CHAPTER-VIII

SOME FIXED POINT THEOREMS

8.1 In this chapter we present some fixed point theorems related to Banach contraction mapping principle in terms of the rational expressions.

In 1974, Cirić[1] proved a theorem the condition

\[(8.1.1) \ d \ (T_x, T_y) \leq q \ max \ \{ d(x,y), d(x,Tx), d(y,Ty),
\frac{1}{2} \ [ d(x,Ty) + d(y,Tx) ] \}\]

where \(q \in (0,1)\) and \(x, y \in X\), then \(T\) has a unique fixed point.

Pal and Maiti[2] have obtained an extension of Cirić’s fixed point theorem for mappings of a orbitally complete metric space into itself. Recently in 1982 Pachpatte[3] proved theorem \(T\)-orbitally complete metric space. There have been several generalizations the contraction mapping principle like Cirić[4], Iseki[5],[6],

[1] Cirić, L.B.
[3] Pachpatte, B.G.
[5] Iseki, K.
[6] Iseki, K.
Istratescu[7], Kannan[8], Maia[9], Pachpatte[10], Roades[11].


THEOREM P.

Let \((x, d)\) be a complete metric space and \(T\) be a mapping of \(X\) into itself such that:

\[
(8.1.2) \quad d(Tx, Ty) \leq q \max \left\{ \frac{d(x, y)}{1+d(x, y)}, \frac{d(y, Ty)[1+d(x, Tx)]}{1+d(x, y)} \right\}
\]

\[
\frac{1}{2} \cdot \frac{d(x, Ty)[1+d(x, Tx) + d(y, Tx)]}{1 + d(x, y)}
\]

for all \(x, y \in X\) and \(q \in (0, 1)\). Then \(T\) has a unique fixed point in \(X\).

In this chapter our object is to extend theorem P.

In fact we prove:

[7] Istratescu, V.I.
[8] Kannan, R.
[10] Pachpatte, B.G.
[12] Pachpatte, B.G.
THEOREM 1:

Let \( (x, d) \) be a complete metric space and \( T \) be a mapping of \( X \) into itself such that

\[
(8.1.3) \quad d(Tx, Ty) \leq q \max \left\{ d(x, y), \frac{d(y, Ty)[1+d(x, Tx)]}{1+d(x, y)}, \frac{d(x, Tx)[1+d(x, Tx)+d(y, Ty)]}{1+d(x, y)}, \frac{d(x, Tx).d(y, Ty)}{d(x, y)}, \frac{d(x, Ty).d(y, Tx)}{d(x, y)} \right\} \quad \ldots \ldots (1)
\]

for all \( x, y \in X \) and \( q \in (0, 1) \), \( x \neq y \). Then \( T \) has a unique fixed point in \( X \).

PROOF:

Let \( x_0 \in X \) be arbitrary and define \( x_1 = Tx_0, \ x_{n+1} = Tx_n, \ n = 1, 2, \ldots \).

Then by (1) we have

\[
d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \\
\leq q \max \left\{ d(x_{n-1}, x_n), \frac{d(x_n, Tx_n)[1+d(x_{n-1}, Tx_{n-1})]}{1+d(x_{n-1}, x_n)}, \frac{d(x_{n-1}, Tx_n)[1+d(x_{n-1}, Tx_{n-1})+d(x_n, Tx_{n-1})]}{1+d(x_{n-1}, x_n)} \right\}
\]
\[
\begin{align*}
\frac{d(x_{n-1}, Tx_{n-1})}{d(x_{n-1}, x_n)} \cdot \frac{d(x_n, Tx_n)}{d(x_n, T\bar{x}_n)} & , \\
& \\
\frac{d(x_{n-1}, Tx_{n-1})}{d(x_{n-1}, x_n)} \cdot \frac{d(x_n, T\bar{x}_n)}{d(x_n, T\bar{x}_{n-1})} & \\
& = q \text{ Max } \left\{ d(x_{n-1}, x_n), \frac{d(x_n, x_{n+1})}{1 + d(x_{n-1}, x_n)} \right\}, \\
& \\
\frac{1}{2} \frac{d(x_{n-1}, x_{n+1})}{1 + d(x_{n-1}, x_n)} & , \\
& \\
\frac{d(x_{n-1}, x_n)}{d(x_{n-1}, x_{n+1})} \cdot \frac{d(x_n, x_{n+1})}{d(x_n, x_n)} & \\
& = q \text{ Max } \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2} d(x_{n-1}, x_{n+1}), d(x_n, x_{n+1}) \right\} \\

