Chapter 3

Balance Index Set of Trees

3.1 Introduction

S. M. Lee, H. H. Su and H. C. Wang [24] found the balance index set of trees of diameter four. In this chapter, the balance index set of caterpillar and lobster graphs of diameter $n$ are obtained.

3.2 Balance index set of caterpillar graphs

For a caterpillar graph $CT(a_1, a_2, a_3, \ldots, a_{n-1})$ of diameter $n$, $a_i$, $i=1, 2, 3, \ldots, n-1$ are the number of vertices adjacent to $i^{th}$ spine vertex.

Reference [32] is based on this chapter.
Denote \( n - 1 \) vertices on the spine as \( u_{a_1}, u_{a_2}, u_{a_3}, \ldots, u_{a_{n-1}} \). Thus, there are \((a_1+a_2+a_3+\cdots+a_{n-1})\) pendant vertices and degrees of \( u_{a_1}, u_{a_2}, u_{a_3}, \ldots, u_{a_{n-1}}\) are \( a_1 + 1, a_2 + 2, a_3 + 2, \ldots, a_{n-2} + 2, a_{n-1} + 1 \) respectively. Also, non-spinal vertices adjacent to \( u_{a_i} \) denoted by \( u_{a_i,1}, u_{a_i,2}, u_{a_i,3}, \ldots, u_{a_i,a_i} \), where \( i = 1, 2, 3, \ldots, n - 1 \).

**Theorem 3.2.1.** The balance index of a caterpillar graph \( CT(a_1, a_2, a_3, \ldots, a_{n-1}) \) of order \( p \) and diameter \( n \) is,

\[
\frac{1}{2} \left( l + \sum_{i=1}^{n-1} (-1)^{f(u_{a_i})} a_i \right), \quad \text{if } p \text{ is even}
\]

\[
\frac{1}{2} \left( l + 1 + \sum_{i=1}^{n-1} (-1)^{f(u_{a_i})} a_i \right), \quad \text{if } p \text{ is odd}
\]

where

\[ l = \begin{cases} 
    n - 2j - 3, & \text{for } j = i, i - 1, i - 2, \text{ where } i = 2, 3, 4, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \\
    n - 3, & \text{for } f(u_{a_i}) = 0, \forall i \text{ or } f(u_{a_i}) = \begin{cases} 
        1, & \text{if } i = 1, n - 1 \\
        0, & \text{elsewhere.}
    \end{cases}
\end{cases}
\]

**Proof.** Consider the caterpillar graph \( CT(a_1, a_2, a_3, \ldots, a_{n-1}) \) of order \( p \) and diameter \( n \).
Case 1. When \( n \) is even.

Subcase 1.1. If \((a_1 + a_2 + a_3 + \cdots + a_{n-1})\) is odd, then the order \( p \) of \( CT(a_1, a_2, a_3, \ldots, a_{n-1}) \) is \( a_1 + a_2 + a_3 + \cdots + a_{n-1} + n - 1 \), which is even.

Let \((a_1 + a_2 + a_3 + \cdots + a_{n-1}) + n - 1 = 2M\). For the friendly labeling, \( M \) vertices are labeled 0 and remaining \( M \) vertices are labeled 1. First consider the case that \( u_{a_1}, u_{a_2}, u_{a_3}, \ldots, u_{a_{n-1}} \) are all labeled 0, i.e. \( n-1 \) spine vertices are partitioned into \((n-1, 0)\). Then \( M-(n-1) \) pendant vertices are labeled 0 and remaining \( M \) pendant vertices are labeled 1.

Thus, by Corollary 2.1.1,
\[
|e_{f^{*}}(0) - e_{f^{*}}(1)| = \frac{1}{2} \left| \left( \sum_{v \in v(0)} \deg(v) - \sum_{v \in v(1)} \deg(v) \right) \right| \\
= \frac{1}{2} |M - (n - 1) + (a_1 + 1) + (a_2 + 2) + (a_3 + 2) + \cdots + (a_{n-1} + 1) - M| \\
= \frac{1}{2} |a_1 + a_2 + a_3 + \cdots + a_{n-1} + n - 3|.
\]
If \( n-1 \) spine vertices are partitioned into \((n-2, 1)\), then \( M-(n-2) \) pendant vertices are labeled 0 and remaining \( M-1 \) pendant vertices are labeled 1.

