CHAPTER-IV
NON CONTRACTION TYPE MAPPING ON
2-BANACH SPACE

4.1 In a paper S. Gahler[1] defines a linear 2-normed space to be a pair \((L, \|\cdot\|, \|\cdot\|')\) where \(L\) is a linear space and \(\|\cdot\|\) is a real valued function defined on \(L\) such that for \(a, b, c \in L\),

\[
(4.1.1)
\]

\[
\|a, b\| = 0 \text{ if and only if } a \text{ and } b \text{ are linearly dependent},
\]

\[
(4.1.2)
\]

\[
\|a, b\| = \|b, a\|,
\]

\[
(4.1.3)
\]

\[
\|a, \beta b\| = |\beta| \|a, b\|, \beta \text{ is real},
\]

\[
(4.1.4)
\]

\[
\|a, b+c\| \leq \|a, b\| + \|a, c\|,
\]

Here \(\|\cdot, \cdot\|\) is called a 2-norm.

It is Know[2] that \(\|\cdot, \cdot\|\) is a Non-Negative function.

Suppose \(X\) denote a Banach space with the norm \(\|\cdot\|\) and let \(K\) be a closed subset of \(X\). The transformation \(F: K \to K\) is called

\[1\] Gahler, S.
\[2\] Gahler, S.
contraction if there exists a constant \(0 \leq \alpha < 1\) such that for arbitrary \(x, y \in k\) the inequality:

\[
||F_x - F_y|| \leq \alpha \ ||x - y||
\]

holds. It is called nonexpansive if the same condition with \(\alpha = 1\) holds. By Banach contraction principle each contraction of \(k\) has exactly one fixed point. The same is true if we assume that only some powers of \(F\) are contraction, but is not true for non expansive mappings. However, Browder[3] has proved that every non expansive mapping of a closed bounded convex subset of a uniformly convex Banach space has at least one fixed point. Kirk[4] proved similar theorems in the space with normal structure. Goebel[5] has given a simple proof of the above result of Browder and Kirk. Kannan[6] has proved a theorem for the mapping which satisfy.

\[
||F_x - F_y|| \leq \frac{1}{2} (||x - F_x|| + ||y - F_y||)
\]

for \(x, y \in k\)

[3] Browder, F.E.  
Where \( k \) is a closed bounded convex subset of reflexive Banach space \( X \). It is to be noted that the reflexivity of the space and the normal structure of \( k \) being consequences of uniform convexity of \( X \). Above mappings (4.1.6) are neither weaker nor stronger than the non expansive mapping (4.1.5), yet it appears that most of the fixed point theorems for non expansive mappings also hold for mapping which are continuous and satisfy (4.1.6).

In a paper Goebel, Kirk and Shimmi[7] proved similar theorem for the mapping satisfying:

\[
\|Fx-Fy\| \leq a_1 \|x-y\| + a_2 \|x-Fx\|
+ a_3 \|y-Fy\| + a_4 \|x-Fy\|
+ a_5 \|y-Fx\|
\]

Where
\[
\sum_{i=1}^{5} a_i = 1
\]

However, Klee[8] showed that even in Hilbert space some convex set admit continuous transformation without fixed points and even such that \( F^2 = I \) where I denote identity mapping. With this object

[8] Klee, V.
In view Goebel and Zlotkiewicz[9] has proved a theorem which generalizes the above mentioned result of Browder.

There are many generalizations of the classical contraction mapping theorem of S. Banach. On the other hand, some fixed point theorems on non-contraction mappings on a Banach space have been obtained by Iseki[10] as under,

**THEOREM KI:**

Let $F$ be a mapping of a Banach space $X$ into itself. If $F$ satisfies conditions

1. $F^2 = I$ where $I$ is the identity mapping,
2. $||Fx-Fy|| \leq \alpha \ ||x-y|| + \beta \ (||x-Fx|| + ||y-Fy||)$

for every $x, y \in X$, where $0 \leq \alpha, \beta$ and $0 < \alpha + 4 \beta < 2$ then $F$ has at least one fixed point.


The object of this chapter is to study some theorems of Non-contraction type mapping on 2-Banach space.

4.2 We shall required some definitions:

[10] Iseki, K.
DEFINITION 4.2.1. A sequence \( \{ x_n \} \) in a linear 2-normed space \( L \) is called a Cauchy sequence if if there exists \( y, z \in L \) such that \( y \) and \( z \) are linearly independent, the
\[
\lim \| x_n - x_m, y \| = 0 \quad \text{and} \quad \lim \| x_n - x_m, z \| = 0
\]

DEFINITION 4.2.2. A sequence \( \{ x_n \} \) in a linear 2-normal space \( L \) is called a convergent sequence if there is an \( x \in L \) such that the
\[
\lim \| x_n - x, y \| = 0 \quad \text{for all} \quad y \in L.
\]

DEFINITION 4.2.3. A linear 2-normed space in which every cauchy sequence is convergent sequence, is called a 2-Banach space.

We shall prove;

THEOREM 1. Let \( F \) be a mapping of 2-Banach space \( X \) into itself. If \( F \) satisfies the conditions.

1. \( F^2 = I \), where \( I \) is the identity mapping.

2. \[
\| Fx - Fy, a \| \leq \alpha \frac{\| x - Fy, a \|}{\| x - y, a \|} + \beta \| x - y, a \|
\]

For every \( x, y \) and \( a \in X \), where \( 0 \leq \alpha \), \( \beta \), \( \alpha + \beta < 1 \) and \( 3 \alpha + \beta < 2 \), then \( F \) has a unique fixed point in \( X \).

