714.62</td>
<td></td>
</tr>
<tr>
<td>$b$</td>
<td>0.15 0.3668 1700.72</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.20 0.3785 1686.25</td>
<td></td>
</tr>
<tr>
<td></td>
<td>120 0.3566 1714.62</td>
<td></td>
</tr>
<tr>
<td>$A$</td>
<td>150 0.4000 1795.17</td>
<td></td>
</tr>
<tr>
<td></td>
<td>180 0.4397 1867.68</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.05 0.3566 1714.62</td>
<td></td>
</tr>
<tr>
<td>$r$</td>
<td>0.10 0.3651 1742.72</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.15 0.3743 1771.20</td>
<td></td>
</tr>
<tr>
<td></td>
<td>30/365 0.35658 1714.62</td>
<td></td>
</tr>
<tr>
<td>$M$</td>
<td>45/365 0.35658 1710.63</td>
<td></td>
</tr>
<tr>
<td></td>
<td>60/365 0.35663 1706.69</td>
<td></td>
</tr>
<tr>
<td></td>
<td>15 $T_4 = 0.3566$ $K_4(T_4) = 1714.62$</td>
<td></td>
</tr>
<tr>
<td>$Q_d$</td>
<td>18 $T_1 = 0.3566$ $K_1(T_1) = 1722.78$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>20 $T_1 = 0.3566$ $K_1(T_1) = 1722.78$</td>
<td></td>
</tr>
</tbody>
</table>
Next, we carry out variations in critical parameter to study effects on decision variable and total cost during $[0, H]$.

It is observed that increase in deterioration rate decreases cycle time and increases total cost of an inventory system during finite planning horizon (Figure 3.3.2.6). The model is very sensitive to changes in the fixed demand ‘$a$’. (Figure 3.3.2.7) Increase in fixed demand increases total cost significantly and decreases cycle time. Increase in demand rate ‘$b$’ increases cycle time and decreases total cost of an inventory system (Figure 3.3.2.8). The model is very sensitive to changes in ordering cost (Figure 3.3.2.9) and inflation rate (Figure 3.3.2.10). The total cost of inventory system decreases if supplier allows longer credit period. The decrease in total cost is because the retailer can earn more interest on the generated sales revenue.

![Figure 3.3.2.6: Variations in $\theta$](image-url)
Figure 3.3.2.7: Variations in $a$

Figure 3.3.2.8: Variations in $b$
Figure 3.3.2.9: Variations in $A$

Figure 3.3.2.10: Variations in $r$
3.3. Deteriorating inventory model for two-level credit-linked demand under permissible delay in payments

In practice, the supplier offers a fixed credit period to the retailer but the retailer does not offer any credit period to the customers, which is unrealistic because of global competition. In real practice, retailer may offer a credit period to its customer in order to boost his own demand. In this section, the impact of two-level credit period on demand is studied when units in inventory deteriorate at a constant rate. An easy-to-use algorithm is developed to determine the optimal credit period and replenishment policy for the retailer. Finally, numerical example is presented to illustrate the theoretical results followed by the sensitivity of various parameters on the optimal solution.

3.3.1 Assumptions and Notations

The proposed mathematical model is based on the following assumptions and notations.

3.3.1.1 Assumptions

1. The inventory system deals with a single item.
2. The supplier offers a credit period $M$ to settle the accounts to the retailer and the retailer, in turn, offers a credit period $N$ to his customers to settle the accounts.
3. The demand rate is a function of the customer’s credit period, $N$; offered by the retailer. The demand function for any $N$ can be represented as a differential difference equation:

$$R(N + 1) - R(N) = r \left[R_m - R(N)\right]$$

Where $R(N)$ : demand for any $N$ per unit time

$R_m$ : Maximum demand

$r$ : Rate of saturation of demand.

The solution of the above differential difference equation, using initial condition that at $N = 0$, $R(0) = R_0$ (initial demand) is given by

$$R(N) = R_0 (1 - r)^N + R_m \left(1 - (1 - r)^N\right)$$

i.e. $R(N) = R_m - (R_m - R_0)(1 - r)^N$

4. Replenishment rate is instantaneous.

5. Shortages are not allowed.

6. Lead – time is zero or negligible.

7. The units in inventory system deteriorate at a constant rate, $\theta$

where ($0 \leq \theta \leq 1$). The deteriorated units can neither be repaired nor replaced during a cycle time.