Two possibilities arise.

(a) When \( f(u_{a_1}) = 1 \),
\[
|e_{f^{*}}(0) - e_{f^{*}}(1)| = \frac{1}{2} |M - (n - 2) - (a_1 + 1) + (a_2 + 2) + (a_3 + 2) + \cdots + (a_{n-1} + 1) - (M - 1)| \\
= \frac{1}{2} |a_1 + a_2 + a_3 + \cdots + a_{n-1} + n - 3|.
\]
Similarly, when \( f(u_{a_{n-1}}) = 1 \),
\[
|e_{f^{*}}(0) - e_{f^{*}}(1)| = \frac{1}{2} |a_1 + a_2 + a_3 + \cdots + a_{n-2} - a_{n-1} + n - 3|.
\]
(b) If one of the spine vertices of degree \(a_i\), \(i = 2, 3, 4, \ldots, n - 2\) is labeled 1, then \(M - (n - 2)\) pendant vertices are labeled 0 and remaining \(M - 1\) pendant vertices are labeled 1.

Thus, \(|e(f^*)(0) - e(f^*)(1)| = \frac{1}{2}|M - (n - 2) + (a_1 + 1) + (a_2 + 2) + (a_3 + 2) + \ldots - (a_i + 2) + \ldots + (a_{n-1} + 1) - (M - 1)| = \frac{1}{2}|a_1 + a_2 + a_3 + \cdots + a_{i-1} - a_i + a_{i+1} + \cdots + a_{n-1} + n - 5|\),

where \(i = 2, 3, 4, \ldots, n - 2\).

If \(n - 1\) spine vertices are partitioned into \((n - i, i)\), where \(i = 2, 3, 4, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\), then \(M - (n - 1 - i)\) pendant vertices are labeled 0 and remaining \(M - i\) pendant vertices are labeled 1.

Three possibilities arise.

(a) When \(f(u_1) = f(u_{a_{n-1}}) = 0\),

\(|e(f^*)(0) - e(f^*)(1)| = \frac{1}{2}|M - (n - 1 - i) + (a_1 + 1) + (a_2 + 2) + (a_3 + 2) + \ldots + (a_{n-2} + 2) + (a_{n-1} + 1) - (M - i)| = \frac{1}{2}|a_1 + a_2 + a_3 + \cdots + a_{n-1} + n - 2i - 3|\), where \(i = 2, 3, 4, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\) and \(i\) coefficients out of \(a_2, a_3, a_4, \ldots, a_{n-2}\) are negative.

(b) When \(f(u_1) = 0\) and \(f(u_{a_{n-1}}) = 1\),

\(|e(f^*)(0) - e(f^*)(1)| = \frac{1}{2}|M - (n - 1 - i) + (a_1 + 1) + (a_2 + 2) + (a_3 + 2) + \ldots + (a_{n-2} + 2) - (a_{n-1} + 1) - (M - i)| = \frac{1}{2}|a_1 + a_2 + a_3 + \cdots + a_{n-2} - a_{n-1} + n - 2i - 1|\), where \(i = 2, 3, 4, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\) and \(i - 1\) coefficients out of \(a_2, a_3, a_4, \ldots, a_{n-2}\) are negative.
(c) When \( f(u_{a_1}) = f(u_{a_{n-1}}) = 1, \)
\[ |e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2}|M - (n - 1 - i) - (a_1 + 1) + (a_2 + 2) + (a_3 + 2) + \ldots + (a_{n-2} + 2) - (a_{n-1} + 1) - (M - i)| \]
\[ = \frac{1}{2}|-2 + a_2 + a_3 + \ldots + a_{n-2} - a_{n-1} + n - 2i + 1|, \]
where \( i = 2, 3, 4, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \) and \( i - 2 \) coefficients out of \( a_2, a_3, a_4, \ldots, a_{n-2} \) are negative.