PROOF:

Let \( x \) be a point of \( X \), and put
\[ y = \frac{1}{2} (F+I)(x), \ z = Fy, \ u = 2y-z. \]

then, by condition (1), (2), we have

\[ ||z-x, a|| = ||Fy-x, a|| = ||Fy-F^2x, a|| \]

\[ \leq a \frac{||y-F^2x, a||}{||y-Fx, a||} \frac{||Fx-Fy, a||}{||Fx-Fy, a||} + \beta \ ||y-Fx, a|| \]

\[ = a \frac{||y-x, a||}{||y-Fx, a||} \frac{||Fx-Fy, a||}{||Fx-Fy, a||} + \beta \ ||y-Fx, a|| \]

\[ \leq a \frac{\frac{1}{2} (F+I)(x)-x, a||}{||\frac{1}{2} (F+I)(x)-Fx, a||} [||Fx-y, a|| + \beta ||y-Fy, a||] \]

\[ + \beta \ ||\frac{1}{2} (F+I)(x)-Fx, a|| \]

\[ = a \frac{\frac{1}{2}||x-Fx, a||}{\frac{1}{2}||x-Fx, a||} [||Fx-\frac{1}{2} (F+I)(x), a|| + \beta ||y-Fy, a||] \]

\[ + \beta \ ||x-Fx, a|| \]

\[ = (\frac{a}{2} + \frac{\beta}{2}) \ ||x-Fx, a|| + \beta \ ||y-Fy, a|| \]

\[ \therefore \ ||z-x, a|| \leq (\frac{a}{2} + \frac{\beta}{2}) \ ||x-Fx, a|| + a \ ||y-Fy, a|| \quad \ldots \ldots \ldots \ldots \quad (1) \]

and

\[ ||u-x, a|| = ||2y-z-x, a|| = ||(F+I)(x)-Fy-x, a|| \]

\[ = ||Fx-Fy, a|| \]
\[ \leq \alpha \frac{||x-Fy, a||}{||x-y, a||} + \beta \frac{||y-Fx, a||}{||x-y, a||} \]

\[ = \alpha \frac{||x-Fy, a||}{||x-\frac{1}{2}(F+I)(x)-Fx, a||} + \beta \frac{||y-Fx, a||}{||x-\frac{1}{2}(F+I)(x), a||} \]

\[ = \alpha \frac{||x-Fy, a||}{\frac{1}{2}||x-Fx, a||} + \beta \frac{||y-Fx, a||}{\frac{1}{2}||x-Fx, a||} \]

\[ \leq \alpha \left( ||y-Fy, a|| + ||y-Fy, a|| \right) \]

\[ + \beta/2 \ ||x-Fx, a|| \]

\[ = \alpha \left( ||x-\frac{1}{2}(F+I)(x), a|| + ||y-Fy, a|| \right) \]

\[ + \beta/2 \ ||x-Fx, a|| \]

\[ = \left( \frac{\alpha}{2} + \frac{\beta}{2} \right) ||x-Fx, a|| + \alpha \ ||y-Fy, a|| \]

Therefore,

\[ ||u-x, a|| \leq \left( \frac{\alpha}{2} + \frac{\beta}{2} \right) ||x-Fx, a|| + \alpha \ ||y-Fy, a|| \ldots \ldots (2) \]

Now

\[ ||z-u, a|| = ||z-x+x-u, a|| \]

\[ \leq ||z-x, a|| + ||u-x, a|| \]

From (1) and (2), we have
\[ ||z-u, a|| \leq \left( \frac{\alpha}{2} + \frac{\beta}{2} \right) ||x-Fx, a|| + \alpha ||y-Fy, a|| \]
\[ + \left( \frac{\alpha}{2} + \frac{\beta}{2} \right) ||x-Fx, a|| + \alpha ||y-Fy, a|| \]

Hence
\[ ||z-u, a|| \leq \left( \alpha + \beta \right) ||x-Fx, a|| + 2 \alpha ||y-Fy, a|| \] .... (3)

On the other hand, we have
\[ ||z-u, a|| = ||Fy-2y-z, a|| = ||Fy-2y + Fy, a|| \]
\[ = 2 ||Fy-y, a|| \] ........ (4)

By (3) and (4), we have
\[ 2||Fy-y, a|| \leq \left( \alpha + \beta \right) ||x-Fx, a|| + 2 \alpha ||y-Fy, a|| \]

Therefore,
\[ 2(1 - \alpha) ||y-Fy, a|| \leq \left( \alpha + \beta \right) ||x-Fx, a|| \]

Hence we have the following inequality.
\[ ||y-Fy, a|| \leq \left( \frac{\alpha + \beta}{2(1-\alpha)} \right) ||x-Fx, a|| \]

or
\[ ||y-Fy, a|| \leq q ||x-Fx, a|| \] ........ (5)

Where \( q = \frac{\alpha + \beta}{2(1-\alpha)} \) < 1

Since, \( 3 \alpha + \beta < 2 \).

