Chapter 5

Friendly Index Set and Friendly Index Number of Graphs

5.1 Introduction

Cahit [5] introduced cordial graph labeling. A function $f$ from $V(G)$ to \{0,1\}, where for each edge $xy$, $f^*(xy) = |f(x) - f(y)|$, is called friendly if $|v_f(1) - v_f(0)| \leq 1$ and a friendly labeling $f$ is called cordial if $|e_f^*(1) - e_f^*(0)| \leq 1$, where $v_f(i)$ is the number of vertices $v$ with $f(v) = i$ and $e_f^*(i)$ is the number of edges $e$ with $f^*(e) = i$.

Reference [31] is based on this chapter.

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Lee and Ng [25] defined the friendly index set of a graph $G$ as $FI(G) = \{|e_{f^*}(1) - e_{f^*}(0)| : f^*$ runs over all friendly labeling $f$ of $G\}$. This concept was extended by W. C. Shiu and Harris Kwong [37] to full friendly index set for the graph $G$, denoted $FFI(G)$, defined as $FFI(G) = \{e_{f^*}(1) - e_{f^*}(0) : f^*$ runs over all friendly labeling $f$ of $G\}$.

Motivated from the above definitions, friendly index number and full friendly index number of graph are defined as follows.

**Definition 5.1.1.** [31] The friendly index number of graph $G$ is defined as the cardinality of its friendly index set and it is denoted as $FIN(G)$.

**Definition 5.1.2.** [31] The full friendly index number of graph $G$ is defined as the cardinality of its full friendly index set and it is denoted as $FFIN(G)$.

Here we introduce two new parameters $e_{Bf^*}(i)$ and $e_{Ff^*}(i)$, which are the number of edges labeled $i$ under balanced labeling and cordial labeling respectively. While proving our results, $FI(G)$ and $FFI(G)$ are used as below:

\[
FI(G) = \{|e_{Ff^*}(1) - e_{Ff^*}(0)| : Ff^* \text{ runs over all friendly labeling } f \text{ of } G\}.
\]

\[
FFI(G) = \{e_{Ff^*}(1) - e_{Ff^*}(0) : Ff^* \text{ runs over all friendly labeling } f \text{ of } G\}.
\]

In this chapter, the relations between balance index set, friendly index set and full friendly index set are established. Also, friendly index set, friendly index number, full friendly index set and full friendly index number of shell graph, crown graph, helm graph and flower graph are obtained.
5.2 Relation between balance index and friendly index

Theorem 5.2.1. Let $G(V, E)$ be a graph with $|E(G)| = q$ and $e_{Bf^*}(i)$ is the number of edges labeled $i$ under the balanced labeling, where $i = 0, 1$. Then

1. $FI(G) = \{q - 2(e_{Bf^*}(0) + e_{Bf^*}(1)) : \text{the partial edge labeling } Bf^* \text{ runs over all friendly labeling } f \text{ of } G\}$.

2. $FFI(G) = \{q - 2(e_{Bf^*}(0) + e_{Bf^*}(1)) : \text{the partial edge labeling } Bf^* \text{ runs over all friendly labeling } f \text{ of } G\}$.

5.3 Friendly index set and friendly index number of graphs

Theorem 5.3.1. The friendly index set of shell graph $S_n, n \geq 4,$

$$FI(S_n) = \begin{cases} 
\{1, 3, 5, \ldots, n - 2\}, & \text{if } n \text{ is odd} \\
\{1, 3, 5, \ldots, n - 1\}, & \text{if } n \text{ is even.}
\end{cases}$$

Proof. The shell graph $S_n, n \geq 4$ contains $n$ vertices and $2n - 3$ edges.

Case 1. When $n$ is odd, for friendly labeling, the possible compositions of $n$ are $\left(\frac{n-1}{2}, \frac{n+1}{2}\right)$ and $\left(\frac{n+1}{2}, \frac{n-1}{2}\right)$. Let the apex vertex be labeled 0.
(a) When the composition of $n$ is $\left(\frac{n-1}{2}, \frac{n+1}{2}\right)$,
\[ e_{Bf^*}(0) = \frac{n-3}{2} + i, \text{ where } i = 0, 1, 2, \ldots, \frac{n-5}{2} \text{ and } \]
\[ e_{Bf^*}(1) = \begin{cases} 
1, 2, 3, \ldots, \frac{n-3}{2}, & \text{for } i = 0 \\
 i + 1, i + 2, i + 3, \ldots, \frac{n-1}{2}, & \text{for } i = 1, 2, 3, \ldots, \frac{n-5}{2}.
\end{cases} \]

(b) When the composition of $n$ is $\left(\frac{n+1}{2}, \frac{n-1}{2}\right)$,
\[ e_{Bf^*}(0) = \frac{n-1}{2} + i, \text{ where } i = 0, 1, 2, \ldots, \frac{n-3}{2} \text{ and } \]
\[ e_{Bf^*}(1) = \begin{cases} 
0, 1, 2, \ldots, \frac{n-3}{2}, & \text{for } i = 0, 1, 2, \ldots, \frac{n-5}{2} \\
 0, 1, 2, \ldots, \frac{n-5}{2}, & \text{for } i = \frac{n-3}{2}.
\end{cases} \]

Thus, taking all possible values of $i$,
\[ |e_{Ff^*}(0) - e_{Ff^*}(1)| = |q - 2(e_{Bf^*}(0) - e_{Bf^*}(1))| = 1, 3, 5, \ldots, n - 2. \]

Therefore, $FI(S_n) = \{1, 3, 5, \ldots, n - 2\}$.

