Chapter 2

Balance Index Set of Graphs

2.1 Introduction

For each binary vertex labeling $f$ of a graph $G$, the partial edge labeling $f^*$ of $G$ is defined as follows: For each edge $uv$ in $G$,

$$f^*(uv) = \begin{cases} 
0, & \text{if } f(u) = f(v) = 0 \\
1, & \text{if } f(u) = f(v) = 1.
\end{cases}$$

Let $v_f(i)$ be the number of vertices of $G$ that are labeled by $i$ under $f$ and $e_{f^*}(i)$ is the number of edges that are labeled by $i$ under $f^*$, where $i = 0, 1$.

References [29, 33] are based on this chapter.
The vertex labeling $f$ is called friendly if $|v_f(0) - v_f(1)| \leq 1$ and friendly labeling is called balanced if $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$.

It is clear that every friendly labeling is not balanced. Therefore, A. N. T. Lee, S. M. Lee and H. K. Ng [20] introduced the balance index set of a graph $G$ as $BI(G) = \{|e_{f^*}(0) - e_{f^*}(1)|, \text{ where } f^* \text{ is the partial edge labeling} \}$ runs over all friendly labelings $f$ of $G$.

Motivated from the definition of balance index set, we have defined the balance index number of a graph.

**Definition 2.1.1.** The balance index number of a graph $G$ is the number of distinct elements in its balance index set and it is denoted as $BIN(G)$.

The balance index set of some families of graph forms an arithmetic progression, but not every graph. Some balanced graphs are considered in [14, 15, 21, 36]. In general, it is difficult to determine the balance index set of a given graph. Most of the existing research work is focussed on some special families of graph with simple structure. For the wonderful work one can see [16, 17, 18, 20, 22, 26].

Kwong and Shiu [19] developed an algebraic approach to find the balance index set. It shows that the balance index set depends on the degree sequence of the graph. It becomes a very powerful tool to deal with balance indexes.

**Lemma 2.1.1.** [24] For any graph $G$,

1. $2e_f(0) + e_f(X) = \sum_{v \in v(0)} \deg v$.
2. $2e_f(1) + e_f(X) = \sum_{v \in v(1)} \deg v$.
3. $2|E(G)| = \sum_{v \in v(G)} \deg v = \sum_{v \in v(0)} \deg v + \sum_{v \in v(1)} \deg v$,

where $e_f(X)$ is the subset of $E(G)$ containing all the unlabeled edges.

**Corollary 2.1.1.** [24] For any friendly labeling $f$, the balance index is

$$e_f(0) - e_f(1) = \frac{1}{2} \left( \sum_{v \in v(0)} \deg v - \sum_{v \in v(1)} \deg v \right).$$

Although many results on balanced labeling and balance index set have already been published, there are still many problems that we can try to solve. In this chapter, results on balanced labeling, balance index set and balance index number of one point union of two complete graphs, Cartesian product of two complete graphs, shell graph, crown graph, helm graph and flower graph are obtained. Also, the balance index number of complement of one point union of two complete graphs is obtained.
2.2 Balanced labeling of $K_m \cdot K_n$

This section begins with the necessary and sufficient condition for friendly labeling of one point union of two graphs. Recall that one point union of two graphs $G$ and $H$, say $G \cdot H$ is obtained by identification of any vertex of graph $G$ with an arbitrary vertex of graph $H$.

**Theorem 2.2.1.** The one point union of two graphs $G$ and $H$, say $G \cdot H$ with $m$ and $n$ vertices respectively, satisfies friendly labeling if and only if

1. $|(m_1 + n_1) - (m_2 + n_2)| = 0 \text{ or } 2$, for $m + n$ even.

2. $|(m_1 + n_1) - (m_2 + n_2)| = 1$, for $m + n$ odd,

where $m$ and $n$ are partitioned into $(m_1, m_2)$ and $(n_1, n_2)$ respectively.

**Proof.** Consider the graph $G \cdot H$, where $|V(G)| = m$ and $|V(H)| = n$.

**Case 1.** If $m + n$ is even, then $|V(G \cdot H)| = m + n - 1$, which is odd. So, every friendly labeling $f$ satisfies the condition $|v_f(0) - v_f(1)| = 1$. This gives, either $v_f(0) = m_1 + n_1 - 1$ and $v_f(1) = m_2 + n_2$ or $v_f(0) = m_1 + n_1$ and $v_f(1) = m_2 + n_2 - 1$.

Thus, $|v_f(0) - v_f(1)| = 1 \iff |(m_1 + n_1) - (m_2 + n_2) + 1| = 1$

$\iff |(m_1 + n_1) - (m_2 + n_2)| = 0 \text{ or } 2$.

**Case 2.** If $m + n$ is odd, then $|V(G \cdot H)| = m + n - 1$, which is even. So, every friendly labeling $f$ satisfies the condition $|v_f(0) - v_f(1)| = 0$. 

This gives, either $v_f(0) = m_1 + n_1 - 1$ and $v_f(1) = m_2 + n_2$ or $v_f(0) = m_1 + n_1$ and $v_f(1) = m_2 + n_2 - 1$.

