Appendix A

A.1 Coaxial Probe Coupling in Circular Waveguide with Perfect Short

The coaxial probe excitation of a circular waveguide having a perfect short at input is shown in Figure A.1. Maximum power is coupled to the circular waveguide from the coaxial line fed probe when the real part of the impedance seen by the probe in the circular waveguide matches with the characteristic impedance of the coaxial line and the reactive part becomes zero.

Detailed expressions are given in [28] to find out the input impedance of a coaxial line fed probe in circular waveguide. Those expressions have been used presently to find out the probe parameters for impedance matching. As described in [28] the expression for impedance offered to the coaxial probe in circular waveguide is given as,

\[ Z_m = \frac{-1}{(I_m)^2} \iiint \int J(\bar{r}) G_\rho(\bar{r}, \bar{r}') J(\bar{r}') ds'd\bar{s} \]  \hspace{1cm} (A.1)

where \( G_\rho(\bar{r}, \bar{r}') \) is the radial component of the electric dyadic Green’s function and \( J(\bar{r}) \) is the current density on the coaxial probe. \( \rho, \phi, z \) is the cylindrical co-ordinate system for the circular waveguide. The expressions for Green’s function [28] are,
\[ G_{\rho}(r,r') = \]
\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2}{\pi \varepsilon_{0} \omega_{mn} \gamma_{m}^{2} \alpha_{mn} \rho \rho'} \sinh[\alpha_{mn} (z' + l)] J_{m} (\gamma_{mn} \rho') J_{m} (\gamma_{mn} \rho) \]
\[ e^{-\alpha_{mn}(z'+l)} \cos(m(\phi-\phi')) + \frac{j}{2 \omega} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_{mn}}{\pi \varepsilon_{0} \omega_{mn} \gamma_{mn}^{2}} \sinh[\alpha_{mn} (z' + l)] \frac{\partial}{\partial \rho} (J_{m} (\gamma_{mn} \rho')) \]
\[ \frac{\partial}{\partial \rho} (J_{m} (\gamma_{mn} \rho)) e^{-\alpha_{mn}(z'+l)} \cos(m(\phi-\phi')) ; z > z' \]

(A.2)

\[ G_{\rho}(r,r') = \]
\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2}{\pi \varepsilon_{0} \omega_{mn} \gamma_{m}^{2} \alpha_{mn} \rho \rho'} e^{-\alpha_{mn}(z'+l)} J_{m} (\gamma_{mn} \rho') J_{m} (\gamma_{mn} \rho) \sinh[\alpha_{mn} (z + l)] \]
\[ \cos(m(\phi-\phi')) + \frac{j}{2 \omega} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_{mn}}{\pi \varepsilon_{0} \omega_{mn} \gamma_{mn}^{2}} e^{-\alpha_{mn}(z'+l)} \frac{\partial}{\partial \rho} (J_{m} (\gamma_{mn} \rho')) \]
\[ \frac{\partial}{\partial \rho} (J_{m} (\gamma_{mn} \rho)) \sinh[\alpha_{mn} (z + l)] \cos(m(\phi-\phi')) ; -l < z < z' \]

(A.3)

The current density on the probe is,
\[ J = \frac{I(\rho)}{2\pi \tau} , \text{ where, } \tau \text{ is the radius of the probe and } I(\rho) = I_{m} \frac{\sin[k(\rho - a + h)]}{\sin(kh)} \text{ is the assumed current distribution on the probe. In this expression of current, } h \text{ is the probe depth, } a \text{ is the radius of circular waveguide, } k \text{ is the free space propagation constant and } l \text{ is the distance of probe from the location of the perfect short.} \]