Let \( G = \frac{1}{2} (F+I) \), then, for any \( x \in X \).
\[ ||G^2x - Gx, a|| = ||Gy - y, a|| \]
\[ = ||\frac{1}{2}(F+I)(y) - y, a|| \]
\[ = \frac{1}{2}||y - Fy, a|| \]

By, (5), we have
\[ ||G^2x - Gx, a|| = \frac{1}{2}||y - Fy, a|| \leq \frac{q}{2} ||x - Fx, a|| \]

Since, \( G = \frac{1}{2}(F+I) \)

\[ \therefore 2G = (F+I) \]

Which implies \( 2Gx = (F+I)x = Fx + x \)

Therefore,
\[ Fx = 2Gx - x \]

Hence,
\[ ||G^2x - Gx, a|| \leq \frac{q}{2} ||x - (2Gx - x), a|| \]

\[ = q ||Gx - x, a|| \]

i.e. \( ||G^2x - Gx, a|| \leq q ||Gx - x, a|| \) \ldots \ldots (6)

By the hypothesis, we have
\[ 0 \leq \frac{\alpha + \beta}{2(1 - \alpha)} = q < 1. \]
Therefore \( \{ G^n x \} \) is a cauchy sequence in \( X \).

By the completeness, \( G^n x \) converges to \( x_0 \in X \).

i.e. \( \lim_{n \to \infty} G^n x = x_0 \).

This implies \( Gx_0 = x_0 \).

Hence \( Fx_0 = x_0 \) i.e. \( x_0 \) is a fixed point of \( F \). For the uniqueness, if possible let \( y_0 \) is also a fixed point.

\[
||x_0 - y_0, a|| = ||Fy_0 - Fx_0, a||
\]

\[
\leq a \cdot \frac{||x_0 - Fy_0, a||}{||x_0 - y_0, a||} + \beta \cdot ||x_0 - y_0, a||
\]

\[
= (a + \beta) \cdot ||x_0 - y_0, a||
\]

\[
\therefore (1 - a - \beta) \cdot ||x_0 - y_0, a|| < 0
\]

Since \( a + \beta < 1 \).

which implies \( 1 - a - \beta > 0 \).

Therefore \( x_0 = y_0 \) Hence the result is proved.

Now we prove;

THEOREM 2.

Let \( F \) be a mapping of 2-Banach space \( X \) into itself. If \( F \) satisfies conditions.

(1) \( F^2 = I \), where \( I \) is the identity mapping.

(2) \[ \|Fx - Fy, a\| \leq \alpha \frac{\|x-Fy, a\| \cdot \|y-Fx, a\|}{\|x-y, a\|} + \beta [\|x-Fx, a\| + \|y-Fy, a\|] + \gamma \|x-y, a\| \]

For every \( x, y, a \in X \), where \( 0 \leq \alpha, \beta, \gamma \) and \( \alpha + \gamma < \), \( 3\alpha + 4\beta + \gamma < 2 \).

Then \( F \) has a unique fixed point in \( X \).

PROOF:

Let \( x \) be a point of \( X \), and put

\[ y = \frac{1}{2} (F+I)(x), \quad z = Fy, \quad u = 2y-z \]

then by conditions (1), (2), we have

\[ \|z-x, a\| = \|Fy-x, a\| \]

\[ = \|Fy-F^2x, a\| \]

\[ \leq \alpha \frac{\|y-F^2x, a\| \cdot \|Fx-Fy, a\|}{\|y-Fx, a\|} + \beta [\|y-Fy,a\| + \|Fx-F^2x, a\|] + \gamma \|y-Fx, a\| \]

\[ = \alpha \frac{\|y-x, a\| \cdot \|Fx-Fy, a\|}{\|y-Fx, a\|} \]
\[ + \beta \left( \| y-Fy, a \| + \| x-Fx, a \| \right) + \gamma \| y-Fx, a \| \leq \alpha \left( \frac{1}{2} \| (F+I)(x) - x, a \| + \| Fx-y, a \| + \| y-Fy, a \| \right) + \| \frac{1}{2}(F+I)(x)-Fx, a \| \]

\[ + \beta \left( \| y-Fy, a \| + \| x-Fx, a \| \right) + \gamma \| \frac{1}{2}(F+I)(x)-Fx, a \| \]

\[ = \alpha \left( \frac{1}{2} \| x-Fx, a \| + \| y-Fy, a \| \right) + \beta \left( \| y-Fy, a \| + \| x-Fx, a \| \right) + \frac{\gamma}{2} \| x-Fx, a \| \]

Therefore,

\[ \| z-x, a \| \leq (\alpha + \beta) \| y-Fy, a \| + \left( \frac{\alpha}{2} + \beta + \frac{\gamma}{2} \right) \| x-Fx, a \| \]

\[ \ldots \ldots (1) \]

and

\[ \| u-x, a \| = \| 2y-z-x, a \| \]

\[ = \| (F+I)(x) - Fy-x, a \| \]

\[ = \| Fx-Fy, a \| \]

\[ \leq \alpha \| x-Fy, a \| + \| y-Fx, a \| \]