Thus, $|v_f(0) - v_f(1)| = 0 \iff |(m_1 + n_1) - (m_2 + n_2) \pm 1| = 0 \iff |(m_1 + n_1) - (m_2 + n_2)| = 1$. 

The following two theorems gives the balancedness of one point union of two complete graphs.

**Theorem 2.2.2.** The graph $K_m \cdot K_n$, $|m - n| \neq 1$ is balanced, when

1. both $m$ and $n$ are even.

2. both $m$ and $n$ are odd with
   
   (a) $\min\{m, n\} = 3$ and $|m - n| \leq 6$.

   (b) $\min\{m, n\} \geq 5$ and $|m - n| \leq 2$.

3. both $m + n$ and $\min\{m, n\}$ are odd with
   
   (a) $\min\{m, n\} = 3$.

   (b) $\min\{m, n\} \geq 5$ and if there exist an integer $z = 2i(n - m)$, where

   $i = \left\lfloor \frac{m - 1}{6} \right\rfloor$, satisfying the condition $m - 3 \leq z \leq m + 1$.

4. $m + n$ is odd, $\min\{m, n\} = \text{even}$ and there exist an integer $z = 2i(n - m)$, where $i = 1, 2, 3, \ldots, \frac{1}{2}(\min\{m, n\})$ satisfying the condition $n - 3 \leq z \leq n + 1$. 

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**Proof.** The graph $K_m \cdot K_n$ contains $m + n - 1$ vertices and \( \frac{m}{2} + \frac{n}{2} \) edges.

Without loss of generality, assume that $m \leq n$.

**Case 1.** If both $m$ and $n$ are even, then $|V(K_m \cdot K_n)|$ is odd. So, every friendly labeling satisfies the condition $|v_f(0) - v_f(1)| = 1$.

Consider two complete blocks $K_m$ and $K_n$ of $K_m \cdot K_n$. If $\frac{m}{2}$ vertices of $K_m$ and $\frac{n}{2}$ vertices of $K_n$ are labeled 1, and remaining vertices are labeled 0, then $|v_f(0) - v_f(1)| = 1$ and $|e_f(0) - e_f(1)| = 0$. Hence, the graph $K_m \cdot K_n$ is strongly edge-balanced.

**Case 2.** If both $m$ and $n$ are odd, then $|V(K_m \cdot K_n)|$ is odd. So, every friendly labeling satisfies the condition $|v_f(0) - v_f(1)| = 1$.

Consider following possibilities.

(a) When $m = 3$ and $n = 2k + 1$, $k \geq 1$.

The graph $K_3 \cdot K_n$ is balanced $\iff |(k + 1) - \binom{k}{2}| \leq 4 \iff k \leq 4$ or $n \leq 9$.

(b) When $m \geq 5$ and $n = 2k + 1$, $k \geq 1$.

The graph $K_m \cdot K_n$ is balanced $\iff |\left[\binom{m+1}{2} + \binom{n-1}{2}\right] - \left[\binom{m-1}{2} + \binom{n+1}{2}\right]| \leq 1$

$\iff |m - n| \leq 2$.

**Case 3.** If $m + n$ is odd, then $|V(K_m \cdot K_n)|$ is even. Thus, every friendly labeling satisfies the condition $|v_f(0) - v_f(1)| = 0$.

Consider following possibilities.

(a) When $m = 3$, label two vertices of $K_3$ by 0 and one vertex by 1, half the number of vertices of $K_n$ by 0 and remaining half by 1.
Thus, \(|e_{f^*}(0) - e_{f^*}(1)| = 1\).

(b) When \(m \geq 5\), \(m\) and \(n\) are partitioned into \(\left(\frac{m-(2i+1)}{2}, \frac{m+(2i+1)}{2}\right)\) and \(\left(\frac{n+i}{2}, \frac{n-i}{2}\right)\) respectively, where \(i = \left\lfloor \frac{m-1}{6} \right\rfloor\).

Thus, the graph \(K_m \cdot K_n\) is balanced

\[
\Leftrightarrow \left| \left[ \left( \frac{m-(2i+1)}{2} \right) + \left( \frac{n+i}{2} \right) \right] - \left[ \left( \frac{m+(2i+1)}{2} \right) + \left( \frac{n-i}{2} \right) \right] \right| \leq 1
\]

\[
\Leftrightarrow \frac{1}{2} |2i(n-m) - (m-1)| \leq 1
\]

\[
\Leftrightarrow m - 3 \leq 2i(n-m) \leq m + 1, \text{ where } i = \left\lfloor \frac{m-1}{6} \right\rfloor.
\]

Hence, there exist an integer \(z = 2i(n-m)\), where \(i = \left\lfloor \frac{m-1}{6} \right\rfloor\) such that \(m - 3 \leq z \leq m + 1\).

**Case 4.** If \(m + n\) is odd, then \(|V(K_m \cdot K_n)|\) is even. So, every friendly labeling satisfies the condition \(|v_f(0) - v_f(1)| = 0\). Therefore, \(m\) and \(n\) are partitioned into \(\left(\frac{m}{2} - i, \frac{m}{2} + i\right)\) and \(\left(\frac{n+(2i+1)}{2} - 1, \frac{n-(2i+1)}{2} + 1\right)\) respectively, where \(i = 1, 2, 3, \ldots, \frac{m}{2}\).