**Case 2.** When $n$ is even, for friendly labeling, $n$ is partitioned into $\left(\frac{n}{2}, \frac{n}{2}\right)$.

So, $e_{Bf^*}(0) = \frac{n}{2} - 1 + i$, where $i = 0, 1, 2, \ldots, \frac{n}{2} - 2$ and
\[ e_{Bf^*}(1) = \begin{cases} 
0, 1, 2, \ldots, \frac{n}{2} - 1, & \text{for } i = 0 \\
 i + 1, i + 2, i + 3, \ldots, \frac{n}{2} - 1, & \text{for } i = 1, 2, 3, \ldots, \frac{n}{2} - 2.
\end{cases} \]

Thus, taking all possible values of $i$,
\[ |e_{Ff^*}(0) - e_{Ff^*}(1)| = |q - 2(e_{Bf^*}(0) - e_{Bf^*}(1))| = 1, 3, 5, \ldots, n - 1. \]

Therefore, $FI(S_n) = \{1, 3, 5, \ldots, n - 1\}$.

Also, if the apex vertex is labeled 1, then $FI(S_n)$ will be same. \qed
Corollary 5.3.1. The shell graph $S_n$ is cordial.

Corollary 5.3.2. The friendly index set of shell graph $S_n$ forms an arithmetic progression with common difference 2.

Corollary 5.3.3. The friendly index number of shell graph $S_n$ is $\left\lfloor \frac{n}{2} \right\rfloor$.

Corollary 5.3.4. The full friendly index set of shell graph $S_n$,

$$FFI(S_n) = \begin{cases} \{-n+6, -n+8, -n+10, \ldots, n-2\}, & \text{if } n \text{ is odd} \\ \{-n+5, -n+7, -n+9, \ldots, n-1\}, & \text{if } n \text{ is even}. \end{cases}$$

Corollary 5.3.5. The full friendly index number of shell graph $S_n$,

$$FFIN(S_n) = \begin{cases} n-3, & \text{if } n \text{ is odd} \\ n-2, & \text{if } n \text{ is even}. \end{cases}$$

Corollary 5.3.6. The full friendly index set of shell graph $S_n$ forms an arithmetic progression with common difference 2.

Example 5.3.1. The friendly index set of the shell graph $S_5$ is $\{1, 3\}$.

Table 5.1: Compositions of vertices of shell graph for friendly labeling and corresponding elements of friendly index set.

<table>
<thead>
<tr>
<th>Compositions of integer $5$</th>
<th>Corresponding friendly Indexes</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2,3)</td>
<td>1, 3</td>
</tr>
<tr>
<td>(3,2)</td>
<td>1, 3</td>
</tr>
</tbody>
</table>
Theorem 5.3.2. The friendly index set of crown graph $C_n \circ K_1$,

$$FI(C_n \circ K_1) = \begin{cases} 
\{0, 4, 8, \ldots, 2n\}, & \text{if } n \text{ is even} \\
\{0, 4, 8, \ldots, 2n - 2\}, & \text{if } n \text{ is odd.}
\end{cases}$$

Proof. The crown graph $C_n \circ K_1$ is of both order and size $2n$.

Case 1. When $n$ is even, for friendly labeling, the possible partitions of number of vertices of cycle and pendant vertices of $C_n \circ K_1$ are $(n - i, i)$ and $(i, n - i)$ respectively, where $i = 0, 1, 2, \ldots, \frac{n}{2}$.

When $i = 0$, $e_{F_f^*}(0) = n$ and $e_{F_f^*}(1) = n$. So, the friendly index is 0.

When $i = 1, 2, 3, \ldots, \frac{n}{2}$,

$e_{B_f^*}(0) = n - i - 1 - j + k$, where $j = 0, 1, 2, \ldots, i - 1$ and $k = 0, 1, 2, \ldots, i$,

$e_{B_f^*}(1) = l + k$, where $l = 0, 1, 2, \ldots, i - 1$ and $k = 0, 1, 2, \ldots, i$ such that $j + l = i - 1$.

Thus, $|e_{F_f^*}(1) - e_{F_f^*}(0)| = |2n - 2[(n - i - 1 - j + k) + (l + k)]| = |4(i - l - k)|$, where $i = 1, 2, \ldots, \frac{n}{2}$, $l = 0, 1, 2, \ldots, i - 1$ and $k = 0, 1, 2, \ldots, i$.

Taking all possible values of $i$, $l$ and $k$, $FI(C_n \circ K_1) = \{0, 4, 8, \ldots, 2n\}$.

Case 2. When $n$ is odd, for friendly labeling, the possible partitions of number of vertices of cycle and pendant vertices of $C_n \circ K_1$ are $(n - i, i)$ and $(i, n - i)$ respectively, where $i = 0, 1, 2, \ldots, \frac{n-1}{2}$.

When $i = 0$, $e_{F_f^*}(0) = n$ and $e_{F_f^*}(1) = n$. So, friendly index is 0.
When \( i = 1, 2, 3, \ldots, \frac{n-1}{2} \),

\( e_{BF^*}(0) = n - i - 1 - j + k \), where \( j = 0, 1, 2, \ldots, i - 1 \) and \( k = 0, 1, 2, \ldots, i \),

\( e_{BF^*}(1) = l + k \), where \( l = 0, 1, 2, \ldots, i - 1 \) and \( k = 0, 1, 2, \ldots, i \) such that \( j + l = i - 1 \).

Thus, \(|e_{FF^*}(1) - e_{FF^*}(0)| = |2n - 2[(n - i - 1 - j + k) + (l + k)] = |4(i - l - k)|\),

where \( i = 1, 2, \ldots, \frac{n-1}{2} \), \( l = 0, 1, 2, \ldots, i - 1 \) and \( k = 0, 1, 2, \ldots, i \).

Taking all possible values of \( i, l \) and \( k \), \( FI(C_n \odot K_1) = \{0, 4, 8, \ldots, 2n - 2\} \).

Also, when the apex vertex is labeled 1, \( FI(C_n \odot K_1) \) will be same. \( \Box \)

**Corollary 5.3.7.** The crown graph \( C_n \odot K_1 \) is cordial.

**Corollary 5.3.8.** The friendly index set of crown graph \( C_n \odot K_1 \) forms an arithmetic progression with common difference 4.