### 3.3.1.2 Notations

- **$Q$** Order quantity (a decision variable)
- **$T$** Cycle time (a decision variable)
- **$l(t)$** Inventory level at any instant of time $t$, $0 \leq t \leq T$
3.3.2 Mathematical Model

The inventory level depletes due to demand and deterioration. The inventory level at any time $t$ during the cycle is governed by the differential equation,

$$
\frac{dI(t)}{dt} = -R(N) - \theta l(t), \quad 0 \leq t \leq T
$$

(3.3.3.1)
with initial condition; \( I(0) = Q \) and boundary condition \( I(T) = 0 \). Then solution of differential equation is,

\[
I(t) = \frac{R(N)}{\theta} \left[ e^{\theta(T-t)} - 1 \right], 0 \leq t \leq T
\]  

(3.3.3.2)

and the order quantity is,

\[
Q = I(0) = \frac{R(N)}{\theta} \left[ e^{\theta T} - 1 \right]
\]  

(3.3.3.3)

The retailer's profit per time unit time compromises of the following components:

(a) Sales revenue; \( SR = \frac{PQ}{T} \)  

(3.3.3.4)

(b) Cost of purchasing; \( PC = \frac{CQ}{T} \)  

(3.3.3.5)

(c) Cost of placing orders; \( OC = \frac{A}{T} \)  

(3.3.3.6)

(d) Inventory holding cost; \( IHC = \frac{CIR(N)}{\theta^2 T} \left[ e^{\theta T} - \theta T - 1 \right] \)  

(3.3.3.7)

The calculation for interest earned and charged will depend upon the lengths of \( T, N \) and \( M \). The following three cases arise:

**Case 1:** \( N \leq M \leq T + N \) (Figure 3.3.3.1)
Here, the retailer generates revenue in \([N, M]\) and earns interest on sales revenue for the time period \((M - N)\). At \(M\), accounts are to be settled and during \([M, T + N]\) interest charges are payable to the supplier by the retailer on the unsold stock. Hence,

(e) Interest charged per time unit is

\[
IC_1 = \frac{C_I e}{T} \int_0^{T + N - M} l(t) dt = \frac{C_I e R(N)}{\theta^2 T} \left[ e^{\theta T} - e^{\theta (M - N)} - \theta (T + N - M) \right]
\]  

(3.3.3.8)

and

(f) Interest earned per unit time is

\[
IE_1 = \frac{P_I e}{T} \int_0^{M - N} R(N) dt = \frac{P_I e R(N) (M - N)^2}{2T}
\]  

(3.3.3.9)

Using equations (3.3.3.4) – (3.3.3.9), the retailer’s profit per unit time is,

\[
\Pi_1(T, N) = SR - PC - OC - IH\text{C} - IC_1 + IE_1
\]  

(3.3.3.10)

Case 2: \(N \leq T + N \leq M\) (Figure 3.3.3.2)

![Figure 3.3.3.2 When \(N \leq M \leq T + N\)](image-url)
Here, the retailer earns interest on the revenue received during the period \((N,T+N)\) and on total sales revenue; \(PQ\) for a period of \((M-T-N)\),

(e) Total interest earned per time unit is,

\[
IE_2 = \frac{PI_e}{T} \int_{N}^{T+N} R(N)dt + \frac{PI_e}{T} (M-T-N)R(N)T
\]

\[
= PI_e R(N) \left( M - N - \frac{T}{2} \right)
\]

(3.3.3.11)

(f) Interest charges are zero. i.e. \(IC_2 = 0\) \hspace{1cm} (3.3.3.12)

As a result, using equation (3.3.3.4) – (3.3.3.7) and (3.3.3.11) – (3.3.3.12), the retailer’s profit per time unit is,

\[
\Pi_2(T,N) = SR - PC - OC - IH - IC_2 + IE_2
\]

(3.3.3.13)

Case 3: \(M \leq N \leq T + N\) (Figure 3.3.3.3)

Figure 3.3.3.3 When \(N \leq M \leq T + N\)