**Subcase 1.2.** If \((a_1 + a_2 + a_3 + \cdots + a_{n-1})\) is even, then the order \( p \) of \( CT(a_1, a_2, a_3, \ldots, a_{n-1}) \) is odd.

Let \((a_1 + a_2 + a_3 + \cdots + a_{n-1}) + n - 1 = 2M + 1.\) For a friendly labeling, without loss of generality, there are \( M + 1 \) vertices labeled 0 and \( M \) vertices labeled 1.

Consider the case that \( u_{a_1}, u_{a_2}, u_{a_3}, \ldots, u_{a_{n-1}} \) are all labeled 0, i.e. \( n - 1 \) spine vertices are partitioned into \((n-1, 0).\) Then \((M+1)-(n-1)\) pendant vertices are labeled 0 and remaining \( M \) pendant vertices are labeled 1.

Thus, \[ |e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2}|(M+1)-(n-1)+(a_1+1)+(a_2+2)+(a_3+2)+\ldots+(a_{n-1}+1)-M| \]
\[ = \frac{1}{2}|a_1 + a_2 + a_3 + \cdots + a_{n-1} + n - 2|. \]

If \( n - 1 \) spine vertices are partitioned into \((n-2, 1),\) then \((M+1)-(n-2)\) pendant vertices are labeled 0 and remaining \( M - 1 \) pendant vertices are labeled 1.
Two possibilities arise.

(a) When \( f(u_{a_1}) = 1 \),
\[
|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2}(M + 1) - (n - 2) - (a_1 + 1) + (a_2 + 2) + (a_3 + 2) + \ldots
+ (a_{n-1} + 1) - (M - 1)\]
\[= \frac{1}{2} |a_1 + a_2 + a_3 + \ldots + a_{n-1} + n - 2|.
\]
Similarly, when \( f(u_{a_{n-1}}) = 1 \),
\[
|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2} |a_1 + a_2 + a_3 + \ldots + a_{n-2} - a_{n-1} + n - 2|.
\]

(b) When one of the spine vertices of degree \( a_i \), \( i = 2, 3, 4, \ldots, n - 2 \) is labeled 1, \( M - (n - 2) \) pendant vertices are labeled 0 and remaining \( M - 1 \) pendant vertices are labeled 1. Thus,
\[
|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2}(M + 1) - (n - 2) + (a_1 + 1) + (a_2 + 2) + (a_3 + 2) + \ldots
- (a_i + 2) + \ldots + (a_{n-1} + 1) - (M - 1)\]
\[= \frac{1}{2} |a_1 + a_2 + a_3 + \ldots + a_{i-1} - a_i + a_{i+1} + \ldots + a_{n-1} + n - 4|,
\]
where \( i = 2, 3, 4, \ldots, n - 2 \).

If \( n - 1 \) spine vertices are partitioned into \( (n - i, i) \), where \( i = 2, 3, 4, \ldots, \lfloor \frac{n}{2} \rfloor \), then \( (M + 1) - (n - 1 - i) \) pendant vertices are labeled 0 and \( M - i \) pendant vertices are labeled 1.

Three possibilities arise.

(a) When \( f(u_{a_1}) = f(u_{a_{n-1}}) = 0 \),
\[
|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2}(M + 1) - (n - 1 - i) + (a_1 + 1) + (a_2 + 2) + (a_3 + 2) + \ldots + (a_{n-2} + 2) + (a_{n-1} + 1) - (M - i)|
\]

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\[ \frac{1}{2} |a_1 + a_2 + a_3 + \cdots + a_{n-1} + n - 2i - 2|, \] where \( i = 2, 3, 4, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \) and \( i \) coefficients out of \( a_2, a_3, a_4, \ldots, a_{n-2} \) are negative.