Thus, \(K_m \cdot K_n\) is balanced

\[
\Leftrightarrow \left| \left[ \left( \frac{n+i}{2} \right) + \left( \frac{n+(2i+1)}{2} - 1 \right) \right] - \left[ \left( \frac{n+i}{2} \right) + \left( \frac{n-(2i+1)}{2} + 1 \right) \right] \right| \leq 1
\]

\[
\Leftrightarrow \frac{1}{2} |2i(n-m) - (n-1)| \leq 1,
\]

\[
\Leftrightarrow n - 3 \leq 2i(n-m) \leq n + 1, i = 1, 2, 3, \ldots, \frac{m}{2}.
\]

Hence, there exist an integer \(z = 2i(n-m)\), where \(i = 1, 2, 3, \ldots, \frac{m}{2}\) such that \(n - 3 \leq z \leq n + 1\). \(\square\)
Theorem 2.2.3. The graph $K_n \cdot K_{n+1}$ is strongly balanced.

Proof. The graph $K_n \cdot K_{n+1}$ is of order $2n$. So, every friendly labeling satisfies the condition $v_f(0) = v_f(1) = n$. If vertices of $K_n$ are labeled 0 and remaining vertices are labeled 1, then $e_{f^*}(0) = e_{f^*}(1) = \binom{n}{2}$.

Thus, $|v_f(0) - v_f(1)| = 0$ and $|e_{f^*}(0) - e_{f^*}(1)| = 0$.

Hence, the graph $K_n \cdot K_{n+1}$ is strongly balanced. \qed

2.3 Balance index set and Balance index number of graphs

(i) Shell graphs

Recall that the shell graph is obtained by taking $n - 3$ concurrent chords in cycle and the vertex at which all the chords are concurrent is called the apex vertex. The shell graph of order $n$ is denoted by $S_n$. Also, $|V(S_n)| = n$ and $|E(S_n)| = 2n - 3$.

Theorem 2.3.1. The balance index set of shell graph $S_n$,

$$BI(S_n) = \begin{cases} \{0, 1, 2, \ldots, \frac{n+1}{2}\}, & \text{if } n \text{ is odd} \\ \{0, 1, 2, \ldots, \frac{n}{2} - 1\}, & \text{if } n \text{ is even.} \end{cases}$$
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Proof. The shell graph $S_n$ contains an apex vertex of degree $n-1$, two vertices of degree 2 and remaining $n-3$ vertices of degree 3. Let the apex vertex be labeled 0.

Case 1. When $n$ is odd, possible compositions of vertices with degrees 2 and 3 for friendly labeling as shown in Table 2.1.

Table 2.1: Possible compositions of vertices of degrees 2 and 3 for friendly labeling.

<table>
<thead>
<tr>
<th>Compositions of 2</th>
<th>Compositions of $n-3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2 - i, i)$</td>
<td>$(i, n - i - 3)$, where $i = 0, 1, 2$.</td>
</tr>
<tr>
<td></td>
<td>$(i - 1, n - i - 2)$, where $i = 1, 2$.</td>
</tr>
</tbody>
</table>

Subcase 1.1. If compositions of 2 and $n-3$ are $(2 - i, i)$ and $(i, n - i - 3)$ respectively, where $i = 0, 1, 2$, then by Corollary 2.1.1,

$$|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2} |(n - 1) + 2(2 - i) + 3i - 2i - 3(n - i - 3)|$$

$$= |n - i - 6|, \text{ where } i = 0, 1, 2.$$

Subcase 1.2. If compositions of 2 and $n-3$ are $(2 - i, i)$ and $(i - 1, n - i - 2)$ respectively, where $i = 1, 2$, then

$$|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2} |(n - 1) + 2(2 - i) + 3(i - 1) - 2i - 3(n - i - 2)|$$

$$= |n - i - 3|, \text{ where } i = 1, 2.$$

Case 2. When $n$ is even, for friendly labeling, possible compositions of 2 and $n-3$ are $(2 - i, i)$ and $(i, n - i - 3)$ respectively, where $i = 0, 1, 2$. Thus, $|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2} |(n - 1) + 2(2 - i) + 3i - 2i - 3(n - i - 3)|$
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\[ |n - i - 6|, \text{ where } i = 0, 1, 2. \]

Therefore, \( BI(S_n) = \{|n - i - 6| : i = 0, 1, 2\} \cup \{|n - i - 3| : i = 1, 2\}. \)

Taking all possible values of \( i \),

\[
BI(S_n) = \begin{cases} 
\{0, 1, 2, \ldots, \frac{n+1}{2}\}, & \text{if } n \text{ is odd} \\
\{0, 1, 2, \ldots, \frac{n}{2} - 1\}, & \text{if } n \text{ is even.}
\end{cases}
\]

Also, when the apex vertex is labeled 1, the balance index set will be same.