**Corollary 5.3.9.** The friendly index number of crown graph \( C_n \odot K_1 \) is \( \left\lfloor \frac{n^2}{2} \right\rfloor + 1 \).

**Corollary 5.3.10.** The full friendly index set of crown graph \( C_n \odot K_1 \),

\[
FFI(C_n \odot K_1) = \begin{cases} 
\{-2n + 4, -2n + 8, -2n + 12, \ldots, 2n\}, & \text{if } n \text{ is even} \\
\{-2n + 6, -2n + 10, -2n + 14, \ldots, 2n - 2\}, & \text{if } n \text{ is odd.}
\end{cases}
\]

**Corollary 5.3.11.** The full friendly index set of crown graph \( C_n \odot K_1 \) forms an arithmetic progression with common difference 4.
Corollary 5.3.12. The full friendly index number of crown graph $C_n \odot K_1$,

$$\text{FFIN}(C_n \odot K_1) = \begin{cases} 
n, & \text{if } n \text{ is even} \\
n - 1, & \text{if } n \text{ is odd}. \end{cases}$$

Example 5.3.2. The friendly index set of crown graph $C_5 \odot K_1$ is $\{0, 4, 8\}$.

Table 5.2: Partitions of vertices of degree 3 and pendant vertices of crown graph for friendly labeling and corresponding elements of friendly index set.

<table>
<thead>
<tr>
<th>Partitions of integers 5 and 5</th>
<th>Corresponding friendly indexes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(5, 0)$ and $(0, 5)$</td>
<td>0</td>
</tr>
<tr>
<td>$(4, 1)$ and $(1, 4)$</td>
<td>0, 4</td>
</tr>
<tr>
<td>$(3, 2)$ and $(2, 3)$</td>
<td>0, 4, 8</td>
</tr>
</tbody>
</table>

Theorem 5.3.3. The friendly index set of helm graph $H_n$,

$$\text{FI}(H_n) = \begin{cases} 
\{1, 3, 5, \ldots, 2n - 1\}, & \text{if } n \text{ is odd} \\
\{0, 2, 4, \ldots, 2n\}, & \text{if } n \text{ is even}. \end{cases}$$

Proof. The helm graph $H_n$ is of order $2n + 1$ and size $3n$.

Case 1. When $n$ is odd.

Let the apex vertex be labeled by 0. For friendly labeling, consider the following compositions of number of rim vertices of wheel and pendant vertices of helm graph.
Subcase 1.1. When the compositions of number of rim vertices of the wheel and pendant vertices of helm are \((n, 0)\) and \((0, n)\) respectively, \(e_{Ff^*}(0) = 2n\) and \(e_{Ff^*}(1) = n\). Thus, \(|e_{Ff^*}(1) - e_{Ff^*}(0)| = n\).

Subcase 1.2. When the compositions of number of rim vertices of wheel and pendant vertices of helm are \((n - i, i)\) and \((i, n - i)\) respectively, where \(i = 1, 2, 3, \ldots, n - 1\),

\[e_{Bf^*}(0) = (n - j) + (n - i) + \min(n - i, i),\]

where

\[j = \begin{cases} 
  i + 1, i + 2, i + 3, \ldots, 2i, & \text{for } i = 1, 2, 3, \ldots, \frac{n-1}{2} \\
  i + 1, i + 2, i + 3, \ldots, n, & \text{for } i = \frac{n-1}{2}, \frac{n+3}{2}, \frac{n+5}{2}, \ldots, n - 1 
\end{cases}\]

and

\[e_{Bf^*}(1) = k + \min(n - i, i),\]

where

\[k = \begin{cases} 
  0, 1, 2, \ldots, i - 1, & \text{for } i = 1, 2, 3, \ldots, \frac{n-1}{2} \\
  2i - n, 2i - (n - 1), 2i - (n - 2), \ldots, i - 1, & \text{for } i = \frac{n+1}{2}, \frac{n+3}{2}, \frac{n+5}{2}, \ldots, n - 1 
\end{cases}\]

Thus, \(|e_{Ff^*}(1) - e_{Ff^*}(0)| = |3n - 2[(n - j) + (n - i) + k + 2l]|\]

= \(|2(i + j - k - 2l) - n|\), where if \(i = 1, 2, 3, \ldots, \frac{n-1}{2}\), then

\[j = i + 1, i + 2, i + 3, \ldots, 2i, \quad k = 0, 1, 2, \ldots, i - 1 \quad \text{and} \quad l = 0, 1, 2, \ldots, i;\]

if \(i = \frac{n+1}{2}, \frac{n+3}{2}, \frac{n+5}{2}, \ldots, n - 1\), then \(j = i + 1, i + 2, i + 3, \ldots, n,\)

\[k = 2i - n, 2i - (n - 1), 2i - (n - 2), \ldots, i - 1 \quad \text{and} \quad l = 0, 1, 2, \ldots, n - i \text{ such that } j + k = 2i.\]
So, $|e_{F^*}(1) - e_{F^*}(0)| = |n + 2i - 4j + 4l|$, where if $i = 1, 2, 3, \ldots, \frac{n-1}{2}$, then $j = i + 1, i + 2, i + 3, \ldots, 2i$ and $l = 0, 1, 2, \ldots, i$;

if $i = \frac{n+1}{2}, \frac{n+3}{2}, \frac{n+5}{2}, \ldots, n - 1$, then $j = i + 1, i + 2, i + 3, \ldots, n$ and $l = 0, 1, 2, \ldots, n - i$.

**Subcase 1.3.** When the compositions of number of rim vertices of wheel and pendant vertices of helm are $(0, n)$ and $(n, 0)$ respectively, $e_{F^*}(0) = n$ and $e_{F^*}(1) = 2n$. Thus, $|e_{F^*}(1) - e_{F^*}(0)| = n$.