(b) When \( f(u_{a_1}) = 0 \) and \( f(u_{a_{n-1}}) = 1 \),
\[ |e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2} |(M + 1) - (n - 1 - i) + (a_1 + 1) + (a_2 + 2) + (a_3 + 2) + \cdots + (a_{n-2} + 2) - (a_{n-1} + 1) - (M - i)| \]
\[ = \frac{1}{2} |a_1 + a_2 + a_3 + \cdots + a_{n-2} - a_{n-1} + n - 2i|, \] where \( i = 2, 3, 4, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \) and \( i - 1 \) coefficients out of \( a_2, a_3, a_4, \ldots, a_{n-2} \) are negative.

(c) When \( f(u_{a_1}) = f(u_{a_{n-1}}) = 1 \),
\[ |e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2} |(M + 1) - (n - 1 - i) + (a_1 + 1) + (a_2 + 2) + (a_3 + 2) + \cdots + (a_{n-2} + 2) - (a_{n-1} + 1) - (M - i)| \]
\[ = \frac{1}{2} |a_1 + a_2 + a_3 + \cdots + a_{n-2} - a_{n-1} + n - 2i + 2|, \]
where \( i = 2, 3, 4, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \) and \( i - 2 \) coefficients out of \( a_2, a_3, a_4, \ldots, a_{n-2} \) are negative.

Case 2. When \( n \) is odd.

Subcase 2.1. If \((a_1 + a_2 + a_3 + \cdots + a_{n-1})\) is odd, then the order \( p \) of \( CT(a_1, a_2, a_3, \ldots, a_{n-1}) \) is \( a_1 + a_2 + a_3 + \cdots + a_{n-1} + n - 1 \) which is odd.

Thus, proof is similar to Subcase 1.2.

Subcase 2.2. If \((a_1 + a_2 + a_3 + \cdots + a_{n-1})\) is even, then the order \( p \) of \( CT(a_1, a_2, a_3, \ldots, a_{n-1}) \) is \( a_1 + a_2 + a_3 + \cdots + a_{n-1} + n - 1 \) which is even.

Thus, proof is similar to Subcase 1.1. \(\square\)
Example 3.2.1. The balance index set of the caterpillar \( CT(2, 1, 1, 2, 1) \) is \( \{0, 1, 2, 3, 4, 5\} \).

Figure 3.1: The caterpillar \( CT(2, 1, 1, 2, 1) \) of diameter 6 and order 12.

Corollary 3.2.1. The balance index set of the graph \( B_{l,m,k} \) is

\[
\left\{ \left\lfloor \frac{l+m+k}{2} \right\rfloor, \left\lfloor \frac{l-m+k}{2} \right\rfloor, \left\lfloor \frac{-l+m+k}{2} \right\rfloor, \left\lfloor \frac{l+m+k-2}{2} \right\rfloor \right\} \bigcup \\
\left\{ \left\lfloor \frac{l+m+k-2i}{2} \right\rfloor, \left\lfloor \frac{l-m+k-2i+2}{2} \right\rfloor, \left\lfloor \frac{-l-m+k-2i+4}{2} \right\rfloor : i = 2, 3, 4, \ldots, \left\lfloor \frac{k+2}{2} \right\rfloor \right\}
\]

Proof. The graph \( B_{l,m,k} \) is a caterpillar \( CT(l, 0, 0, \ldots, m) \) of diameter \( k + 2 \). Therefore, substituting \( n = k + 2, a_1 = l, a_{n-1} = m \) and \( a_2 = a_3 = a_4 = \ldots = a_{n-2} = 0 \) in Theorem 3.2.1, we get the balance index set of \( B_{l,m,k} \). \( \square \)

Example 3.2.2. The balance index set of \( B_{3,3,3} \) is \( \{0, 1, 2, 3, 4\} \).

Figure 3.2: The graph \( B_{3,3,3} \) of diameter 5 and order 10.