Equation (A.1) can be expressed as,
\[ Z_{m} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (R_{mn} + JX_{mn}) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (R_{mn} + JX_{mn}) \]

where, \( R_{mn} + JX_{mn} \) are the input impedance due to TE_{mn} mode, while \( R_{mn} + JX_{mn} \) are due to TM_{mn} mode. The cross-sectional dimension of the circular waveguide only allows TE_{11} mode to propagate. Thus, all \( R_{mn}, R_{mn} \) vanish except \( R_{11} \) which is a real part of impedance offered to probe due to propagating TE_{11} mode in circular waveguide. The expressions for real and imaginary parts of impedance are given in [28]. If the real part of the impedance offered to the probe have to be matched
with the characteristic impedance \( Z_0 \) of the coaxial line, then in this condition the reactance part of \( Z_m \) should be zero. The design dimensions of the coaxial probe for impedance matching can be found from the expressions,

\[
Z_0 = \frac{\omega \mu \sin^2(\beta_{11}'l)}{\pi \omega l_{11}'^2 \sin^2(kh)} X \left\{ \int_0^{\pi} J_{l_{11}'}[\nu_{11}'(a - y/k)] \sin(kh - y) \frac{dy}{ka - y} \right\}^2
\]  
(A.5)

\[
0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} X_{mn}(l, h) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} X_{mn}(l, h)
\]  
(A.6)

Equations (A.5) and (A.6) give unknowns such as height ‘\( h \)’ of probe and the position ‘\( l \)’ of the probe from the perfect short for matching where real part of the impedance matches with the characteristic impedance of the coaxial line and the reactance due to all modes vanishes.

Using the expressions of equations (A.5) and (A.6) as described in [28], the input impedance and the return loss as can be computed. A circular waveguide of diameter 32.54 mm, excited with a probe of diameter 1.6 mm was analyzed. The diameter of inner and outer conductor of coaxial line was selected as 1.6 mm and 6.5 mm respectively. The distance of the perfect short from the probe and the probe depth were varied and it was found out that a short distance of 19 mm from the probe position gave optimum matching at 6.6 GHz. The return loss was of the order of -30 dB at 6.6 GHz and the -10 dB return loss bandwidth was of the order of 12%. This distance of the perfect short from the probe is nearly quarter of the guide wavelength which is equivalent to 90 degree phase. Also, probe height of 11 mm (nearly quarter of the wavelength) was found to be exactly matching with the coaxial line TEM mode impedance (83 Ohms). Thus the analysis for probe coupling shows that impedance matching for maximum coupling can be obtained for a particular depth of coaxial line fed probe and its location from the short.

**A.2 Coaxial Probe Coupling in Circular Waveguide with Tapered Short**

In case of multi-frequency mode transducer (see Figure 2.1), the probe depth and its location with respect to the tapered section should also be determined for maximum power coupling to the circular waveguide. Since, the tapered waveguide section provides a virtual short to the coaxial probe signal,
the probe depth can be taken as that computed from equations A.5 and A.6. For this depth of probe, approximate location of the probe for impedance matching can be found out. Figure A.2, shows a circular waveguide of radius $r_0$ terminated to a cut off waveguide of radius $r_1$ through a tapered waveguide section. The maximum power from the coaxial probe can be coupled in the circular waveguide if the probe is located such that the direct power coupled to circular waveguide is in phase with the power reflected from the tapered section. To find out the location of the probe, the phase contribution by tapered section of waveguide should be known.

Let the location of the probe with respect to cutoff position is given by $X_t + X_s$ where, $X_t$ is the distance in tapered section and is $X_s$ the distance in straight waveguide section. Corresponding to these distances total phase can be given as

$$\text{Total phase} = \int_0^{X_t} \frac{2\pi}{\lambda_g(x)} \, dx + \frac{2\pi}{\lambda_s} X_s$$  \hspace{1cm} (A.7)

$$\lambda_g(x) = \frac{\lambda}{1 - (\lambda / \lambda_c(x))^2}^{1/2}$$

$$\lambda_c(x) = 2\pi r(x) / 1.841 = 3.41 r(x)$$

$r(x) = r_c + x \tan(\theta)$

where,
\( \lambda_g(x) \) is the guide wavelength in the tapered section and \( \lambda_c(x) \) is the cutoff wavelength in the tapered section. \( \lambda_g \) is the guide wavelength of straight waveguide section in which probe is protruding. \( r_c \) is the cut off radius for TE_{11} mode in the tapered section. \( r(x) \) is the radius of waveguide in the tapered section, \( x \) is the distance from cut off location and \( \theta \) is the taper angle. To achieve maximum coupling in the waveguide with a tapered section, the total phase computed from equation (A.7) can be taken as \( \frac{\pi}{2} \).