Corollary 2.3.1. If \( n \) is odd, then the shell graph \( S_n \) is strongly-edge balanced and if \( n \) is even, then \( S_n \) is strongly balanced.

Corollary 2.3.2. The balance index set of the shell graph \( S_n \) forms an arithmetic progression with common difference 1.

Corollary 2.3.3. The balance index number of the shell graph \( S_n \),

\[
BIN(S_n) = \begin{cases} 
\frac{n+3}{2}, & \text{if } n \text{ is odd} \\
\frac{n}{2}, & \text{if } n \text{ is even.}
\end{cases}
\]

Example 2.3.1. The balance index set of the shell graph \( S_5 \) is \( \{0, 1, 2, 3\} \).

(ii) Helm graphs

The helm graph is obtained from a wheel by attaching a pendant edge to each rim vertex. Also, \( |V(H_n)| = 2n + 1 \) and \( |E(H_n)| = 3n \).
Theorem 2.3.2. The balance index set of the helm graph \( H_n \) (where \( n \geq 3 \)),

\[
BI(H_n) = \{2n - 3i : i = 0, 1, 2, \ldots, n\} \cup \{2n - 3i - 1 : i = 1, 2, 3, \ldots, n\}.
\]

Proof. The helm graph \( H_n \) contains \( n \) pendant vertices, \( n \) rim vertices of degree 4 and an apex vertex of degree \( n \).

Let the apex vertex be labeled 0.

For friendly labeling, possible compositions of vertices of degree 4 and pendant vertices are given in Table 2.2.

Table 2.2: Possible compositions of vertices of degree 4 and pendant vertices for friendly labeling.

<table>
<thead>
<tr>
<th>Compositions of vertices of degree 4</th>
<th>Compositions of pendant vertices</th>
</tr>
</thead>
<tbody>
<tr>
<td>((n - i, i))</td>
<td>((i, n - i)), where ( i = 0, 1, 2, \ldots, n ).</td>
</tr>
<tr>
<td></td>
<td>((i - 1, n - i + 1)), where ( i = 1, 2, 3, \ldots, n ).</td>
</tr>
</tbody>
</table>

Case 1. If compositions of vertices of degree 4 and pendant vertices are \((n - i, i)\) and \((i, n - i)\) respectively, where \( i = 0, 1, 2, \ldots, n \), then

\[
|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2} |n + 4(n - i) + i - 4i - (n - i)|
= |2n - 3i|, \text{ where } i = 0, 1, 2, \ldots, n.
\]

Case 2. If compositions of vertices of degree 4 and pendant vertices are \((n - i, i)\) and \((i - 1, n - i + 1)\) respectively, where \( i = 1, 2, 3, \ldots, n \), then

\[
|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2} |n + 4(n - i) + i - 1 - 4i - (n - i + 1)|
= |2n - 3i - 1|, \text{ where } i = 1, 2, 3, \ldots, n.
\]
Thus, the balance index set of the helm graph,

\[ BI(H_n) = \{\lvert 2n - 3i \rvert : i = 0, 1, 2, \ldots, n \} \cup \{\lvert 2n - 3i - 1 \rvert : i = 1, 2, 3, \ldots, n \}. \]

When the apex vertex is labeled 1, the balance index set will be same. \( \square \)

**Corollary 2.3.4.** The helm graph \( H_n \) (where \( n \geq 3 \)) is balanced. If \( n \) is even and \( 2n \equiv 0, 1 (\text{mod } 3) \), then \( H_n \) is strongly balanced.

**Example 2.3.2.** The balance index set of \( H_5 \) is \{0, 1, 2, 3, 4, 5, 6, 7, 10\}.

(iii) **Flower graphs**

The flower graph is obtained from the helm graph by joining each pendant vertex to the apex of the helm. Also, \( |V(Fl_n)| = 2n + 1 \) and \( |E(Fl_n)| = 4n \).

**Theorem 2.3.3.** The balance index set of the flower graph,

\[ BI(Fl_n) = \{2(n - i) : i = 0, 1, 2, \ldots, n\}. \]

**Proof.** The flower graph \( Fl_n \) contains, \( n \) vertices of degree 2, \( n \) rim vertices of degree 4 and an apex vertex of degree 2n.

Let the apex vertex be labeled 0. The possible compositions of vertices with degrees 4 and 2 for friendly labeling as shown in Table 2.3.

**Table 2.3:** Possible compositions of vertices with degrees 4 and 2 for friendly labeling.

<table>
<thead>
<tr>
<th>Compositions of vertices of degree 4</th>
<th>Compositions of vertices of degree 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>((n - i, i))</td>
<td>( (i, n - i) ), where ( i = 0, 1, 2, \ldots, n ).</td>
</tr>
<tr>
<td></td>
<td>( (i - 1, n - i + 1) ), where ( i = 1, 2, 3, \ldots, n ).</td>
</tr>
</tbody>
</table>
Case 1. If compositions of vertices with degrees 4 and 2 are \((n - i, i)\) and \((i, n - i)\) respectively, where \(i = 0, 1, 2, \ldots, n\), then
\[
|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2} |2n + 4(n - i) + 2i - 4i - 2(n - i)| \\
= |2(n - i)|, \text{ where } i = 0, 1, 2, \ldots, n.
\]
But, all balance indexes are positive, thus \(|e_{f^*}(0) - e_{f^*}(1)| = 2(n - i)|.