**Subcase 1.4.** When the compositions of number of rim vertices of wheel and pendant vertices of helm are $(n - (i + 1), i + 1)$ and $(i, n - i)$ respectively, where $i = 0, 1, 2, \ldots, n - 1$,

$e_{B^*}(0) = (n - j) + (n - (i + 1)) + \min(n - (i + 1), i)$, where

$$j = \begin{cases} 
  i + 2, i + 3, i + 4, \ldots, 2(i + 1), & \text{for } i = 0, 1, 2, \ldots, \frac{n-3}{2} \\
  i + 2, i + 3, i + 4, \ldots, n, & \text{for } i = \frac{n-1}{2}, \frac{n+1}{2}, \frac{n+3}{2}, \ldots, n - 1
\end{cases}$$

and $e_{B^*}(1) = k + \min(n - i, i + 1)$, where

$$k = \begin{cases} 
  0, 1, 2, \ldots, i, & \text{for } i = 0, 1, 2, \ldots, \frac{n-3}{2} \\
  2(i + 1) - n, 2(i + 1) - (n - 1), 2(i + 1) - (n - 2), \ldots, i, & \text{for } i = \frac{n-1}{2}, \frac{n+1}{2}, \frac{n+3}{2}, \ldots, n - 1.
\end{cases}$$

Thus, $|e_{F^*}(1) - e_{F^*}(0)| = |3n - 2[(n - j) + (n - (i + 1)) + k + 2l + 1]|$

where if $i = 0, 1, 2, \ldots, \frac{n-3}{2}$, then $j = i + 2, i + 3, i + 4, \ldots, 2(i + 1)$,
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\[ k = 0, 1, 2, \ldots, i \text{ and } l = 0, 1, 2, \ldots, i; \text{ if } i = \frac{n-1}{2}, \frac{n+1}{2}, \frac{n+3}{2}, \ldots, n-1, \]

then \( j = i + 2, i + 3, i + 4, \ldots, n, \) \( k = 2(i + 1) - n, 2(i + 1) - (n - 1), \)

\( 2(i + 1) - (n - 2), \ldots, i \) and \( l = 0, 1, 2, \ldots, n - (i + 1) \) such that \( j + k = 2(i + 1). \)

So, \(|e_{Ff^*}(1) - e_{Ff^*}(0)| = |n + 2i - 4j + 4l + 4|, \)

where if \( i = 0, 1, 2, \ldots, \frac{n-3}{2}, \) then \( j = i + 2, i + 3, i + 4, \ldots, 2(i + 1) \) and \( l = 0, 1, 2, \ldots, i; \) if \( i = \frac{n-1}{2}, \frac{n+1}{2}, \frac{n+3}{2}, \ldots, n - 2, \) then \( j = i + 2, i + 3, i + 4, \ldots, n \)

and \( l = 0, 1, 2, \ldots, n - (i + 1). \)

**Subcase 1.5.** When the compositions of number of rim vertices of wheel and pendant vertices of helm are \((0, n)\) and \((n - 1, 1)\) respectively, \( e_{Ff^*}(0) = n + 1 \)

and \( e_{Ff^*}(1) = 2n - 1. \) Thus, \(|e_{Ff^*}(1) - e_{Ff^*}(0)| = n - 2. \)

So, considering all the above subcases and possible values of \( i, j \) and \( l, \)

\( BI(H_n) = \{1, 3, 5, \ldots, 2n - 1\}. \)

**Case 2.** When \( n \) is even.

Let the apex vertex be labeled by 0.

**Subcase 2.1.** When the compositions of number of rim vertices of wheel and pendant vertices of helm are \((n, 0)\) and \((0, n)\) respectively, \( e_{Ff^*}(0) = 2n \)

and \( e_{Ff^*}(1) = n. \) Thus, \(|e_{Ff^*}(1) - e_{Ff^*}(0)| = n. \)

**Subcase 2.2.** When the compositions of number of rim vertices of wheel and pendant vertices of helm are \((n - i, i)\) and \((i, n - i)\) respectively, where \( i = 1, 2, 3, \ldots, n - 1, \)

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\[ e_{BF^*}(0) = (n - j) + (n - i) + \min(n - i, i), \text{ where} \]

\[
j = \begin{cases} 
  i + 1, i + 2, i + 3, \ldots, 2i, & \text{for } i = 1, 2, 3, \ldots, \frac{n}{2} \\
  i + 1, i + 2, i + 3, \ldots, n, & \text{for } i = \frac{n}{2} + 1, \frac{n}{2} + 2, \frac{n}{2} + 3, \ldots, n - 1 
\end{cases}
\]

and \( e_{BF^*}(1) = k + \min(n - i, i) \), where

\[
k = \begin{cases} 
  0, 1, 2, \ldots, i - 1, & \text{for } i = 1, 2, 3, \ldots, \frac{n}{2} \\
  2i - n, 2i - (n - 1), 2i - (n - 2), \ldots, i - 1, & \text{for } i = \frac{n}{2} + 1, \frac{n}{2} + 2, \frac{n}{2} + 3, \ldots, n - 1.
\end{cases}
\]

So, \( |e_{BF^*}(1) - e_{BF^*}(0)| = |3n - 2[(n - j) + (n - i) + k + 2l]| \)

\[ = |2(i + j - k - 2l) - n|, \text{ where} \]

if \( i = 1, 2, 3, \ldots, \frac{n}{2} \), then \( j = i + 1, i + 2, i + 3, \ldots, 2i \), \( k = 0, 1, 2, \ldots, i - 1 \) and \( l = 0, 1, 2, 3, \ldots, i \); if \( i = \frac{n}{2} + 1, \frac{n}{2} + 2, \frac{n}{2} + 3, \ldots, n - 1 \), then

\[ j = i + 1, i + 2, i + 3, \ldots, n, \quad k = 2i - n, 2i - (n - 1), 2i - (n - 2), \ldots, i - 1 \]

and \( l = 0, 1, 2, \ldots, n - i \) such that \( j + k = 2i \).