A circular waveguide of diameter 32.54 mm and a probe of depth 11.6 mm was taken. Circular waveguide was terminated at its input with a tapered waveguide section of taper angle 5.7°. The probe was kept at a location where the total phase becomes 90 degrees with respect to the cut off diameter (26.8 mm at 6.6 GHz) in the tapered section of waveguide. Taper angle of 5.7° yielded a distance of 28 mm between the cut off diameter and the coaxial probe for achieving a total phase of 90 degrees as computed using equation (A.7). The complete geometry was simulated on ansoft HFSS. With tapered short, the probe location for optimum coupling (or impedance matching) has been found to be 28.4 mm in simulation against the computed value of 28 mm from equation (A.7). For tapered short, the simulated return loss was of the order of -25 dB at 6.6 GHz and the -10 dB return loss bandwidth was of the order of 5 %. This simulation showed that the return loss and coupling performance of a circular waveguide terminated with a tapered section can be obtained if the height of the probe is taken as same as that computed for a perfectly shorted circular waveguide and the probe location computed using equation (A.7).

A.3 General Expression of Dyadic Green’s Function

The expressions for dyadic Green’s functions \( \overline{G}_{e1}, \overline{G}_{m1}, \overline{G}_{e2} \) and \( \overline{G}_{m2} \) for cylindrical waveguide [33] are,

\[
\overline{G}_{e2}(\vec{R}, \vec{R}') = -\frac{1}{k^2} \frac{e^{ik\vec{R}' \cdot \vec{R}}}{\sqrt{\pi}} + \sum_{n,m} \left[ c_{o} \overline{N}_{o_n}(\pm k_{p} \vec{R}) \overline{N}_{o_{n'}}(\mp k_{p} \vec{R}') + c_{o} \overline{M}_{o_{n}l}(\pm k_{p} \vec{R}) \overline{M}_{o_{n'}l}(\mp k_{p} \vec{R}') \right] \hat{Z}_c \hat{Z}' \tag{A.8}
\]

\[
\overline{G}_{e1}(\vec{R}, \vec{R}') = -\frac{1}{k^2} \frac{e^{ik\vec{R}' \cdot \vec{R}}}{\sqrt{\pi}} + \sum_{n,m} \left[ c_{o} \overline{M}_{o_{n}l}(\pm k_{p} \vec{R}) \overline{M}_{o_{n'}l}(\mp k_{p} \vec{R}') + c_{o} \overline{N}_{o_{n}l}(\pm k_{p} \vec{R}) \overline{N}_{o_{n'}l}(\mp k_{p} \vec{R}') \right] \hat{Z}_c \hat{Z}' \tag{A.9}
\]

\[
\overline{G}_{m1}(\vec{R}, \vec{R}') = \sum_{n,m} \left[ c_{o} \overline{N}_{o_{n}l}(\pm k_{p} \vec{R}) \overline{N}_{o_{n'}l}(\mp k_{p} \vec{R}') + c_{o} \overline{M}_{o_{n}l}(\pm k_{p} \vec{R}) \overline{M}_{o_{n'}l}(\mp k_{p} \vec{R}') \right] \hat{Z}_c \hat{Z}' \tag{A.10}
\]

\[
\overline{G}_{m2}(\vec{R}, \vec{R}') = \sum_{n,m} \left[ c_{o} \overline{N}_{o_{n}l}(\pm k_{p} \vec{R}) \overline{N}_{o_{n'}l}(\mp k_{p} \vec{R}') + c_{o} \overline{M}_{o_{n}l}(\pm k_{p} \vec{R}) \overline{M}_{o_{n'}l}(\mp k_{p} \vec{R}') \right] \hat{Z}_c \hat{Z}' \tag{A.11}
\]
The dyadic Green's functions in equations A.8 to A.11, have been expressed in terms of cylindrical vector wave functions $\vec{M}_{o\mu}$ and $\vec{N}_{o\lambda}$ as

$$\vec{M}_{o\mu}(h) = \nabla \times \left[ J_n(\mu r) \cos n\phi \ e^{i\phi z} \right]$$

$$\vec{N}_{o\lambda}(h) = \frac{1}{\kappa_{\lambda}} \nabla \times \nabla \times \left[ J_n(\lambda r) \cos n\phi \ e^{i\phi z} \right]$$