Case 2. If compositions of vertices with degrees 4 and 2 are \((n - i, i)\) and \((i - 1, n - i + 1)\) respectively, where \(i = 1, 2, 3, \ldots, n\), then
\[
|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2} |2n + 4(n - i) + 2(i - 1) - 4i - 2(n - i + 1)| \\
= |2n - 2i - 2|, \text{ where } i = 1, 2, 3, \ldots, n.
\]
Therefore, the balance index set of the flower graph,
\[
BI(Fl_n) = \{2(n - i) : i = 0, 1, 2, \ldots, n\} \cup \{|2(n - i - 1)| : i = 1, 2, 3, \ldots, n\}.
\]
Taking all possible values of \(i\), \(BI(Fl_n) = \{2(n - i) : i = 0, 1, 2, \ldots, n\}\).

When the apex vertex is labeled 1, the balance index set will be same. \(\square\)

**Corollary 2.3.5.** If \(n\) is odd, then the flower graph \(Fl_n\) is strongly edge-balanced and if \(n\) is even, then it is strongly balanced.

**Corollary 2.3.6.** The balance index set of the flower graph \(Fl_n\) forms an arithmetic progression with common difference 2.

**Corollary 2.3.7.** The balance index number of the flower graph,
\[
BIN(Fl_n) = n + 1.
\]

**Example 2.3.3.** The balance index set of \(Fl_5\) is \(\{0, 2, 4, 6, 8, 10\}\).
(iv) **One point union of two complete graphs**

The one point union of complete graphs $K_m$ and $K_n$, denoted as $K_m \cdot K_n$ is obtained by identification of any vertex of complete graph $K_m$ with an arbitrary vertex of complete graph $K_n$.

Also, $|V(K_m \cdot K_n)| = m + n - 1$ and $|E(K_m \cdot K_n)| = \binom{m}{2} + \binom{n}{2}$.

**Theorem 2.3.4.** The balance index set of one point union of two complete graphs $K_m$ and $K_n$ (where $m \leq n$),

$$BI(K_m \cdot K_n) = \begin{cases} \left\{ \frac{1}{2} |(m - n)(m - 2i)| : i = 0, 1, 2, \ldots, m - 1 \right\} \cup \\
\left\{ \frac{1}{2} |(m - n)(m - 2i) + 2(n - 1)| : i = 0, 1, 2, \ldots, m - 1 \right\}, \\
\left\{ \frac{1}{2} |(m - n)(m - 2i) + (n - 1)| : i = 0, 1, 2, \ldots, m - 1 \right\}, \\
\text{if } m + n \text{ is even} \\
\text{if } m + n \text{ is odd.} \end{cases}$$

**Proof.** The graph $K_m \cdot K_n$ contains $m - 1$ vertices of degree $m - 1$, $n - 1$ vertices of degree $n - 1$ and an identified vertex of degree $m + n - 2$.

Without loss of generality, assume that $m \leq n$.

Let the identified vertex be labeled 0.

**Case 1.** Let $m + n$ be even.

Possible compositions of vertices of degree $m - 1$ and $n - 1$ for friendly labeling as shown in Table 2.4.
Table 2.4: Possible compositions of \( m - 1 \) and \( n - 1 \) for friendly labeling.

<table>
<thead>
<tr>
<th>( m - 1 )</th>
<th>( n - 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((m - 1 - i, i))</td>
<td>((\frac{n-m}{2} + i - 1, \frac{n+m}{2} - i), where ( i = 0, 1, 2, \ldots, m - 1 ).</td>
</tr>
<tr>
<td>((\frac{n-m}{2} + i, \frac{n+m}{2} - i - 1), where ( i = 0, 1, 2, \ldots, m - 1 ).</td>
<td></td>
</tr>
</tbody>
</table>

(a) Consider the compositions \((m - 1 - i, i)\) and \((\frac{n-m}{2} + i - 1, \frac{n+m}{2} - i)\) of \( m - 1 \) and \( n - 1 \) respectively, where \( i = 0, 1, 2, \ldots, m - 1 \).

Then, 
\[
|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2} |m + n - 2 + (m - 1)(m - 1 - i) + (n - 1)(\frac{n-m}{2} + i - 1) - i(m - 1) - (n - 1)(\frac{n+m}{2} - i)|
\]
\[
= \frac{1}{2} |(m - n)(m - 2i)|, \text{ where } i = 0, 1, 2, \ldots, m - 1.
\]

(b) If \( m-1 \) and \( n-1 \) are partitioned into \((m - 1 - i, i)\) and \((\frac{n-m}{2} + i, \frac{n+m}{2} - i - 1)\) respectively, where \( i = 0, 1, 2, \ldots, m - 1 \), then

\[
|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2} |m + n - 2 + (m - 1)(m - 1 - i) + (n - 1)(\frac{n-m}{2} + i) - i(m - 1) - (n - 1)(\frac{n+m}{2} - i - 1)|
\]
\[
= \frac{1}{2} |(m - n)(m - 2i) + 2(n - 1)|, \text{ where } i = 0, 1, 2, \ldots, m - 1.
\]

Case 2. Let \( m + n \) be odd.