In this case,

(e) Interest charged per unit time is,
\[ IC_3 = \frac{C_l R(N)}{\theta^2 T} \left[(N - M)\theta(e^{\theta T} - 1) + e^{\theta T} - \theta T - 1\right] \quad (3.3.3.14) \]

Hence, using equation (3.3.3.4) – (3.3.3.7) and (3.3.3.14), the retailer's profit per unit time is,

\[ \Pi_3(T, N) = SR - PC - OC - IHC - IC_3 \quad (3.3.3.15) \]

Therefore, the retailer’s profit per unit time \( \Pi(T, N) \) is

\[
\Pi(T, N) = \begin{cases} 
\Pi_1(T, N), & \text{if } N \leq M \leq T + N \\
\Pi_2(T, N), & \text{if } N \leq T + N \leq M \\
\Pi_3(T, N), & \text{if } M \leq N \leq T + N 
\end{cases} \quad (3.3.3.16)
\]

which is a function of two variables \( T \) and \( N \), where, \( T \) is continuous and \( N \) is discrete.

To obtain closed form of the solution, we write series expansion containing term up to \( \theta \) under the assumption that \( 0 \leq \theta \leq 1 \). Hence,

\[
\Pi_1(T, N) = (P - C)R(N) + \frac{(P - C)R(N)\theta T}{2} - \frac{C(I + I_e)R(N)T}{2} - \frac{A}{T} - \frac{(C_l - P_l)R(N)(M - N)^2}{2T} \]

\[
\Pi_2(T, N) = (P - C)R(N) + \frac{(P - C)R(N)\theta T}{2} - \frac{CIR(N)T}{2} - \frac{A}{T} + P_l R(N) \left(M - N - \frac{T}{2}\right) \]

\[
\Pi_3(T, N) = (P - C)R(N) + \frac{(P - C)R(N)\theta T}{2} - \frac{CIR(N)T}{2} - \frac{A}{T} + P_l R(N) \left(M - N - \frac{T}{2}\right) \]
\[
\Pi_3(T, N) = (P - C)R(N) + \frac{(P - C)R(N)\theta T}{2} - \frac{CIR(N)T}{2} - \frac{A}{T}Cl_cR(N)\left(\frac{T}{2} + N - M\right)
\]

3.3.3 Solution Methodology

Our aim is to determine the optimal values of \( T \) and \( N \) which maximizes \( \Pi(T, N) \). For fixed \( N \), take the first and second order derivatives of \( \Pi_i(T, N) \), \( i = 1, 2, 3 \) gives,

\[
\frac{\partial \Pi_1}{\partial T} = \frac{(P - C)R(N)\theta}{2} - \frac{C(l + I_c)R(N)}{2} + \frac{A}{T^2} + \frac{(Cl_c - Pl_e)R(N)(M - N)^2}{2T^2} \tag{3.3.3.17}
\]

\[
\frac{\partial^2 \Pi_1}{\partial T^2} = -\frac{2A}{T^3} - \frac{(Cl_c - Pl_e)R(N)(M - N)^2}{T^3} \tag{3.3.3.18}
\]

\[
\frac{\partial \Pi_2}{\partial T} = \frac{(P - C)R(N)\theta}{2} - \frac{CIR(N)}{2} + \frac{A}{T^2} - \frac{Pl_eR(N)}{2} \tag{3.3.3.19}
\]

\[
\frac{\partial^2 \Pi_2}{\partial T^2} = -\frac{2A}{T^3} \tag{3.3.3.20}
\]

\[
\frac{\partial \Pi_3}{\partial T} = \frac{(P - C)R(N)\theta}{2} - \frac{C(l + I_c)R(N)}{2} + \frac{A}{T^2} \tag{3.3.3.21}
\]

and \( \frac{\partial^2 \Pi_3}{\partial T^2} = -\frac{2A}{T^3} \) \( \tag{3.3.3.22} \)