(A.12)  

(A.13)

where $\mu = \frac{q_{nm}}{a}$ and $\lambda = \frac{p_{nm}}{a}$

$$\kappa_{\lambda}^2 = \lambda^2 + h^2$$

$$\kappa_{\mu}^2 = \mu^2 + h^2$$

$$k_{\lambda}^2 = k^2 - \lambda^2$$

$$k_{\mu}^2 = k^2 - \mu^2$$

The general expressions given in equations A.8 to A.11, for $\vec{G}_{el}$, $\vec{G}_{ml}$, $\vec{G}_{e2}$ and $\vec{G}_{m2}$, the different components of dyadic Green's function in circular waveguide have been derived.

### A.4 Expression for Components of Green's Function

The derived expressions for $G_{rr}$, $G_{rz}$, $G_{r\phi}$, $G_{zr}$, $G_{z\phi}$, $G_{rz}$, $G_{r\phi}$, $G_{z\phi}$ are as follows.

(a) Component of $\vec{G}_{el}$ for computing electric field in circular waveguide due to electric current $J$ on the post or probe

$$G_{rr}(r,r') = \sum_{nm} C_{\mu} \left[ \frac{-nJ_n(\mu r) \cos(n\phi) e^{ikz}}{r} \right] \left[ \frac{-nJ_n(\mu r') \cos(n\phi') e^{-ikz'}}{r'} \right]$$

$$+ \sum_{nm} \frac{kC_{\mu}}{\kappa_{\lambda}^2} \left[ \frac{-ik_n J_n(\lambda r) \cos(n\phi) e^{ikz}}{r} \right] \left[ \frac{-ik_n J_n(\lambda r') \cos(n\phi') e^{-ikz'}}{r'} \right]$$

(A.14)

(b) Component of $\vec{G}_{m2}$ for computing magnetic field in circular waveguide due to electric current $J$ on the post or probe

$$G_{r\phi}(r,r') = \sum_{nm} kC_{\mu} \left[ \frac{1}{\kappa_{\mu}} \left[ \frac{-ik_n J_n(\mu r) \sin(n\phi) e^{ikz}}{r} \right] \right] \left[ \frac{-nJ_n(\mu r') \sin(n\phi') e^{-ikz'}}{r'} \right]$$

$$+ \sum_{nm} \frac{kC_{\mu}}{\kappa_{\lambda}^2} \left[ \frac{-\partial J_n(\lambda r) \cos(n\phi) e^{ikz}}{\partial r} \right] \left[ \frac{-\partial J_n(\lambda r') \cos(n\phi') e^{-ikz'}}{\partial r'} \right]$$

$$G_{zr}(r,r') = \sum_{nm} kC_{\mu} \left[ \frac{1}{\kappa_{\mu}} \left[ \frac{\mu^2 J_n(\mu r)}{1} \cos(n\phi) e^{ikz} \right] \right] \left[ \frac{-nJ_n(\mu r') \sin(n\phi') e^{-ikz'}}{r'} \right]$$

(A.15)  

(A.16)
(c) Component of $G_{m1}$ for computing electric field in circular waveguide due to magnetic current M in coaxial aperture (i.e., due to coaxial aperture field)

\[
G_{m}(r, r') = \sum_{nm} kC_{m} \left[ \frac{1}{\kappa_{m}} \left\{ -\frac{nJ_{n}(\mu r)}{r} \cos \phi e^{ikr} \right\} \left\{ \frac{ik_{n}nJ_{n}(\mu r')}{r'} \cos \phi e^{-ikr'} \right\} \right]
+ \sum_{nm} kC_{m} \left[ \left\{ \frac{\mu^{2}J_{n}(\mu r)}{1} \cos \phi e^{ikr} \right\} \left\{ \frac{\mu^{2}J_{n}(\mu r')}{1} \cos \phi e^{-ikr'} \right\} \right]
\]

(A.17)