\[ + \beta \left( \| x-Fx, a \| + \| y-Fy, a \| \right) + \gamma \| x-y, a \| \]
\[
\begin{align*}
= \alpha & \left( ||x-Fy, a|| + \frac{1}{2}(F+1)(x)-Fx, a|| \right. \\
& \left. ||x-\frac{1}{2}(F+1)(x), a|| \right) \\
+ \beta & \left( ||x-Fx, a|| + ||y-Fy, a|| \right) \\
+ \gamma & \left( ||x-\frac{1}{2}(F+1)(x), a|| \right) \\
\leq \alpha & \left( ||x-y, a|| + ||y-Fy, a|| \right) \\
+ \beta & \left( ||x-Fx, a|| + ||y-Fy, a|| \right) \\
+ \frac{\gamma}{2} & \left( ||x-Fx, a|| \right) \\
= & \alpha \left( ||x-\frac{1}{2}(F+1)(x), a|| + ||y-Fy, a|| \right) \\
+ \beta & \left( ||x-Fx, a|| + ||y-Fy, a|| \right) \\
+ \frac{\gamma}{2} & \left( ||x-Fx, a|| \right) \\
= & (\alpha + \beta) ||y-Fy, a|| \\
+ & \left( \frac{\alpha}{2} + \beta + \frac{\gamma}{2} \right) ||x-Fx, a|| \quad \text{......... (2)}
\end{align*}
\]

Now,
\[
||z-u, a|| \leq ||z-x, a|| + ||u-x, a||
\]

From (1) and (2), we have,
\[
||z-u, a|| \leq \left[ (\alpha + \beta) ||y-Fy, a|| + \left( \frac{\alpha}{2} + \beta + \frac{\gamma}{2} \right) ||x-Fx, a|| \right]
\]
\[ + [(\alpha + \beta)||y-Fy, a||
+ (\frac{\alpha}{2} + \beta + \frac{\gamma}{2}) ||x-Fx, a||\]

Hence,

\[||z-u,a|| \leq (2\alpha + 2\beta)||y-Fy, a||
+ (\alpha + 2\beta + \gamma)||x-Fx, a|| \]

\[\ldots (3)\]

On the other hand, we have

\[||z-u, a|| = ||z-(2y-z), a|| = ||2z-2y, a||
= ||2Fy-2y, a||
= 2||Fy-y, a|| = 2||y-Fy, a|| \]

\[\ldots (4)\]

By (3) and (4), we have

\[2||y-Fy, a|| \leq (2\alpha + 2\beta)||y-Fy, a||
+ (\alpha + 2\beta + \gamma)||x-Fx, a||\]

Therefore,

\[(2-2\alpha-2\beta)||y-Fy, a|| \leq (\alpha + 2\beta + \gamma)||x-Fx, a||\]

Hence we have the following inequality.

\[||y-Fy, a|| \leq \frac{\frac{\alpha}{2} + 2\beta + \gamma}{2(1-\alpha-\beta)} ||x-Fx, a||\]

or,

\[||y-Fy, a|| \leq q||x-Fx, a||\]
where \[ q = \frac{(\alpha + 2\beta + \gamma)}{2(1 - \alpha - \beta)} < 1 \]

Since,

\[ 3\alpha + 4\beta + \gamma < 2. \]

Let \( G = \frac{1}{3} (F+I) \), then, for any \( x \in X \)

\[ ||G^2x - Gx, a|| = ||Gy - y, a|| \]
\[ = ||\frac{1}{3}(F+I)y - y, a|| \]
\[ = \frac{1}{3}||y - Fy, a|| \]
\[ < \frac{q}{2} ||x - Fx, a|| \]
\[ = \frac{q}{2} ||x - (2Gx - x), a|| \]
\[ = q ||Gx - x, a|| \]

By the hypothesis, we have

\[ 0 < \frac{\alpha + 2\beta + \gamma}{2(1 - \alpha - \beta)} \quad q < 1 \]

Since, \( 3\alpha + 4\beta + \gamma < 2. \)

Therefore, \( \{ G^nx \} \) is a cauchy sequence in \( X \). By the completeness,

\( \{ G^nx \} \) converges to some element \( x_0 \) in \( X \).

i.e. \( \lim_{n \to \infty} G^nx = x_0 \)
This implies \( Gx_0 = x_0 \)

Hence \( Fx_0 = x_0 \) i.e. \( x_0 \) is a Fixed point of \( F \). For the uniqueness, if possible let \( y_0 \) is also a fixed point.

\[
||x_0 - y_0, a|| = ||Fx_0 - Fy_0, a||
\]

\[
\leq \alpha \frac{||x_0 - Fy_0, a|| \cdot ||y_0 - Fx_0, a||}{||x_0 - y_0, a||}
\]

\[
+ \beta [||x_0 - Fx_0, a|| + ||y_0 - Fy_0, a||] + \gamma ||x_0 - y_0, a||
\]

\[
= (\alpha + \gamma) ||x_0 - y_0, a||
\]

\[
\therefore (1 - \alpha - \gamma) ||x_0 - y_0, a|| \leq 0
\]

Since \( \alpha + \gamma < 1 \) which implies \( 1 - \alpha - \gamma > 0 \).

Therefore,

\[
x_0 = y_0.
\]

Hence the result is proved.

**THEOREM 3.**

Let \( F \) be a mapping of a 2-Banach spaces \( X \) into itself.

If satisfies conditions

(1) \( F^2 = I \), where \( I \) is the identity mapping.
(2) \[ \|F(x-Fy, a)\| \leq a \frac{\|x-Fy, a\| \cdot \|y-Fx, a\|}{\|x-y, a\|} \]

\[ + \beta \left( \|x-Fx, a\| + \|y-Fy, a\| \right) \]

\[ + \gamma \left( \|x-Fy, a\| + \|y-Fx, a\| \right) \]

\[ + \delta \|x-y, a\| \]

For every \( x, y, a \in X \),

Where \( a, \beta, \gamma, \delta \geq 0 \)

\[ a + 2 \gamma + \delta < 1 \quad \text{and} \quad 3a + 4\beta + 4\gamma + \delta < 2 \]

Then \( F \) has a unique fixed point in \( X \).