For fixed \( N \), Equations (3.3.3.20) and (3.3.3.22) indicate that \( \Pi_4(T, N) \) and \( \Pi_3(T, N) \) are concave for all \( T > 0 \). However, \( \Pi_1(T, N) \) is concave for all \( T > 0 \)
if \( Cl_e > Pl_e \). Thus, there exists a unique value of \( T = T_1 \) which maximizes \( \Pi_1(T) \). It is given by equating equation (3.3.3.17) to be zero. We get,

\[
T_1 = \frac{2A + (Cl_e - Pl_e)R(N)(M - N)^2}{\sqrt{C(\theta + I + I_e)R(N) - P\theta R(N)}} \tag{3.3.3.23}
\]

\( T_1 \) would satisfy the condition \( 0 \leq M - N \leq T \) provided,

\[
2A - \left[ C(\theta + I) + Pl_e \right]R(N)(M - N)^2 \geq 0 \tag{3.3.3.24}
\]

Similarly, there exists a unique value of \( T = T_2 \) which maximizes \( \Pi_2(T) \). It is given by equating equation (3.3.3.19) to be zero.

We get, \( T_2 = \frac{2A}{\sqrt{(C(\theta + I) + Pl_e)R(N)}} \) \tag{3.3.3.25} \]

\( T_2 \) would satisfy the condition \( 0 \leq T \leq (M - N) \) provided,

\[
2A - \left[ C(\theta + I) + Pl_e \right]R(N)(M - N)^2 \geq 0 \tag{3.3.3.26}
\]

and \( T_3 = \frac{2A}{\sqrt{C(\theta + I + I_e)R(N)}} \) \tag{3.3.3.27} \]

maximizes profit \( \Pi_3(T) \). \( T_3 \) would satisfy the condition \( (M - N) \leq 0 \leq T \) provided \( 2A - \left[ C(\theta + I + I_e) \right]R(N)(M - N)^2 \geq 0 \) \tag{3.3.3.28} \]

Combining the three possible cases, we have following theorem:

**Theorem 1**: For a fixed value of \( N \)

(i) If \( 2A - \left[ C(\theta + I) + Pl_e \right]R(N)(M - N)^2 \geq 0 \) then \( T^* = T_1 \).
If $2A - [C(\theta + I) + PI_e]R(N)(M - N)^2 \leq 0$ then $T^* = T_2$.

(iii) If $2A - C[\theta + I + I_e]R(N)(M - N)^2 \geq 0$ and $(M - N) < 0$ then $T^* = T_3$

**Proof:** It immediately follows from (3.3.3.24), (3.3.3.26) and (3.3.3.28).

### 3.3.4 Computational Algorithm

In order to optimize $T$ and $N$ simultaneously, we have following steps:

**Step 1:** Start with $N = 1$

**Step 2:** Determine the optimal value of $T$ using Theorem 1

**Step 3:** If $0 \leq M - N \leq T$ then calculate $\Pi_1(T,N)$ otherwise go to Step 5.

**Step 4:** If $\Pi_1(T,N) > \Pi_1(T,N - 1)$, increment $N$ by $N + 1$ and go to Step 2 otherwise current value of $N$ is optimal. Compute $Q$ and $\Pi(T,N)$.

**Step 5:** If $0 \leq T \leq (M - N)$ then calculate $\Pi_2(T,N)$ otherwise go to Step 7.

**Step 6:** If $\Pi_2(T,N) > \Pi_2(T,N - 1)$, increment $N$ by $N + 1$ and go to Step 2 otherwise current value of $N$ is optimal. Compute $T$ and $\Pi(T,N)$.

**Step 7:** If $(M - N) \leq 0 \leq T$ then calculate $\Pi_3(T,N)$.

**Step 8:** If $\Pi_3(T,N) > \Pi_3(T,N - 1)$, increment $N$ by $N + 1$ and go to Step 2 otherwise current value of $N$ is optimal. Compute $T$ and $\Pi(T,N)$. 
3.3.5 Numerical Example

Let maximum demand \( (R_m) = 100 \) units/day, minimum demand \( (R_o) = 30 \) units/day, rate of saturation of demand \( (r) = 12 \% \), \( A = $1000/\text{order} \), \( N = 45 \) days, \( C = $ 30/\text{unit} \), \( P = $ 40/\text{unit} \), \( I = 15 \% \) per year, \( I_c = 15 \% \) per year, \( I_e = 10 \% \) per year (Jaggi et. al. (2008)) and \( \theta = 5 \% \).