For friendly labeling, consider the compositions \((m - 1 - i, i)\) and \((\frac{n-m-1}{2} + i, \frac{n+m-1}{2} - i)\) of \( m - 1 \) and \( n - 1 \) respectively, where \( i = 0, 1, 2, \ldots, m - 1 \).

Then, 
\[
|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2} |m + n - 2 + (m - 1)(m - 1 - i) + (n - 1)(\frac{n-m-1}{2} + i) - i(m - 1) - (n - 1)(\frac{n+m-1}{2} - i)|
\]
\[
= \frac{1}{2} |(m - n)(m - 2i) + (n - 1)|, \text{ where } i = 0, 1, 2, \ldots, m - 1.
\]
Thus, the balance index set of $K_m \cdot K_n$ (where $m \leq n$),

$$\text{BI}(K_m \cdot K_n) = \begin{cases} \left\{ \frac{1}{2} |(m - n)(m - 2i)| : i = 0, 1, 2, \ldots, m-1 \right\} \cup \\ \left\{ \frac{1}{2} |(m - n)(m - 2i) + 2(n - 1)| : i = 0, 1, 2, \ldots, m-1 \right\} \\ \left\{ \frac{1}{2} |(m - n)(m - 2i) + (n - 1)| : i = 0, 1, 2, \ldots, m-1 \right\} \end{cases},$$

if $m + n$ is even

$$\left\{ \frac{1}{2} |(m - n)(m - 2i) + (n - 1)| : i = 0, 1, 2, \ldots, m-1 \right\} \text{,}$$

if $m + n$ is odd.

Also, when the identified vertex is labeled 1, the balance index set will be same. \hfill \square

**Corollary 2.3.8.** The balance index set of $K_n \cdot K_{n+1}$ is $\{0, 1, 2, \ldots, n-1\}$, which forms a ring with respect to addition modulo $n$ and if $n$ is a prime number then this set forms a field.

**Corollary 2.3.9.** If $n$ is even, then the balance index set of $K_n \cdot K_{3n-2}$ is $\{0, (n - 1), 2(n - 1), \ldots, (n + 1)(n - 1)\}$ and this set forms an arithmetic progression with common difference $n - 1$.

**Corollary 2.3.10.** If $n$ is even, then the balance index set of $K_n \cdot K_{2n-1}$ is $\{0, (n-1), 2(n-1), \ldots, \frac{n}{2}(n-1)\}$ and this set forms an arithmetic progression with common difference $n - 1$. 
Corollary 2.3.11. For \( m \leq n \),

\[
\max\{BI(K_m \cdot K_n)\} = \begin{cases} 
\frac{mn-m(m-2)}{2} - 1, & \text{if } m \text{ and } n \text{ are odd} \\
\left(\frac{m}{2} - 1\right)(n - m) + (n - 1), & \text{if } m \text{ and } n \text{ are even} \\
\left\lfloor \frac{(m-1)(n-m+1)}{2} \right\rfloor, & \text{if } m \text{ is odd and } n \text{ is even} \\
\left\lfloor \frac{(m-2)(n-m)+(n-1)}{2} \right\rfloor, & \text{if } m \text{ is even and } n \text{ is odd}.
\end{cases}
\]

Corollary 2.3.12. For \( m \leq n \),

\[
\min\{BI(K_m \cdot K_n)\} = \begin{cases} 
\left\lfloor \frac{(m-2)n-m^2}{2} + 1 \right\rfloor, & \text{if } m \text{ and } n \text{ are odd} \\
0, & \text{if } m \text{ and } n \text{ are even}.
\end{cases}
\]

Corollary 2.3.13. The balance index number of one point union of two complete graphs (for \( m < n \)),

\[
BIN(K_m \cdot K_n) = \begin{cases} 
\frac{3m}{2} + 1, & \text{if } m \text{ is even, } n \text{ is even and } n \neq 3m - 2 \\
m + 2, & \text{if } m \text{ is even and } n = 3m - 2 \\
m, & \text{if } m \text{ is odd and } n \text{ is even} \\
m, & \text{if } m \text{ is even, } n \text{ is odd and } n \neq 2m - 1 \\
\frac{m}{2} + 1, & \text{if } m \text{ is even and } n = 2m - 1.
\end{cases}
\]

Corollary 2.3.14. The balance index number of \( K_m \cdot K_m \) is 2.

Proof. By Theorem 2.3.4, \( BI(K_m \cdot K_m) = \{0, m - 1\} \).
Thus, \( BIN(K_m \cdot K_m) = 2 \).
Example 2.3.4. Balance index set of graph $K_3 \cdot K_5 = \{1, 3, 5\}$ is shown in Figure 2.1.

![Graphs with labels and differences](image)

Figure 2.1: Possible friendly labelings of $K_3 \cdot K_5$.