**PROOF:**

Let \( x \) be a point of \( X \), and put

\[ y = \frac{1}{2} (F+I)(x), \quad z = Fy, \quad u = 2y-z \]

then by condition (1), (2), we have

\[ \|z-x, a\| = \|Fy-F^2x, a\| \]
\begin{align*}
\leq & \, \|y-F^2x, a\| \cdot \, \|Fx-Fy, a\| \\
& \quad \|y-Fx, a\| \\
& \quad + \beta [\|y-Fy, a\| + \|Fx-F^2x, a\|] \\
& \quad + \gamma [\|y-F^2x, a\| + \|Fx-Fy, a\|] \\
& \quad + \delta \|y-Fx, a\| \\
= & \, \alpha \left[ \frac{1}{2}(F+I)(x)-x, a \right] \cdot \|Fx-Fy, a\| \\
& \quad \|\frac{1}{2}(F+I)(x)-Fx, a\| \\
& \quad + \beta [\|y-Fy, a\| + \|x-Fx, a\|] \\
& \quad + \gamma [\|\frac{1}{2}(F+I)(x)-x, a\| + \|Fx-Fy, a\|] \\
& \quad + \delta \|\frac{1}{2}(F+I)(x)-F(x), a\| \\
= & \, \alpha [\|Fx-y, a\| + \|y-Fy, a\|] \\
& \quad + \beta [\|y-Fy, a\| + \|x-Fx, a\|] \\
& \quad + \gamma [\frac{1}{2} \|x-Fx, a\| + \|Fx-y, a\| + \|y-Fy, a\|] \\
& \quad + \frac{\delta}{2} \|x-Fx, a\| \\
= & \, (\alpha + \beta + \gamma) \|y-Fy, a\| + \left(\frac{\alpha}{2} + \beta + \gamma + \frac{\delta}{2}\right) \|x-Fx, a\| \\
\end{align*}

Therefore,

\[ \|z-x, a\| \leq (\alpha + \beta + \gamma) \|y-Fy, a\| + \left(\frac{\alpha}{2} + \beta + \gamma + \frac{\delta}{2}\right) \|x-Fx, a\| \]

and

\[
\|U-x, a\| = \|2y-z-x, a\| = \|(F+I)(x)-Fy-x, a\| \\
= \|Fx-Fy, a\| \\
\leq & \, \alpha \left[ \|x-Fy, a\| \cdot \|y-Fx, a\| \\
& \quad \|x-y, a\| \\
& \quad + \beta [\|x-Fx, a\| + \|y-Fy, a\|] \\
\]

\[ \text{---------(1)} \]

\[ \|U-x, a\| = \|2y-z-x, a\| = \|(F+I)(x)-Fy-x, a\| \\
= \|Fx-Fy, a\| \\
\leq & \, \alpha \left[ \|x-Fy, a\| \cdot \|y-Fx, a\| \\
& \quad \|x-y, a\| \\
& \quad + \beta [\|x-Fx, a\| + \|y-Fy, a\|] \\
\]
\[ + \gamma \left[ \|x-Fy, a\| + \|y-Fx, a\| \right] + \delta \left[ \|x-y, a\| \right] \]

\[ = \alpha \left[ \|x-Fy, a\| \cdot \|\frac{1}{2}(F+I)(x)-Fx, a\| \right] \]

\[ + \beta \left[ \|x-Fx, a\| + \|y-Fy, a\| \right] \]

\[ + \gamma \left[ \|x-y, a\| + \|y-Fy, a\| \right] \]

\[ + \delta \left[ \|x-\frac{1}{2}(F+I)(x), a\| \right] \]

\[ \leq \alpha \left[ \|x-y, a\| + \|y-Fy, a\| \right] \]

\[ + \beta \left[ \|x-Fx, a\| + \|y-Fy, a\| \right] \]

\[ + \gamma \left[ \|x-y, a\| + \|y-Fy, a\| + \frac{1}{2} \|x-Fx, a\| \right] \]

\[ + \frac{\delta}{2} \left[ \|x-Fx, a\| \right] \]

\[ = \alpha \left[ \|x-\frac{1}{2}(F+I)(x), a\| + \|y-Fy, a\| \right] \]

\[ + \beta \left[ \|x-Fx, a\| + \|y-Fy, a\| \right] \]

\[ + \gamma \left[ \|x-\frac{1}{2}(F+I)(x), a\| + \|y-Fy, a\| + \frac{1}{2} \|x-Fx, a\| \right] \]

\[ + \frac{\delta}{2} \left[ \|x-Fx, a\| \right] \]

\[ = (\alpha + \beta + \gamma) \|y-Fy, a\| + (\frac{\alpha}{2} + \beta + \gamma + \frac{\delta}{2}) \|x-Fx, a\| \]

Therefore

\[ \|U-x, a\| \leq (\alpha + \beta + \gamma) \|y-Fy, a\| + \left( \frac{\alpha}{2} + \beta + \gamma + \frac{\delta}{2} \right) \|x-Fx, a\| \]

\[ \text{(2)} \]

Now

\[ \|z-u, a\| \leq \|z-x, a\| + \|u-x, a\| \]