Corollary 3.2.2. The balance index set of coconut tree is

\[
\left\{ \left\lfloor \frac{l+m-2}{2} \right\rfloor, \left\lfloor \frac{-l+m-2i}{2} \right\rfloor, \left\lfloor \frac{l+m-4}{2} \right\rfloor \right\} \bigcup \\
\left\{ \left\lfloor \frac{l+m-2i-2}{2} \right\rfloor, \left\lfloor \frac{-l+m-2i}{2} \right\rfloor, \left\lfloor \frac{l-m-2i+2}{2} \right\rfloor : i = 2, 3, 4, \ldots, \left\lfloor \frac{m}{2} \right\rfloor \right\}
\]
Proof. The coconut tree is a caterpillar $CT(0, 0, 0, \ldots, m)$ of diameter $m$. Therefore, substituting $n = m$, $a_{n-1} = l$ and $a_1 = a_2 = a_3 = \cdots = a_{n-2} = 0$ in Theorem 3.2.1, we get the balance index set of coconut tree.

Example 3.2.3. The balance index set of the coconut tree of diameter 5 and order 9 is $\{0, 1, 2, 3, 5\}$.

![Figure 3.3: The coconut tree of diameter 5 and order 9.](image)

3.3 **Balance index set of lobster graphs**

In a caterpillar $CT(a_1, a_2, a_3, \ldots, a_{n-1})$, if $a_i \neq 0$ for $i = 2, 3, \ldots, n-2$, then $a_i$, $i = 2, 3, 4, \ldots, n-2$ number of $P_3$ paths contained the vertex $u_{a_i}$. Since $P_3$ is of length 2, after adding more adjacent edges and vertices to the end vertices of these paths, the new graph is a lobster graph of diameter $n$.

Denote this graph as

$CT(a_1, a_2, a_3, \ldots, a_{n-1})(u_{a_2}(t_{2,1}, t_{2,2}, t_{2,3}, \ldots, t_{2,a_2}), (u_{a_3}(t_{3,1}, t_{3,2}, t_{3,3}, \ldots, t_{3,a_3})), u_{a_4}(t_{4,1}, t_{4,2}, t_{4,3}, \ldots, t_{4,a_4}), \ldots, u_{a_{n-2}}(t_{n-2,1}, t_{n-2,2}, t_{n-2,3}, \ldots, t_{n-2,a_{n-2}})),$

where $t_{i,j}$ is the number of edges and vertices added to the vertex $u_{a_{i,j}}$,

$i = 2, 3, 4, \ldots, n-2$ and $j = 1, 2, \ldots, a_i$. 

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In order to write the results in an uniform manner, name this lobster as

$$CT(d_1, d_2, d_3, \ldots, d_{n-2}, d_0) \ (u_{a2}(d_{n-1}, d_n, d_{n+1}, \ldots, d_{d_2+n-2}), u_{a3}(d_{d_2+n-1}, d_{d_2+n}, d_{d_2+n+1}, \ldots, d_{d_2+d_3+n-2}), u_{a4}(d_{d_2+d_3+n-1}, d_{d_2+d_3+n}, d_{d_2+d_3+n+1}, \ldots, d_{d_2+d_3+d_4+n-2}), \ldots, u_{a_{n-2}}(d_{d_2+d_3+\ldots+d_{n-3}+n-1}, d_{d_2+d_3+\ldots+d_{n-3}+n}, d_{d_2+d_3+\ldots+d_{n-3}+n+1}, \ldots, d_{d_2+d_3+\ldots+d_{n-2}+n-2}).$$

Also, denote $n-1$ spine vertices by $v_1, v_2, v_3, \ldots, v_{n-2}, v_0$, the vertices adjacent to $v_2$ by $v_{n-1}, v_n, v_{n+1}, \ldots, v_{d_2+n-2}$, adjacent to $v_3$ by $v_{d_2+n-1}, v_{d_2+n}, v_{d_2+n+1}, \ldots, v_{d_2+d_3+n-2}$, etc. and adjacent to $v_{n-2}$ by $v_{d_2+d_3+d_4+\ldots+d_{n-3}+n-1}$.

Thus, this lobster contains $d_0 + d_1 + \sum_{i=2}^{n-1} d_i$ pendant vertices. Also, the degree of $v_i$ for $n-1 \leq i \leq \left(\sum_{j=2}^{n-2} d_j\right) + n - 2$ is $d_i + 1$ and the degree of $v_k$ is $d_k + 2$ for $k = 2, 3, 4, \ldots, n - 2$.