(d) Component of $G_{e2}$ for computing magnetic field in circular waveguide due to magnetic current M in coaxial aperture (i.e., due to coaxial aperture field)

\[
G_{e2}(r, r') = \sum_{nm} kC_{m} \left[ \frac{1}{\kappa_{m}} \left\{ -\frac{nJ_{n}(\mu r)}{r} \sin \phi e^{ikr} \right\} \left\{ \frac{\mu^{2}J_{n}(\mu r')}{1} \cos \phi e^{-ikr'} \right\} \right]
\]

(A.18)

A.5 Mode Matching Technique for Step Junction Discontinuity in Circular Waveguide

A schematic of step junction between two uniform circular waveguide sections as shown in Figure A.3 has been taken into consideration to explain the mode matching technique. In this technique, the
total modal field is matched at the junction between uniform waveguide sections. The amplitudes of separate modes at the output of junction are computed in terms of the modes at the input of the junction. These amplitudes are expressed in the form of a scattering matrix.

![Diagram of a step junction between two smooth-walled waveguides of different radius.](image)

**Figure A.3** Step junction between two smooth-walled waveguides of different radius.

Let the transverse electric and magnetic modal functions \([35], [42]\) corresponding to \(M\) modes on the left side of the junction are represented by \(\vec{e}_{m1}, \vec{h}_{m1}\) and corresponding to \(N\) modes on the right side of the junction are represented by \(\vec{e}_{n2}, \vec{h}_{n2}\). The transverse electric and magnetic fields on the left side are represented as \([35]\).

\[
\vec{E}_1 = \sum_{m=1}^{M} \{A_m \exp(-\gamma_m z) + B_m \exp(\gamma_m z)\}\vec{e}_{m1} \tag{A.23}
\]

\[
\vec{H}_1 = \sum_{m=1}^{M} \{A_m \exp(-\gamma_m z) - B_m \exp(\gamma_m z)\}\vec{h}_{m1} \tag{A.24}
\]

where, \(A_m\) and \(B_m\) are the forward and reflected amplitude coefficients of mode \(m\) on the left-hand side of the junction.

The fields on the right side are,

\[
\vec{E}_2 = \sum_{n=1}^{N} \{C_n \exp(-\gamma_n z) + D_n \exp(\gamma_n z)\}\vec{e}_{n2} \tag{A.25}
\]

\[
\vec{H}_2 = \sum_{n=1}^{N} \{C_n \exp(-\gamma_n z) - D_n \exp(\gamma_n z)\}\vec{h}_{n2} \tag{A.26}
\]

where \(C_n\) and \(D_n\) are the forward and reflected amplitude coefficients of mode \(n\) on the right-hand side of the junction.

Matching the total transverse fields across the junction at \(z = 0\), gives
The continuity of the fields and the orthogonality relationship between modes results into a pair of simultaneous equations,

\[
\begin{align*}
\sum_{m=1}^{M} (A_m + B_m) \vec{e}_{m1} &= \sum_{n=1}^{N} (C_n + D_n) \vec{e}_{n2} \quad (A.27) \\
\sum_{m=1}^{M} (A_m - B_m) \vec{h}_{m1} &= \sum_{n=1}^{N} (C_n + D_n) \vec{e}_{n2} \quad (A.28)
\end{align*}
\]

where \([A]\) and \([B]\) are \(M\)-element column matrices in the section on the left-hand of the junction containing the unknown modal coefficients \(A_1\) to \(A_M\) and \(B_1\) to \(B_M\). Similarly, \([C]\) and \([D]\) are \(N\)-element column matrices on the right-hand side of the junction containing the unknown modal coefficients \(C_1\) to \(C_N\) and \(D_1\) to \(D_N\). \([P]\) is an \(N\times M\) matrix whose elements are integrals representing the mutual coupled power between mode \(i\) on the left-hand side and mode \(j\) on the right hand-side. \([P]^T\) is transpose of \([P]\), i.e., rows and column are interchanged. \([Q]\) is an \(N\times N\) diagonal matrix whose elements are integrals representing the self coupled power on the right hand-side of the junction. \([R]\) is an \(M\times N\) diagonal matrix whose elements are integrals representing the self coupled power on the left hand-side of the junction. The elements of these matrices are given as \([35], [42]\).