Thus, \( |e_{BF^*}(1) - e_{BF^*}(0)| = |n + 2i - 4j + 4l| \), where

if \( i = 1, 2, 3, \ldots, \frac{n}{2} \), then \( j = i + 1, i + 2, i + 3, \ldots, 2i \) and \( l = 0, 1, 2, 3, \ldots, i \);

if \( i = \frac{n}{2} + 1, \frac{n}{2} + 2, \frac{n}{2} + 3, \ldots, n - 1 \), then \( j = i + 1, i + 2, i + 3, \ldots, n \)

and \( l = 0, 1, 2, \ldots, n - i \).

**Subcase 2.3.** When the compositions of number of rim vertices of wheel and pendant vertices of helm are \((0, n)\) and \((n, 0)\) respectively, \( e_{BF^*}(0) = n \)
and \( e_{Ff^*}(1) = 2n \). Thus, \(|e_{Ff^*}(0) - e_{Ff^*}(1)| = n\).

**Subcase 2.4.** When the compositions of number of rim vertices of wheel and pendant vertices of helm are \((n - (i + 1), i + 1)\) and \((i, n - i)\) respectively, where \(i = 0, 1, 2, \ldots, n - 1\),

\[
e_{Bf^*}(0) = (n - j) + (n - (i + 1)) + \min(n - (i + 1), i), \quad \text{where}
\]

\[
j = \begin{cases} 
  i + 2, i + 3, i + 4, \ldots, 2(i + 1), & \text{for } i = 0, 1, 2, \ldots, \frac{n}{2} - 1 \\
  i + 2, i + 3, i + 4, \ldots, n, & \text{for } i = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, n - 1
\end{cases}
\]

and \( e_{Bf^*}(1) = k + \min(n - i, i + 1) \), where

\[
k = \begin{cases} 
  0, 1, 2, \ldots, i, & \text{for } i = 0, 1, 2, \ldots, \frac{n}{2} - 1 \\
  2(i + 1) - n, 2(i + 1) - (n - 1), 2(i + 1) - (n - 2), \ldots, i, & \text{for } i = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, n - 1
\end{cases}
\]

Thus, \(|e_{Ff^*}(1) - e_{Ff^*}(0)| = |3n - 2[(n - j) + (n - (i + 1)) + k + 2l + 1]|\),

where if \(i = 0, 1, 2, \ldots, \frac{n}{2} - 1\), then \(j = i + 2, i + 3, i + 4, \ldots, 2(i + 1)\),

\(k = 0, 1, 2, \ldots, i\) and \(l = 0, 1, 2, \ldots, i\); if \(i = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, n - 1\), then \(j = i + 2, i + 3, i + 4, \ldots, n, k = 2(i + 1) - n, 2(i + 1) - (n - 1), 2(i + 1) - (n - 2), \ldots, i\)

and \(l = 0, 1, 2, \ldots, n - (i + 1)\) such that \(j + k = 2(i + 1)\).

So, \(|e_{Ff^*}(1) - e_{Ff^*}(0)| = |n + 2i - 4j + 4l + 4|\), where if \(i = 0, 1, 2, \ldots, \frac{n}{2} - 1\),

then \(j = i + 2, i + 3, i + 4, \ldots, 2(i + 1)\) and \(l = 0, 1, 2, \ldots, i\); if \(i = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, n - 2\), then \(j = i + 2, i + 3, i + 4, \ldots, n\) and \(l = 0, 1, 2, \ldots, n - (i + 1)\).
Subcase 2.5. When the compositions of number of rim vertices of wheel and pendant vertices of helm are \((0, n)\) and \((n - 1, 1)\) respectively, \(e_{FF^*}(0) = n + 1\) and \(e_{FF^*}(1) = 2n - 1\). Thus, \(|e_{FF^*}(1) - e_{FF^*}(0)| = n - 2\).

Considering all the above subcases and possible values of \(i, j\) and \(l\),

\[ BI(H_n) = \{0, 2, 4, \ldots, 2n\}. \]

Also, when an apex vertex is labeled 1, the friendly index set will be same. \(\square\)

**Corollary 5.3.13.** The helm graph \(H_n\) is cordial.

**Corollary 5.3.14.** The friendly index set of helm graph \(H_n\) forms an arithmetic progression with common difference 2.

**Corollary 5.3.15.** The friendly index number of helm graph \(H_n\),

\[ FIN(H_n) = \begin{cases} 
  n, & \text{if } n \text{ is odd} \\
  n + 1, & \text{if } n \text{ is even.} 
\end{cases} \]

**Corollary 5.3.16.** The full friendly index set of helm graph \(H_n\),

\[ FFI(H_n) = \begin{cases} 
  \{-2n + 5, -2n + 7, -2n + 9, \ldots, 2n - 1\}, & \text{if } n \text{ is odd} \\
  \{-2n + 6, -2n + 8, -2n + 10, \ldots, 2n\}, & \text{if } n \text{ is even.} 
\end{cases} \]

**Corollary 5.3.17.** The full friendly index set of helm graph \(H_n\) forms an arithmetic progression with common difference 2.

**Corollary 5.3.18.** The full friendly index number of helm graph \(H_n\) is \(2n - 2\).
Example 5.3.3. The friendly index set of helm graph $H_5$ is $\{1, 3, 5, 7, 9\}$.

Table 5.3: Compositions of number of rim vertices of wheel and pendent vertices of $H_5$ for friendly labeling and corresponding elements of friendly index set.