**Theorem 3.3.1.** The balance index of a lobster graph of diameter $n$ and order $p$ is

$$e_f.(0) - e_f.(1) = \begin{cases} \pm \frac{1}{2} \left[ 1 + \sum_{i=0}^{\left(\sum_{j=2}^{n-2} d_j\right) + n - 2} (-1)^{f(v_i)} d_i + \sum_{i=2}^{n-2} (-1)^{f(v_i)} \right], & \text{if } p \text{ is odd.} \\
\frac{1}{2} \left[ \sum_{i=0}^{\left(\sum_{j=2}^{n-2} d_j\right) + n - 2} (-1)^{f(v_i)} d_i + \sum_{i=2}^{n-2} (-1)^{f(v_i)} \right], & \text{if } p \text{ is even.} \end{cases}$$
Proof. Consider the lobster graph

\[ \text{CT}(d_1, d_2, d_3, \ldots, d_{n-2}, d_0) (u_{a_2}(d_{n-1}, d_n, d_{n+1}, \ldots, d_{d_2+n-2}), u_{a_3}(d_{d_2+n-1}, d_{d_2+n}, d_{d_2+n+1}, \ldots, d_{d_2+d_3+n-2}), u_{a_4}(d_{d_2+d_3+n-1}, d_{d_2+d_3+n}, d_{d_2+d_3+n+1}, \ldots, d_{d_2+d_3+\ldots+d_{n-3}+n-2}), \ldots, d_{d_2+d_3+\ldots+d_{2(n-3)+n-2)}) \]

of order \( p \) and diameter \( n \).

Case 1. When \( n \) is even.

Subcase 1.1. If \( \sum_{i=0}^{n-2} d_i \) is odd, then the number of vertices equal to \( \left[ \sum_{j=0}^{n-2} d_j \right] + n - 1 \) is even. Let it be \( 2M \). For a friendly labeling, there are \( M \) vertices labeled 0 and remaining \( M \) vertices labeled 1.

Let \( v_i \) for all \( i \geq 0 \) be labeled 0. Then there are \( M - (n-1) - \sum_{i=2}^{n-2} d_i \) pendant vertices labeled 0 and remaining \( M \) pendant vertices labeled 1.

Thus, \( e_{f^*}(0) - e_{f^*}(1) = \)

\[
\frac{1}{2} \left[ M - (n-1) + 2(n-3) + d_0 + 1 + d_1 + 1 + \left( \sum_{i=n-1}^{n-2} (d_i + 1) \right) - M \right]
\]

\[
= \frac{1}{2} \left[ \left( \sum_{j=0}^{n-2} d_j \right) + n - 3 \right]
\]
\[
\frac{1}{2} \left[ \left( \left( \sum_{j=2}^{n-2} d_j \right) + n - 2 \sum_{i=0}^{n-2} d_i \right) + n - 3 \right].
\]

Similarly, if \( k \) vertices among \( v_i \) for all \( 0 \leq i \leq \left( \sum_{j=2}^{n-2} d_j \right) + n - 2 \) labeled 0, then there are \( M - k \) pendant vertices labeled 0 and remaining \( M - \left( \left( \sum_{j=2}^{n-2} d_j \right) + n - 1 - k \right) \) pendant vertices labeled 1. Define \( P \) and \( N \) to be the sets containing all 0-vertices and 1-vertices among \( v_i \) respectively, for all \( 0 \leq i \leq \left( \sum_{j=2}^{n-2} d_j \right) + n - 2 \).