\[
\begin{align*}
P_{mn} &= \int (\vec{e}_{m1} \times \vec{h}_{n2}) \cdot d_s, \quad (A.31) \\
Q_{mn} &= \int (\vec{e}_{n2} \times \vec{h}_{n2}) \cdot d_s, \quad (A.32) \\
R_{mn} &= \int (\vec{e}_{m1} \times \vec{h}_{m1}) \cdot d_s, \quad (A.34)
\end{align*}
\]

where \(s_1\) and \(s_2\) are the cross-sectional areas of the waveguides of radius \(a_1\) and \(a_2\) respectively.

Equations (A.29) and (A.30) can be rearranged into the scattering matrix \([S]\) formulation as,

\[
[Y] = [S][X] \quad (A.35)
\]

where,

\[
[S] = \begin{bmatrix}
[S_{11}] & [S_{12}] \\
[S_{21}] & [S_{22}]
\end{bmatrix} \quad (A.36)
\]
The elements of $[S]$ in terms of matrices $[R], [P], [Q]$ are described in [35] and given as,

$$
[Y] = \begin{bmatrix} B \\ D \end{bmatrix}
$$

$$
[X] = \begin{bmatrix} A \\ C \end{bmatrix}
$$

(C.37)

Cascading of Scattering Matrices

A mode transducer geometry having tapered waveguide section between two uniform waveguides of different cross-sections is shown in Figure A.4. The tapered section is approximated as series of incremental step junctions. Scattering matrix for incremental straight waveguide section between two step junctions should be known. This scattering matrix will be an $N \times N$ matrix having only diagonal elements as $V_{im} = \exp(-\gamma \kappa l)$, where $l$ is the length of the incremental length of waveguide between junctions. The scattering matrix due to step junction discontinuities and due to the incremental lengths is cascaded progressively to obtain an overall scatter matrix for the geometry shown in Figure A.4.

Figure A.4  Smooth wall conical waveguide approximated by incremental step junctions to apply mode matching using cylindrical modes.
If there are two scattering matrices \([S^a]\) and \([S^b]\) corresponding to two step junctions, then the expressions for cascaded scattering matrix \([S^c]\) are [35],

\[
[S^c] = \begin{bmatrix}
[S^c_{11}] & [S^c_{12}] \\
[S^c_{21}] & [S^c_{22}]
\end{bmatrix}
\]

where,

\[
[S^c_{11}] = [S^a_{12}] [I] - [S^b_{11}] [I][S^a_{22}]^{-1} [S^b_{11}] [S^a_{21}] + [S^a_{11}]
\]

(A.43)

\[
[S^c_{12}] = [S^a_{12}] [I] - [S^b_{11}] [I][S^a_{22}]^{-1} [S^b_{12}]
\]

(A.44)

\[
[S^c_{21}] = [S^b_{21}] [I] - [S^a_{22}] [I][S^b_{11}]^{-1} [S^a_{21}]
\]

(A.45)

\[
[S^c_{22}] = [S^b_{21}] [I] - [S^a_{22}] [I][S^b_{12}]^{-1} [S^a_{22}] + [S^b_{22}]
\]

(A.46)

Similar to cascading of two matrices, one can cascade number of scattering matrices corresponding to different junctions in order to obtain resultant scattering matrix \([S]\). From the resultant matrix \([S]\) the characteristics of any conical horn can be found out from,

\[
\begin{bmatrix}
[B] \\
[D]
\end{bmatrix} = \begin{bmatrix}
[A] \\
[C]
\end{bmatrix}
\]

(A.47)

where, \([A]\) and \([B]\) are column matrices containing the forward and reflected coefficients of all the mode looking into the horn from source side. \([C]\) and \([D]\) are column matrices containing the forward and reflected coefficients of all modes looking into the aperture of the horn from the outside. The reflection coefficient of the horn is \([B] = [S_{11}] [A]\). The transmission coefficient of the horn is \([D] = [S_{21}] [A]\), where, \([A]\) is the matrix corresponding to incident mode which is assumed to be \(\text{TE}_{11}\) mode of circular waveguide.