<table>
<thead>
<tr>
<th>Compositions of integers 5 and 5</th>
<th>Corresponding friendly indexes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(5, 0)$ and $(0, 5)$</td>
<td>5</td>
</tr>
<tr>
<td>$(4, 1)$ and $(1, 4)$</td>
<td>1, 3</td>
</tr>
<tr>
<td>$(3, 2)$ and $(2, 3)$</td>
<td>1, 3, 5, 7</td>
</tr>
<tr>
<td>$(2, 3)$ and $(3, 2)$</td>
<td>1, 3, 5, 9</td>
</tr>
<tr>
<td>$(1, 4)$ and $(4, 1)$</td>
<td>3, 7</td>
</tr>
<tr>
<td>$(0, 5)$ and $(5, 0)$</td>
<td>5</td>
</tr>
<tr>
<td>$(4, 1)$ and $(0, 5)$</td>
<td>1</td>
</tr>
<tr>
<td>$(3, 2)$ and $(1, 4)$</td>
<td>1, 3, 5</td>
</tr>
<tr>
<td>$(2, 3)$ and $(2, 3)$</td>
<td>1, 3, 5, 7</td>
</tr>
<tr>
<td>$(1, 4)$ and $(3, 2)$</td>
<td>1, 5</td>
</tr>
<tr>
<td>$(0, 5)$ and $(4, 1)$</td>
<td>3</td>
</tr>
</tbody>
</table>

Theorem 5.3.4. The friendly index set of flower graph $Fl_n$,

$$FI(Fl_n) = \begin{cases} 
\{0, 4, 8, \ldots, 2n - 2\}, & \text{if } n \text{ is odd} \\
\{0, 4, 8, \ldots, 2n\}, & \text{if } n \text{ is even}.
\end{cases}$$
Proof. The flower graph $Fl_n$ is of order $2n + 1$ and size $4n$. Also, it contains an apex vertex of degree $2n$, $n$ vertices of degree 4 and remaining $n$ vertices of degree 2.

Case 1. When $n$ is odd.

Let the apex vertex be labeled 0. For friendly labeling, consider the following compositions of number of vertices with degrees 4 and 2.

Subcase 1.1. When the compositions of number of vertices with degrees 4 and 2 are $(n, 0)$ and $(0, n)$ respectively, $e_{F^*}(0) = 2n$ and $e_{F^*}(1) = 2n$. Thus, $|e_{F^*}(0) - e_{F^*}(1)| = 0$.

Subcase 1.2. When the compositions of number of vertices with degrees 4 and 2 are $(n - i, i)$ and $(i, n - i)$ respectively, where $i = 1, 2, 3, \ldots, n - 1$, $e_{B^*}(0) = (n - j) + (n - i) + i + \min(n - i, i)$, where

$$j = \begin{cases} 
  i + 1, i + 2, i + 3, \ldots, 2i, & \text{for } i = 1, 2, 3, \ldots, \frac{n-1}{2} \\
  i + 1, i + 2, i + 3, \ldots, n, & \text{for } i = \frac{n+1}{2}, \frac{n+3}{2}, \frac{n+5}{2}, \ldots, n - 1
\end{cases}$$

and $e_{B^*}(1) = k + \min(n - i, i)$, where

$$k = \begin{cases} 
  0, 1, 2, \ldots, i - 1, & \text{for } i = 1, 2, 3, \ldots, \frac{n-1}{2} \\
  2i - n, 2i - (n - 1), 2i - (n - 2), \ldots, i - 1, & \text{for } i = \frac{n+1}{2}, \frac{n+3}{2}, \frac{n+5}{2}, \ldots, n - 1
\end{cases}$$

So, $|e_{F^*}(1) - e_{F^*}(0)| = |4n - 2((n - j) + (n - i) + i + k + 2l)|$, where if $i = 1, 2, 3, \ldots, \frac{n-1}{2}$, then $j = i + 1, i + 2, i + 3, \ldots, 2i$, $k = 0, 1, 2, \ldots, i - 1$ and $l = 0, 1, 2, \ldots, i$; if $i = \frac{n+1}{2}, \frac{n+3}{2}, \frac{n+5}{2}, \ldots, n - 1$, then $j = i + 1, i + 2, i + 3, \ldots, n$. 


Chapter 5. Friendly Index Set and Friendly Index Number of Graphs

\[ k = 2i - n, 2i - (n - 1), 2i - (n - 2), \ldots, i - 1 \text{ and } l = 0, 1, 2, \ldots, n - i \]

such that \( j + k = 2i \).

Thus, \(|e_{FF^*}(0) - e_{FF^*}(1)| = 4|i - j + l|\), where if \( i = 1, 2, 3, \ldots, \frac{n-1}{2} \), then \( j = i+1, i+2, i+3, \ldots, 2i \) and \( l = 0, 1, 2, \ldots, i \); if \( i = \frac{n+1}{2}, \frac{n+3}{2}, \frac{n+5}{2}, \ldots, n-1 \), then \( j = i+1, i+2, i+3, \ldots, n \) and \( l = 0, 1, 2, \ldots, n-i \).

**Subcase 1.3.** When the compositions of number of vertices with degrees 4 and 2 are \((0, n)\) and \((n, 0)\) respectively, \( e_{FF^*}(0) = 2n \) and \( e_{FF^*}(1) = 2n \). Thus, \(|e_{FF^*}(1) - e_{FF^*}(0)| = 0 \).