Thus, \( e_{f^*}(0) - e_{f^*}(1) = \)

\[
\frac{1}{2} \left[ M - k + \sum_{v \in P} \text{deg}(v) - M - \left( \left( \sum_{j=2}^{n-2} d_j \right) + n - 1 - k \right) + \sum_{v \in N} \text{deg}(v) \right]\]

\[
= \frac{1}{2} \left[ M - k + \left( \sum_{v \in P} \text{deg}(v) - 1 \right) + 1 \right] - M + \left( \sum_{j=2}^{n-2} d_j \right)\]

\[
+ \left[ n - 1 - k - \left( \sum_{v \in N} \text{deg}(v) - 1 \right) + 1 \right]\]

\[
= \frac{1}{2} \left[ M - k + \left( \sum_{v \in P} \text{deg}(v) - 1 \right) \right] + k - M + \left( \sum_{j=2}^{n-2} d_j \right)\]

\[
+ \left[ n - 1 - k - \left( \sum_{v \in N} \text{deg}(v) - 1 \right) \right] - \left( \left( \sum_{j=2}^{n-2} d_j \right) + n - 1 - k \right)\]

\[
= \frac{1}{2} \left[ \sum_{v \in P} \text{deg}(v) - 1 - \sum_{v \in N} \text{deg}(v) - 1 \right].
\]

Also,

\[
\text{deg}(v) - 1 = \begin{cases} 
  d_i, & \text{for } i = 0, 1 \text{ and } n - 3 \leq i \leq \left( \sum_{j=2}^{n-2} d_j \right) + n - 2 \\
  d_i + 1, & \text{for } 2 \leq i \leq n - 2.
\end{cases}
\]
Therefore, $e_{f^*}(0) - e_{f^*}(1) =$

$$\frac{1}{2} \left[ (-1)^{f(v_0)} d_0 + (-1)^{f(v_1)} d_1 + \sum_{i=n-3}^{n-2} (-1)^{f(v_i)} d_i + \sum_{i=2}^{n-2} (-1)^{f(v_i)} (d_i + 1) \right]$$

$$= \frac{1}{2} \left[ \left( \sum_{j=2}^{n-2} d_j \right)^{+n-2} \right]$$

Subcase 1.2. If $\sum_{i=0}^{n-2} d_i$ is even, then the number of vertices equal to $\left( \sum_{i=0}^{n-2} d_i \right) + n - 1$ is odd. Let it be $2M + 1$. For friendly labeling, there are $M + 1$ vertices labeled 0 and remaining $M$ vertices labeled 1.

If $v_i$ for all $i \geq 0$ are labeled 0, then there are $(M + 1) - (n - 1) - \sum_{i=2}^{n-2} d_i$ pendant vertices labeled 0 and remaining $M$ pendant vertices labeled 1.

Thus, $e_{f^*}(0) - e_{f^*}(1) =$

$$\frac{1}{2} \left[ (M + 1) - (n - 1) + 2(n - 3) + d_0 + 1 + d_1 + 1 + \left( \sum_{i=n-1}^{n-2} (d_i + 1) \right) - M \right]$$

$$= \frac{1}{2} \left[ \left( \sum_{i=n-1}^{n-2} d_i \right) + \left( \sum_{i=0}^{n-2} d_i \right) + n - 2 \right]$$
\[
\frac{1}{2} \left[ \left( \sum_{j=2}^{n-2} d_j \right) + n - 2 \right] - \left( \sum_{i=0}^{\left( \sum_{j=2}^{n-2} d_j \right) + n - 2} d_i \right) + n - 2
\]

Similarly, when \( k \) vertices among \( v_i \) for all \( 0 \leq i \leq \left( \sum_{j=2}^{n-2} d_j \right) + n - 2 \) are labeled 0, there are \( M + 1 - k \) pendant vertices labeled 0 and remaining \( M - \left( \sum_{j=2}^{n-2} d_j \right) + n - 1 - k \) pendant vertices labeled 1.

Define \( P \) and \( N \) to be the sets containing 0-vertices and 1-vertices among \( v_i \) respectively, for all \( 0 \leq i \leq \left( \sum_{j=2}^{n-2} d_j \right) + n - 2 \).