Using algorithm, optimal cycle time is 29.35 days, optimal credit period \( (N) \) offered by the retailer to the customer is 43 days and profit per day is $929.

The sensitivity analysis on \( r, M, A, I_c \) and \( \theta \) is exhibited in Table 3.3.3.1–3.3.3.5, respectively.

<table>
<thead>
<tr>
<th>Table 3.3.3.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Effect of changes in ( r ) on the optimal solution.</td>
</tr>
<tr>
<td>( r )</td>
</tr>
<tr>
<td>0.09</td>
</tr>
<tr>
<td>0.12</td>
</tr>
<tr>
<td>0.15</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 3.3.3.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Effect of changes in ( M ) on the optimal solution.</td>
</tr>
<tr>
<td>( M )</td>
</tr>
<tr>
<td>30</td>
</tr>
<tr>
<td>45</td>
</tr>
<tr>
<td>60</td>
</tr>
</tbody>
</table>
Table 3.3.3.3
Effect of changes in $A$ on the optimal solution.

<table>
<thead>
<tr>
<th>$A$</th>
<th>$T^*(\text{days})$</th>
<th>$N^*(\text{days})$</th>
<th>$\Pi(T^<em>,N^</em>)$ in $$</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>20.75</td>
<td>45</td>
<td>949</td>
</tr>
<tr>
<td>1000</td>
<td>29.35</td>
<td>43</td>
<td>929</td>
</tr>
<tr>
<td>1200</td>
<td>32.16</td>
<td>42</td>
<td>922</td>
</tr>
</tbody>
</table>

Table 3.3.3.4
Effect of changes in $c_i$ on the optimal solution.

<table>
<thead>
<tr>
<th>$c_i$</th>
<th>$T^*(\text{days})$</th>
<th>$N^*(\text{days})$</th>
<th>$\Pi(T^<em>,N^</em>)$ in $$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.12</td>
<td>31.03</td>
<td>45</td>
<td>933</td>
</tr>
<tr>
<td>0.15</td>
<td>29.35</td>
<td>43</td>
<td>929</td>
</tr>
<tr>
<td>0.18</td>
<td>27.89</td>
<td>43</td>
<td>925</td>
</tr>
</tbody>
</table>

Table 3.3.3.5
Effect of changes in $\theta$ on the optimal solution.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$T^*(\text{days})$</th>
<th>$N^*(\text{days})$</th>
<th>$\Pi(T^<em>,N^</em>)$ in $$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.03</td>
<td>28.99</td>
<td>45</td>
<td>933</td>
</tr>
<tr>
<td>0.05</td>
<td>29.35</td>
<td>43</td>
<td>929</td>
</tr>
<tr>
<td>0.08</td>
<td>29.87</td>
<td>40</td>
<td>921</td>
</tr>
</tbody>
</table>

From Table 3.3.3.1, it is observed that the rate of saturation of demand is very sensitive parameter. It decreases credit period offered by the retailer to the customer. The retailer's profit increases.
The similar changes are observed in Table 3.3.3.2 when credit period offered to the retailer increases. The negative impact in optimal profit and credit period offered to the customer when ordering cost (Table 3.3.3.3) and interest charged (Table 3.3.3.4) increases. In Table 3.3.3.5, deterioration rate is varied. Increase in deterioration rate reduces retailer’ profit.

Conclusions

An EOQ model (3.1) is developed under inflation when demand of a product is decreasing in the market to determine the optimal ordering policy when the supplier provides a trade credit linked to order quantity. The effect of the values of the parameters on the optimal solution is studied to illustrate the developed theoretical results.

In Model (3.2), optimal policy is derived for deteriorating items when the supplier provides a permissible delay in payments if ordered units are more than pre – specified number by the supplier. The effect inflation is incorporated. The proposed model can be extended to a two – parameter Weibull distribution. It can be generalized to allow for shortages. The comparison of quantity discounts and trade credit is also an interesting future scope of research.

In Model (3.3), the effect of credit – linked demand on the retailer's optimal profit is studied. The retailer's profit is maximized with respect to the cycle time and the credit period offered by the retailer. It is observed that the
credit period offered to customer has significant positive impact on the unrealized demand.