From (1) and (2), we have

\[ \|z-u, a\| \leq (2\alpha + 2\beta + 2\gamma) \|y-Fy, a\| + \left( \alpha + 2\beta + 2\gamma + \delta \right) \|x-Fx, a\| \]

\[ \text{(3)} \]
On the other hand, we have

\[ ||z-u, a|| = ||z-(2y-z), a|| \]
\[ = 2||z-y, a|| \]
\[ ||z-u, a|| = 2||Fy-y, a|| \] --------- (4)

By (3) and (4), we have

\[ 2||Fy-y|| \leq (2a + 2b + 2y)||y-Fy, a|| \]
\[ + (a + 2b + 2 \gamma + \delta)||x-Fx, a|| \]

Therefore

\[ (2-2a-2b-2y)||y-Fy, a|| \leq (a + 2b + 2y + \delta)||x-Fx, a|| \]

Hence we have the following inequality

\[ ||y-Fy, a|| \leq \frac{(a + 2b + 2y + \delta)}{2(1-a-b-y)} ||x-Fx, a|| \]

or \[ ||y-Fy, a|| \leq q ||x-Fx, a|| \]

Where

\[ q = \frac{a + 2b + 2y + \delta}{2(1-a-b-y)} \leq 1 \]

Since,

\[ 3a + 4b + 4y + \delta < 2 \]

Let \( G = \frac{1}{2}(F+I) \), then, for any \( x \in X \).

\[ ||G^2x-Gx, a|| = ||Gy-y, a|| \]
\[ = ||\frac{1}{2}(F+I)(y)-y, a|| \]
\[ = \frac{1}{2}||y-Fy, a|| \]
\[ < \frac{q}{2} ||x-Fx, a|| \]
\[ = \frac{q}{2} ||x-(2Gx-x), a|| \]
\[ = q ||Gx-x, a|| \]
By the hypothesis, we have

\[ 0 \leq q < 1 \]

Therefore, \( \{G^n x\} \) is a Cauchy sequence in \( X \).

By the completeness, \( \{G^n x\} \) converges to some point \( X_0 \) in \( X \).

i.e. \( \lim_{n \to \infty} G^n x = X_0 \)

This implies that \( Gx_0 = X_0 \)

Hence \( Fx_0 = X_0 \) i.e. \( X_0 \) is a fixed point of \( F \).

For the uniqueness, if possible let \( Y_0 \) is also a fixed point.

\[ ||X_0 - Y_0, a|| = ||F_x^0 - F_{Y_0}, a|| \]

\[ \leq d \frac{||X_0 - F_{Y_0}, a|| + ||Y_0 - F_{X_0}, a||}{||X_0 - Y_0, a||} \]

\[ + \beta \left( ||X_0 - F_{X_0}, a|| + ||Y_0 - F_{Y_0}, a|| \right) \]

\[ + \gamma \left( ||X_0 - F_{Y_0}, a|| + ||Y_0 - F_{X_0}, a|| \right) \]

\[ + \delta ||X_0 - Y_0, a|| \]

\[ = (\alpha + 2\gamma + \delta) ||X_0 - Y_0, a|| \]

Therefore,

\[ (1 - \alpha - 2\gamma - \delta) ||X_0 - Y_0, a|| \leq 0 \]

Since \( \gamma + 2\gamma + \delta < 1 \)

Which implies \( 1 - \alpha - 2\gamma - \delta > 0 \)

Therefore,

\[ X_0 = Y_0 \]

Hence the result is proved.
In 1982, Khan[3], Sharma and Bajaj[4] proved a theorem in Banach space for commuting mapping, on the basis of that, now we prove the theorem which generalizes the theorem 1 in 2-Banach space:

THEOREM 2: Let $K$ be a closed and Convex subset of a 2-Banach space $X$.

Let $F : K + K$, $G : K + K$

satisfy the following conditions.

(1.3) $F$ and $G$ commutes

(1.4) $F^2 = I$ and $G^2 = I$, where $I$ is identity mapping

\[
||F x - F y, a|| \leq \frac{||G x - F y, a||}{||G x - G y, a||} + \beta \left( ||G x - F x, a|| + ||G y - F y, a|| \right) \\
+ \gamma \left( ||G x - F y, a|| + ||G y - F x, a|| \right) + \delta ||G x - G y, a||
\]

For every $x, y, a \in K$ and $0 \leq \alpha, \beta, \gamma, \delta$ and $3\alpha + 4\beta + 4\gamma + \delta < 2$

Then there exists at least one fixed point $x_0 \in K$, such that

$F x_0 = G x_0$. Further is $\alpha + 2\gamma + \beta < 1$,

Then $x_0$ is the unique fixed point of $F$ and $G$.