Thus, \( e_f^\star(0) - e_f^\star(1) = \)
\[
\frac{1}{2} \left[ \left( M + 1 - k + \sum_{v \in P} \deg(v) \right) - \left( M - \left( \sum_{j=2}^{n-2} d_j \right) + n - 1 - k \right) + \sum_{v \in N} \deg(v) \right]
\]
\[
= \frac{1}{2} \left[ M + 1 - k + \left( \sum_{v \in P} (\deg(v) - 1) + 1 \right) - M + \left( \sum_{j=2}^{n-2} d_j \right) \right]
\]
\[
+ \left[ n - 1 - k - \left( \sum_{v \in N} (\deg(v) - 1) + 1 \right) \right]
\]
\[
= \frac{1}{2} \left[ M + 1 - k + \left( \sum_{v \in P} (\deg(v) - 1) \right) + k - M + \left( \sum_{j=2}^{n-2} d_j \right) \right]
\]
\[
+ \left[ n - 1 - k - \left( \sum_{v \in N} (\deg(v) - 1) \right) - \left( \sum_{j=2}^{n-2} d_j \right) + n - 1 - k \right]
\]
\[
= \frac{1}{2} \left[ 1 + \sum_{v \in P} (\deg(v) - 1) - \sum_{v \in N} (\deg(v) - 1) \right].
\]

Also,
\[
\deg(v) - 1 = \begin{cases} 
  d_i, & \text{for } i = 0, 1 \text{ and } n - 3 \leq i \leq \left( \sum_{j=2}^{n-2} d_j \right) + n - 2 \\
  d_i + 1, & \text{for } 2 \leq i \leq n - 2.
\end{cases}
\]
Chapter 3. Balance Index Set of Trees

So, \( e_f^\ast(0) - e_f^\ast(1) = \)
\[
\frac{1}{2} \left[ 1 + (-1)^{f(v_0)}d_0 + (-1)^{f(v_1)}d_1 + \sum_{i=n-3}^{n-2} (-1)^{f(v_i)}d_i + \sum_{i=2}^{n-2} (-1)^{f(v_i)}(d_i + 1) \right]
\]
\[
= \frac{1}{2} \left[ 1 + \sum_{i=0}^{n-2} (-1)^{f(v_i)}d_i + \sum_{i=2}^{n-2} (-1)^{f(v_i)} \right].
\]

When a friendly labeling satisfy \( v_f(1) > v_f(0) \), it produces the negative values of the above balance indexes.

Therefore, \( e_f^\ast(0) - e_f^\ast(1) = \pm \frac{1}{2} \left[ 1 + \sum_{i=0}^{n-2} (-1)^{f(v_i)}d_i + \sum_{i=2}^{n-2} (-1)^{f(v_i)} \right] \).

**Case 2.** When \( n \) is odd.

**Subcase 2.1.** If \( \sum_{i=0}^{n-2} d_i \) is odd, then the number of vertices equal to
\[
\left[ \frac{(n-2)}{2} \right] + n - 1 \text{ is odd and proof is similar to Subcase 1.2.}
\]
Subcase 2.2. If \( \sum_{i=0}^{n-2} d_i \) is even, then the number of vertices equal to 
\[
\left( \sum_{j=2}^{n-2} d_j \right) + n - 2 
\]
and \( \sum_{i=0}^{n-2} d_i \) + \( n - 1 \) is even and proof is similar to Subcase 1.1.

Hence, for a lobster graph of diameter \( n \) and order \( p \), the balance index is
\[
e_{f^*}(0) - e_{f^*}(1) = \begin{cases} 
\frac{1}{2} \left[ \left( \sum_{j=2}^{n-2} d_j \right) + n - 2 \right] \sum_{i=0}^{n-2} (-1)^{f(v_i)} d_i + \sum_{i=2}^{n-2} (-1)^{f(v_i)} , & \text{if } p \text{ is even} \\
\pm \frac{1}{2} \left[ \left( \sum_{j=2}^{n-2} d_j \right) + n - 2 \right] 1 + \sum_{i=0}^{n-2} (-1)^{f(v_i)} d_i + \sum_{i=2}^{n-2} (-1)^{f(v_i)} , & \text{if } p \text{ is odd}.
\end{cases}
\]