A. N. T. Lee, S. M. Lee and H. K. Ng [20] found the balance index set of complete bipartite graphs.
Theorem 2.3.5. [20] The balance index set of complete bipartite graph, $BI(K_{m,n})$ is

1. $\{(n-m)(i-m^2)/2 : i = 0, 1, 2, \ldots, m\}$, if $m+n$ is even

2. $\{(n-m)(i-m^2)/2 - m^2/2, |(n-m)(i-m^2)/2 + m^2/2| : i = 0, 1, 2, \ldots, m\}$, if $m+n$ is odd.

Corollary 2.3.15. [20] $BI(K_{n,n}) = \{0\}$.

The graph $K_m \cdot K_n$ is isomorphic to $K_1 \cup K_{m-1,n-1}$. Thus, to obtain $BI(K_m \cdot K_n)$, it is enough to find the balance index set of $K_{m-1,n-1}$. So, Theorem 2.3.5 and Corollary 2.3.15 gives the following results.

Theorem 2.3.6. The balance index number of complement one point union of two complete graphs (for $m < n$),

$$BIN(K_m \cdot K_n) = \begin{cases} 
\frac{m+1}{2}, & \text{if } m \text{ and } n \text{ are odd} \\
\frac{m}{2}, & \text{if } m \text{ and } n \text{ are even} \\
m, & \text{if } m \text{ is odd and } n \text{ is even} \\
m, & \text{if } m \text{ is even and } n \text{ is odd \quad (n \neq 2m - 1)} \\
\frac{m}{2} + 1, & \text{if } m \text{ is even and } n = 2m - 1.
\end{cases}$$

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Proof. Consider the graph $K_m \cdot K_n$, which is isomorphic to $K_1 \cup K_{m-1,n-1}$.

**Case 1.** When both $m$ and $n$ are odd,

$$BI(K_m \cdot K_n) = \left\{ \frac{1}{2}(i - 1)|n - m| : i = 1, 3, 5, \ldots, m \right\}$$

and this set contains $\frac{m+1}{2}$ distinct elements. Thus, $BIN(K_m \cdot K_n) = \frac{m+1}{2}$.

**Case 2.** When both $m$ and $n$ are even,

$$BI(K_m \cdot K_n) = \left\{ \frac{1}{2}(i - 1)|n - m| : i = 2, 4, \ldots, m \right\}$$

and this set contains $\frac{m}{2}$ distinct elements. So, $BIN(K_m \cdot K_n) = \frac{m}{2}$.

**Case 3.** When $m$ is odd and $n$ is even,

$$BI(K_m \cdot K_n) = \left\{ \frac{1}{2}|(n-m)(2i-m+1) \pm (m-1)| : i = 0, 1, 2, \ldots, m-1 \right\}.$$ 

This set contains $2m$ elements and the elements are repeating twice. Therefore,

$$BIN(K_m \cdot K_n) = m.$$

**Case 4.** When $m$ is even and $n$ is odd,

$$BI(K_m \cdot K_n) = \left\{ \frac{1}{2}|(n-m)(2i-m+1) \pm (m-1)| : i = 0, 1, 2, \ldots, m-1 \right\}.$$ 

This set contains $2m$ elements and the elements are repeating twice. Thus,

$$BIN(K_m \cdot K_n) = m.$$

**Case 5.** When $m$ is even and $n = 2m-1$,

$$BI(K_m \cdot K_n) = \left\{ im : i = 0, 1, 2, \ldots, \frac{m}{2} \right\}$$

and this set contains $\frac{m}{2} + 1$ distinct elements. So, $BIN(K_m \cdot K_n) = \frac{m}{2} + 1$.

**Theorem 2.3.7.** The balance index number of $K_n \cdot K_n$ is 1.

Proof. Since $BI(K_{n,n}) = \{0\}$, $K_n \cdot K_n = 1$. 

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(v) Cartesian product of two complete graphs

The Cartesian product of two complete graphs denoted by $K_m \Box K_n$, is a graph with the vertex set $V(K_m) \times V(K_n)$ and the edge set $E(K_m \Box K_n) = \{((u_1, v_1), (u_2, v_2)) : v_1 = v_2 \text{ and } (u_1, u_2) \in E(K_m) \text{ or } u_1 = u_2 \text{ and } (v_1, v_2) \in E(K_n)\}$. Also, $|V(K_m \Box K_n)| = mn$ and $|E(K_m \Box K_n)| = \frac{mn(m+n-2)}{2}$.

**Theorem 2.3.8.** The balance index set of Cartesian product of two complete graphs,

$$BI(K_m \Box K_n) = \begin{cases} \{0\}, & \text{if } mn \text{ is even} \\ \{\frac{m+n}{2} - 1\}, & \text{if } mn \text{ is odd.} \end{cases}$$

**Proof. Case 1.** When $mn$ is even, for friendly labeling, half the number of vertices are labeled by 0 and the remaining half by 1. This can be achieved in $\binom{mn}{\frac{mn}{2}}$ ways. Also, the graph $K_m \Box K_n$ contains $m$ copies of $K_n$ and $n$ copies of $K_m$. Label the vertices of $n$ copies of $K_m$ as follows: In a copy of $K_m$, consider all possible partitions $(m_{10}, m_{11})$ of $m$. Similar manner, partition $m$ in remaining $n - 1$ copies in such a way that $\sum m_{i0} = \sum m_{i1}$, where $i=1, 2, \ldots, n$. So, the edges of $n$ copies of $K_m$ satisfies $e_{f^*}(0) = e_{f^*}(1)$. The remaining edges of $K_m \Box K_n$ gives us $m$ copies of $K_n$. Also, these edges satisfies the condition $e_{f^*}(0) = e_{f^*}(1)$. Thus, in $K_m \Box K_n$, $e_{f^*}(0) = e_{f^*}(1)$. Hence, $BI(K_m \Box K_n) = \{0\}$. 