PROOF: From (1.3) and (1.4) it follows that $(FG)^2 = I$ and (1.4) and (1.5) we have

\[
||FG x - FG y, a|| \leq \frac{||G G^2 x - FG^2 y, a||}{||G G^2 x - G G^2 y, a||} + \beta \left( ||G G^2 x - FG^2 y, a|| + ||G G^2 y - FG^2 y, a|| \right) \\
+ \gamma \left( ||G G^2 x - F G^2 y, a|| + ||G G^2 y - FG^2 x, a|| \right)
\]
\[ + \delta \left| \left| G \cdot C^2 x - G \cdot C^2 y, a \right| \right| \]
\[ = a \left| \left| Gx-FG \cdot Gy, a \right| \right| \cdot \left| \left| Gy-FG \cdot Gx, a \right| \right| \]
\[ + \beta \left( \left| \left| Gx-FG \cdot Gy, a \right| \right| + \left| \left| Gy-FG \cdot Gy, a \right| \right| \right) \]
\[ + \gamma \left( \left| \left| Gx-FG \cdot Gy, a \right| \right| + \left| \left| Gy-FG \cdot Gx, a \right| \right| \right) \]
\[ + \delta \left| \left| Gx-Gy, a \right| \right| \]

Now if we put \( Gx = Z \), and \( Gy = w \) we get
\[ \left| \left| FGz-FGw, a \right| \right| \]
\[ \leq a \left| \left| Z-FGw, a \right| \right| \cdot \left| \left| w-FGz, a \right| \right| \]
\[ + \beta \left( \left| \left| Z-FGw, a \right| \right| + \left| \left| w-FGw, a \right| \right| \right) \]
\[ + \gamma \left( \left| \left| Z-FGw, a \right| \right| + \left| \left| w-FGz, a \right| \right| \right) \]
\[ + \delta \left| \left| Z-w, a \right| \right| \]

When \( (FG)^2 = I \) and \( 3a + 4\beta + 4\gamma + \delta < 2 \).
So by theorem 1, \( FG \) has at least one fixed point.

Say \( X_0 \) in \( K \) i.e.
\[ FGx_0 = X_0 \quad ---- (A) \]

and so,
\[ FFGx_0 = Fx_0 \]
or
\[ Gx_0 = Fx_0 \quad ---- (B) \]

Now
\[ \left| \left| Fx_0 - X_0, a \right| \right| = \left| \left| Fx_0 - F^2 x_0, a \right| \right| = \left| \left| Fx_0 - F(Fx_0), a \right| \right| \]
\[ < a \frac{\left| \left| Gx_0 - FFx_0, a \right| \right| \cdot \left| \left| GFx_0 - Fx_0, a \right| \right|}{\left| \left| Gx_0 - GFx_0, a \right| \right|} \]
\[ + \beta \left( \left| \left| Gx_0 - Fx_0, a \right| \right| + \left| \left| GFx_0 - FFx_0, a \right| \right| \right) \]
\[ + \gamma \| Gx_0 - FFx_0, a \| + \| GFx_0 - Fx_0, a \| \]
\[ + \delta \| Gx_0 - GFx_0, a \| \]
\[ = \alpha \frac{\| Gx_0 - X_0, a \| \| GFx_0 - Fx_0, a \|}{\| Gx_0 - GFx_0, a \|} \]
\[ + \beta \| Gx_0 - Fx_0, a \| + \| GFx_0 - X_0, a \| \]
\[ + \gamma \| Gx_0 - X_0, a \| + \| GFx_0 - Fx_0, a \| \]
\[ + \delta \| Gx_0 - GFx_0, a \| \]
\[ = \alpha \frac{\| Gx_0 - X_0, a \| \| X_0 - Fx_0, a \|}{\| Gx_0 - X_0, a \|} \]
\[ + \gamma \| Gx_0 - X_0, a \| + \| X_0 - Fx_0, a \| \]
\[ + \delta \| Gx_0 - X_0, a \| \]
\[ = (\alpha + 2\gamma + \delta) \| X_0 - Fx_0, a \| \]

Hence \( \| X_0 - Fx_0, a \| \leq (\alpha + 2\gamma + \delta) \| X_0 - Fx_0, a \| \)

Since \( \alpha + 2\gamma + \delta < 1 \), it follows \( Fx_0 = X_0 \) i.e. \( X_0 \) is the Fixed point of \( F \), but \( Fx_0 = Gx_0 \) and so we have \( Gx_0 = X_0 \) i.e. \( X_0 \) is the common Fixed point of \( F \) and \( G \).

Now to show that \( X_0 \) is unique common fixed point of \( F \) and \( G \), Let us consider \( Y_0 \) be another common fixed point of \( F \) and \( G \).

Now, using (1.3), (1.4) and (A), (B) we have
\[ \| X_0 - Y_0, a \| = \| F^2x_0 - F^2y_0, a \| = \| FFx_0 - FFy_0, a \| \]
\[ \leq \alpha \frac{\| GFx_0 - FFy_0, a \| \| GFy_0 - Fx_0, a \|}{\| GFx_0 - GFy_0, a \|} \]
\[ + \beta \left[ \|GF_x - FF_y, a\| + \|GF_y - FF_y, a\| \right] + \gamma \left[ \|GF_x - FF_y, a\| + \|GF_y - FF_x, a\| \right] + \delta \|GF_x - GF_y\| \]

\[ = \alpha \frac{\|x_o - y_o, a\| \|y_o - x_o, a\|}{\|x_o - y_o, a\|} \]

\[ + \beta \left[ \|x_o - x_o, a\| + \|y_o - y_o, a\| \right] + \gamma \left[ \|x_o - y_o, a\| + \|y_o - x_o, a\| \right] + \delta \|x_o - y_o, a\| \]

\[ = (\alpha + 2 \gamma + \delta) \|x_o - y_o, a\| \]

Since \( \alpha + 2 \gamma + \delta < 1 \), it follows \( x_o = y_o \) proving the uniqueness of \( x_o \). This completes the proof of the theorem.

Now we shall prove a theorem for three mappings.

In fact we prove:

THEOREM 5.