**Subcase 1.4.** When the compositions of number of vertices with degrees 4 and 2 are \((n - (i + 1), i + 1)\) and \((i, n - i)\), where \( i = 0, 1, 2, \ldots, n-1 \), respectively, \( e_{BF^*}(0) = (n - j) + (n - (i + 1)) + i + \min(n - (i + 1), i) \), where

\[ j = \begin{cases} 
  i + 2, i + 3, i + 4, \ldots, 2(i + 1), & \text{for } i = 0, 1, 2, \ldots, \frac{n-3}{2} \\
  i + 2, i + 3, i + 4, \ldots, n, & \text{for } i = \frac{n-1}{2}, \frac{n+1}{2}, \frac{n+3}{2}, \ldots, n-1
\end{cases} \]

and \( e_{BF^*}(1) = k + \min(n - i, i + 1) \), where

\[ k = \begin{cases} 
  0, 1, 2, \ldots, i, & \text{for } i = 0, 1, 2, \ldots, \frac{n-3}{2} \\
  2(i + 1) - n, 2(i + 1) - (n - 1), 2(i + 1) - (n - 2), \ldots, i & \text{for } i = \frac{n-1}{2}, \frac{n+1}{2}, \frac{n+3}{2}, \ldots, n-1.
\end{cases} \]
So, \( |e_{F^*}(0) - e_{F^*}(1)| = |4n - 2((n - j) + (n - (i + 1)) + i + 2l + k + 1)|, \)
where if \( i = 0, 1, 2, \ldots, \frac{n-3}{2}, \) then \( j = i + 2, i + 3, i + 4, \ldots, 2(i + 1), \) \( k = 0, 1, 2, \ldots, i \) and \( l = 0, 1, 2, \ldots, i; \) if \( i = \frac{n-1}{2}, \frac{n+1}{2}, \frac{n+3}{2}, \ldots, n - 1, \) then \( j = i + 2, i + 3, i + 4, \ldots, n, \) \( k = 2(i+1) - n, 2(i+1) - (n-1), 2(i+1) - (n-2), \ldots, i \)
and \( l = 0, 1, 2, \ldots, n - (i + 1) \) such that \( j + k = 2(i + 1). \)

Thus, \( |e_{F^*}(0) - e_{F^*}(1)| = |4(i - j + l + 1)|, \) where if \( i = 0, 1, 2, \ldots, \frac{n-3}{2}, \)
then \( j = i + 2, i + 3, i + 4, \ldots, 2(i + 1) \) and \( l = 0, 1, 2, \ldots, i; \)
if \( i = \frac{n-1}{2}, \frac{n+1}{2}, \frac{n+3}{2}, \ldots, n - 2, \) then \( j = i + 2, i + 3, i + 4, \ldots, n \) and \( l = 0, 1, 2, \ldots, n - (i + 1). \)

**Subcase 1.5.** When the compositions of number of vertices with degrees 4 and 2 are \((0, n)\) and \((n-1, 1)\) respectively, \( e_{F^*}(0) = 2n \) and \( e_{F^*}(1) = 2n. \)

Thus, \( |e_{F^*}(1) - e_{F^*}(0)| = 0. \)

Considering all the above subcases and all possible values of \( i, j \) and \( l, \)
\( FI(Fl_n) = \{0, 4, 8, \ldots, 2n - 2\}. \)

**Case 2.** When \( n \) is even.

Let the apex vertex be labeled by 0. For friendly labeling, consider the following compositions of number of vertices with degrees 4 and 2.

**Subcase 2.1.** When the compositions of number of vertices with degrees 4 and 2 are \((n, 0)\) and \((0, n)\) respectively, \( e_{F^*}(0) = 2n \) and \( e_{F^*}(1) = 2n. \)

Thus, \( |e_{F^*}(0) - e_{F^*}(1)| = 0. \)
Subcase 2.2. When the compositions of number of vertices with degrees 4 and 2 are \((n - i, i)\) and \((i, n - i)\) respectively, where \(i = 1, 2, 3, \ldots, n - 1\),
\[
e_{BF^*}(0) = (n - j) + (n - i) + i + \min(n - i, i),
\]
where
\[
j = \begin{cases} 
  i + 1, i + 2, i + 3, \ldots, 2i, & \text{for } i = 1, 2, 3, \ldots, \frac{n}{2} \\
  i + 1, i + 2, i + 3, \ldots, n, & \text{for } i = \frac{n}{2} + 1, \frac{n}{2} + 2, \frac{n}{2} + 3, \ldots, n - 1
\end{cases}
\]
and \(e_{BF^*}(1) = k + \min(n - i, i)\), where
\[
k = \begin{cases} 
  0, 1, 2, \ldots, i - 1, & \text{for } i = 1, 2, 3, \ldots, \frac{n}{2} \\
  2i - n, 2i - n + 1, 2i - n + 2, \ldots, i - 1, & \text{for } i = \frac{n}{2} + 1, \frac{n}{2} + 2, \frac{n}{2} + 3, \ldots, n - 1
\end{cases}
\]
So, \(|e_{F^*}(0) - e_{F^*}(1)| = |4n - 2((n - j) + (n - i) + i + k + 2l)|\), where if \(i = 1, 2, 3, \ldots, \frac{n}{2}\), then \(j = i + 1, i + 2, i + 3, \ldots, 2i\), \(k = 0, 1, 2, \ldots, i - 1\) and \(l = 0, 1, 2, \ldots, i\); if \(i = \frac{n}{2} + 1, \frac{n}{2} + 2, \frac{n}{2} + 3, \ldots, n - 1\), then \(j = i + 1, i + 2, i + 3, \ldots, n\), \(k = 2i - n, 2i - (n - 1), 2i - (n - 2), \ldots, i - 1\) and \(l = 0, 1, 2, \ldots, n - i\) such that \(j + k = 2i\).

Thus, \(|e_{F^*}(0) - e_{F^*}(1)| = 4|i - j + l|\), where if \(i = 1, 2, 3, \ldots, \frac{n}{2}\), then \(j = i + 1, i + 2, i + 3, \ldots, 2i\) and \(l = 0, 1, 2, \ldots, i\); if \(i = \frac{n}{2} + 1, \frac{n}{2} + 2, \frac{n}{2} + 3, \ldots, n - 1\), then \(j = i + 1, i + 2, i + 3, \ldots, n\) and \(l = 0, 1, 2, \ldots, n - i\).

Subcase 2.3. When the compositions of number of vertices with degrees 4 and 2 are \((0, n)\) and \((n, 0)\) respectively, \(e_{F^*}(0) = 2n\) and \(e_{F^*}(1) = 2n\).