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Case 2. If $mn$ is odd, then to satisfy friendly labeling, either $\frac{mn-1}{2}$ vertices are labeled by 0 and remaining $\frac{mn+1}{2}$ vertices are labeled by 1 or vice versa. This can be achieved in $\frac{mn}{2} \times \frac{mn}{2}$ ways. Label the vertices of $n$ copies of $K_m$ as follows: In a copy of $K_m$, all possible partition of $m$ as $(m_{10}, m_{11})$. Similar manner, partition $m$ in remaining $n - 1$ copies in such a way that $|\sum m_{i0} - \sum m_{i1}| = 1$, where $i = 1, 2, \ldots, n$. These edges in $n$ copies of $K_m$ gives $|e_f^*(0) - e_f^*(1)| = \frac{m-1}{2}$. The remaining edges of $K_m \Box K_n$ gives $m$ copies of $K_n$ and these edges satisfies the condition $|e_f^*(0) - e_f^*(1)| = \frac{n-1}{2}$. Thus, the edges of $K_m \Box K_n$ satisfies $|e_f^*(0) - e_f^*(1)| = \frac{m+n}{2} - 1$.

Hence, $\text{BI}(K_m \Box K_n) = \left\{ \frac{m+n}{2} - 1 \right\}$. 

Corollary 2.3.16. If $mn$ is odd and $m + n \leq 4$, then the graph $K_m \Box K_n$ is balanced and if $mn$ is even, then this graph is strongly balanced.

Corollary 2.3.17. The balance index number of Cartesian product of two complete graphs, $\text{BIN}(K_m \Box K_n) = 1$.

Example 2.3.5. The balance index set of $K_4 \Box K_3$ is $\{0\}$. 

![Figure 2.2: Cartesian product of $K_4$ and $K_3$.](image)
Table 2.5: Partitions of integers 4 and 3 for friendly labeling.

<table>
<thead>
<tr>
<th>All distinct friendly labelings of 3 copies of $K_4$</th>
<th>Corresponding friendly labelings of 4 copies of $K_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(4, 0), (0, 4), (2, 2)</td>
<td>(2, 1), (2, 1), (1, 2), (1, 2).</td>
</tr>
<tr>
<td>(4, 0), (1, 3), (1, 3)</td>
<td>(3, 0), (1, 2), (1, 2), (1, 2); (2, 1), (2, 1), (1, 2), (1, 2).</td>
</tr>
<tr>
<td>(3, 1), (1, 3), (2, 2)</td>
<td>(3, 0), (1, 2), (1, 2), (0, 3); (3, 0), (1, 2), (1, 2), (1, 2); (2, 1), (2, 1), (1, 2), (1, 2).</td>
</tr>
<tr>
<td>(2, 2), (2, 2), (2, 2)</td>
<td>(3, 0), (0, 3), (2, 1), (1, 2); (3, 0), (3, 0), (0, 3), (0, 3); (2, 1), (2, 1), (1, 2), (1, 2).</td>
</tr>
</tbody>
</table>

(vi) Crown graphs

Recall that the crown graph is obtained by joining a pendant edge to each vertex of cycle. Also, $|V(C_n \odot K_1)| = |E(C_n \odot K_1)| = 2n$.

**Theorem 2.3.9.** The balance index set of the crown graph,

$BI(C_n \odot K_1) = \{n - 2i : i = 0, 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor \}$.

**Proof.** The crown graph $C_n \odot K_1$ contains $n$ pendant vertices and remaining $n$ vertices of degree 3. To satisfy friendly labeling, number of pendant vertices and vertices of degree 3 are partitioned into $(n-i, i)$ and $(i, n-i)$ respectively, where $i = 0, 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$.

So, $|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2}[(n - i) + 3i - i - 3(n - i)] = (n - 2i)$,

where $i = 1, 2, 3, \ldots, \lfloor \frac{n}{2} \rfloor$.

Thus, $BI(C_n \odot K_1) = \{n - 2i : i = 0, 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor \}$. \qed
Corollary 2.3.18. The crown graph $C_n \odot K_1$ is balanced and it is strongly balanced when $n$ is even.

Corollary 2.3.19. The balance index set of the crown graph $C_n \odot K_1$ forms an arithmetic progression with common difference 2.

Corollary 2.3.20. The balance index number of crown graph,

$BIN(C_n \odot K_1) = \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Example 2.3.6. Balance index set of crown graph $C_5 \odot K_1$ is $\{1, 3, 5\}$.