Let \( F, G, H \) are three mappings of a 2- Banach space \( X \) into itself such that

1. \( FG = GF, GH = HG, \) and \( FH = HF \)
2. \( F^2 = I, G^2 = I \) and \( H^2 = I \), where \( I \) is the identity mapping
3. \( \|Fx - Fy, a\| \leq \alpha \frac{\|GHx - Fy, a\| \|GHy - Fx, a\|}{\|GHx - GHy, a\|} \)
+ β ||GHx - GHy, a||

For every x, y, a ∈ X and 0 ≤ α, β such that 3α + β < 2. Then there exists at least one fixed point x₀ ∈ X, such that Fx₀ = GHx₀ and FGx₀ = Hx₀. Further if α + β < 1 then x₀ is the unique common fixed point of F, G and H.

PROOF:

From (1) and (2) it follows that

(FGH)² = I and by (2) and (3), we write.

||FGH.Gx - FGH.Gy, a||

≤ α ||(GH)².Gx - FGH.Gy, a|| + ||(GH)².Gy - FGH.Gx, a||

≤ α ||(GH)².Gx - (GH)².Gy, a||

+ β ||(GH)².Gx - (GH)².Gy, a||

= α ||Gx - FGH(Gy), a|| ||Gy - FGH.Gx, a||

||Gx - Gy, a||

Now if we put Gx = z, and Gy = w, we have,

||FGHz - FGHw, a||
\[
\leq \alpha \frac{||z-FGHw, a|| \cdot ||w-FGHz, a||}{||z-w, a||} + \beta ||z-w, a||
\]

When \((FGH)^2 = I\) and \(3 \alpha + \beta < 2\).

So by theorem 1, \(FGH\) has at least one fixed point.

Say \(x_0 \in X\). Thus

(4) \(FGHx_0 = x_0\) and so

(5) \(GH(FGHx_0) = GHx_0\)

or

\(Fx_0 = GHx_0\)

Also

(6) \(H(FGH)(x_0) = Hx_0\)

or

\(FGx_0 = Hx_0\)

Now using (1) to (6), we have

\[
||Hx_0 - x_0, a|| = ||FGx_0 - F^2x_0, a||
\]

\[
= \frac{||F(Gx_0) - F(Fx_0), a||}{||GH.Gx_0 - GH.Fx_0, a||}
\]

\[
\leq \alpha \frac{||GH.Gx_0 - Fx_0, a|| \cdot ||GH(Fx_0) - FGx_0, a||}{||GH.Gx_0 - GH.Fx_0, a||} + \beta ||GHGx_0 - GH.Fx_0, a||
\]
\[
\begin{align*}
\alpha \frac{\|G^2H_xo - F^2x_o, a\|}{\|H_xo - FGH_xo, a\|} & \|GHF_xo - FGx_o, a\| \\
+ \beta \|G^2H_xo - FGH_xo, a\| \\
= \alpha \frac{\|H_xo - x_o, a\|}{\|H_xo - x_o, a\|} + \beta \|H_xo - x_o, a\| \\
= (\alpha + \beta) \|x_o - H_xo, a\| \\
\end{align*}
\]

Hence, \(\|H_xo - x_o, a\| \leq (\alpha + \beta) \|H_xo - x_o, a\|\)

Since \(\alpha + \beta < 1\), it follows that \(H_xo = x_o\).

Hence \(x_o\) is the fixed point of \(H\).

Thus we have by (5) that

\(Gx_o = Fx_o\)

Again

\[\|Fx_o - x_o, a\| = \|Fx_o - F^2x_o, a\|\]

\[= \|Fx_o - F(Fx_o), a\|\]

\[\leq \alpha \frac{\|GHx_o - F(Fx_o), a\|}{\|GHx_o - GH(Fx_o), a\|} + \beta \|GHx_o - GH(Fx_o), a\|\]

\[= \alpha \frac{\|Fx_o - x_o, a\|}{\|Fx_o - x_o, a\|} + \beta \|Fx_o - x_o, a\|\]
\[ = (\alpha + \beta) \Vert Fx_o - x_o, a \Vert \]

Hence
\[ \Vert Fx_o - x_o, a \Vert \leq (\alpha + \beta) \Vert Fx_o - x_o, a \Vert \]

Since \( \alpha + \beta < 1 \).

Hence,
\[ Fx_o = x_o \quad \text{but} \quad Fx_o = Gx_o \]

Therefore
\[ Fx_o = Gx_o = Hx_o = x_o \]

Hence \( x_o \) is the fixed point of \( F, G, \) and \( H \).

To prove uniqueness,
\[ \Vert x_o - y_o, a \Vert = \Vert F^2 x_o - F^2 y_o, a \Vert = \Vert F(Fx_o) - FFy_o, a \Vert \]
\[ \leq \alpha \\Vert GHF x_o - F Fy_o, a \Vert \\Vert GFH y_o - FFx_o, a \Vert \]
\[ + \beta \\Vert GHI H x_o - GHI y_o, a \Vert \]
\[ = \alpha \frac{\Vert x_o - y_o, a \Vert \Vert y_o - x_o, a \Vert}{\Vert x_o - y_o, a \Vert} + \beta \Vert x_o - y_o, a \Vert \]
\[ \therefore \Vert x_o - y_o, a \Vert \leq (\alpha + \beta) (x_o - y_o, a) \]

Since \( \alpha + \beta < 1 \) \quad \text{Thus} \quad x_o = y_o

This completes the prove of the theorem.