Thus, \(|e_{F^*}(0) - e_{F^*}(1)| = 0\).
Subcase 2.4. When the compositions of number of vertices with degrees 4 and 2 are \((n-(i+1), i+1)\) and \((i, n-i)\) respectively, where \(i = 0, 1, 2, \ldots, n-1\),

\[ e_{Bf^*}(0) = (n - j) + (n - (i + 1)) + i + \min(n - (i + 1), i), \]

where

\[ j = \begin{cases} 
  i + 2, i + 3, i + 4, \ldots, 2(i + 1), & \text{for } i = 0, 1, 2, \ldots, \frac{n}{2} - 1 \\
  i + 2, i + 3, i + 4, \ldots, n, & \text{for } i = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, n - 1
\end{cases} \]

and \(e_{Bf^*}(1) = k + \min(n - i, i + 1)\), where

\[ k = \begin{cases} 
  0, 1, 2, \ldots, i, & \text{for } i = 0, 1, 2, \ldots, \frac{n}{2} - 1 \\
  2(i + 1) - n, 2(i + 1) - (n - 1), 2(i + 1) - (n - 2), \ldots, i, & \text{for } i = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, n - 1
\end{cases} \]

So, \(|e_{f^*}(0) - e_{f^*}(1)| = |4n - 2((n - j) + (n - (i + 1)) + i + 2l + k + 1)|\),

where if \(i = 0, 1, 2, \ldots, \frac{n}{2} - 1\), then \(j = i + 2, i + 3, i + 4, \ldots, 2(i + 1), \)

\(k = 0, 1, 2, \ldots, i\) and \(l = 0, 1, 2, \ldots, i\); if \(i = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, n - 1\), then \(j = i + 2, i + 3, i + 4, \ldots, n, k = 2(i + 1) - n, 2(i + 1) - (n - 1), 2(i + 1) - (n - 2), \ldots, i\)

and \(l = 0, 1, 2, \ldots, n - (i + 1)\) such that \(j + k = 2(i + 1)\).

Thus, \(|e_{f^*}(0) - e_{f^*}(1)| = |4(i - j + l + 1)|\), where if \(i = 0, 1, 2, \ldots, \frac{n}{2} - 1, \)

then \(j = i + 2, i + 3, i + 4, \ldots, 2(i + 1)\) and \(l = 0, 1, 2, \ldots, i; \)

if \(i = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, n - 2, \) then \(j = i + 2, i + 3, i + 4, \ldots, n\) and \(l = 0, 1, 2, \ldots, n - (i + 1)\).
**Subcase 2.5.** When the compositions of number of vertices with degrees 4 and 2 are \((0, n)\) and \((n - 1, 1)\) respectively, \(e_{F^*}(0) = 2n\) and \(e_{F^*}(1) = 2n\). Thus, \(|e_{F^*}(1) - e_{F^*}(0)| = 0\).

Considering all the above subcases, \(FI(Fl_n) = \{0, 4, 8, \ldots, 2n\}\).

When the apex vertex is labeled 1, the friendly index set will be same.

**Corollary 5.3.19.** The flower graph \(Fl_n\) is cordial.

**Corollary 5.3.20.** The friendly index set of the flower graph \(Fl_n\) forms an arithmetic progression with common difference 4.

**Corollary 5.3.21.** The friendly index number of flower graph \(Fl_n\) is \(\left\lfloor \frac{n}{2} \right\rfloor + 1\).

**Corollary 5.3.22.** The full friendly index set of flower graph \(Fl_n\),

\[
FFI(Fl_n) = \begin{cases} 
\{-2n + 6, -2n + 10, -2n + 14, \ldots, 2n - 2\}, & \text{if } n \text{ is odd} \\
\{-2n + 4, -2n + 8, -2n + 12, \ldots, 2n\}, & \text{if } n \text{ is even.}
\end{cases}
\]

**Corollary 5.3.23.** The full friendly index set of the flower graph \(Fl_n\) forms an arithmetic progression with common difference 4.

**Corollary 5.3.24.** The full friendly index number of flower graph \(Fl_n\),

\[
FFIN(Fl_n) = \begin{cases} 
n - 1, & \text{if } n \text{ is odd} \\
n, & \text{if } n \text{ is even.}
\end{cases}
\]
Example 5.3.4. The friendly index set of flower graph $F_{l_5}$ is $\{0, 4, 8\}$.

Table 5.4: Compositions of number of rim vertices of wheel and degree 2 vertices of $F_{l_5}$ for friendly labeling and corresponding friendly indexes.

<table>
<thead>
<tr>
<th>Compositions of integers 5 and 5</th>
<th>Corresponding friendly indexes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(5, 0)$ and $(0, 5)$</td>
<td>0</td>
</tr>
<tr>
<td>$(4, 1)$ and $(1, 4)$</td>
<td>0, 4</td>
</tr>
<tr>
<td>$(3, 2)$ and $(2, 3)$</td>
<td>0, 4, 8</td>
</tr>
<tr>
<td>$(2, 3)$ and $(3, 2)$</td>
<td>0, 4, 8</td>
</tr>
<tr>
<td>$(1, 4)$ and $(4, 1)$</td>
<td>0, 4</td>
</tr>
<tr>
<td>$(0, 5)$ and $(5, 0)$</td>
<td>0</td>
</tr>
<tr>
<td>$(4, 1)$ and $(0, 5)$</td>
<td>4</td>
</tr>
<tr>
<td>$(3, 2)$ and $(1, 4)$</td>
<td>0, 4, 8</td>
</tr>
<tr>
<td>$(2, 3)$ and $(2, 3)$</td>
<td>0, 4, 8</td>
</tr>
<tr>
<td>$(1, 4)$ and $(3, 2)$</td>
<td>0, 4</td>
</tr>
<tr>
<td>$(0, 5)$ and $(4, 1)$</td>
<td>0</td>
</tr>
</tbody>
</table>