\text{Since } d(x_n, x_{n+1}) \leq q d(x_n, x_{n+1}) \text{ is impossible as } (q < 1), \text{ one has} \\

d(x_n, x_{n+1}) \leq q \text{ max } \left\{ d(x_{n-1}, x_n), \frac{1}{2} d(x_{n-1}, x_{n+1}) \right\}
\end{align*}
\]
If \( \max \{ d(x_{n-1}, x_n), \frac{1}{2} d(x_{n-1}, x_{n+1}) \} \)

\[ = d(x_{n-1}, x_n) \quad \ldots \ldots (2) \]

then \( d(x_n, x_{n+1}) \leq q \ d(x_{n-1}, x_n) \quad \ldots \ldots (3) \)

If max of the two numbers in (2) is

\[ \frac{1}{2} d(x_{n-1}, x_{n+1}), \text{then} \]

\[ d(x_n, x_{n+1}) \leq \frac{1}{2} \ q \ d(x_{n-1}, x_{n+1}) \]

\[ \leq \frac{1}{2} \ q \ [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \]

which implies

\[ d(x_n, x_{n+1}) \leq \frac{q}{2q-1} \ d(x_{n-1}, x_n) \leq q \ d(x_{n-1}, x_n) \quad \ldots (4) \]

From (3) and (4) we have

\[ d(x_n, x_{n+1}) \leq q \ d(x_{n-1}, x_n) \quad \ldots \ldots (5) \]

Proceeding in this manner we obtain

\[ d(x_n, x_{n+1}) \leq q \ d(x_{n-1}, x_n) \leq q^2 \ d(x_{n-2}, x_{n-1}) \]

\[ \vdots \]

\[ \leq q^n \ d(x_0, x_1), \]

and hence for \( m > n. \)

\[ d(x_n, x_m) \leq q^n [1 + q + q^2 + \ldots + q^{m-n-1}] \ d(x_0, x_1) \ldots \ldots (6) \]

\[ \leq \frac{q^n}{1-q} \ d(x_0, x_1) \]

Since \( q < 1, \quad q^n \to 0, \text{as} \quad n \to \infty. \)
So from (6) it follows that the sequence \( \{ x_n \} \) is a cauchy sequence since \((x, d)\) is complete, there is a point \( u \in X \) such that-

\[
u = \lim_{n \to \infty} x_n \quad \ldots \quad (7)
\]

We shall now prove that \( u \) is a fixed point of \( T \). By (1) and triangle inequality we have

\[
d(u, Tu) \leq d(u, Tx_n) + d(Tx_n, Tu)
\]

\[
\leq d(u, x_{n+1}) + q \max \left\{ d(x_n, u), \frac{d(u, Tu)[1+d(x_n, x_{n+1})]}{1 + d(x_n, u)}, \frac{d(x_n, Tu)[1+d(x_n, x_{n+1})+d(u, x_{n+1})]}{1 + d(x_n, u)} \right\}
\]

\[
\leq \frac{d(x_n, x_{n+1}) d(u, Tu)}{d(x_n, u)} + \frac{d(x_n, u)}{d(x_n, u)}
\]

from the above inequality we observe that \( d(u, Tu) \) tends to zero as \( n \) tends to infinity, and hence we obtain \( Tu = u \), So we have proved \( u \) is a fixed point of \( T \).

Now, we shall show that \( u \) is a unique fixed point of \( T \).

Let \( V \) be another fixed point of \( T \), then \( d(u, v) = d(Tu, Tv) \)
\[
q \max \left\{ d(u, v), \frac{d(v, Tv) \cdot [1 + d(u, Tu)]}{1 + d(u, v)}, \frac{d(u, Tu) \cdot d(v, Tv)}{d(u, v)}, \frac{d(u, Tv) \cdot d(v, Tu)}{d(u, v)} \right\}
\]

from the above inequality we observe that \( d(u, v) = 0 \), i.e. \( u = v \). Hence \( u \) is unique fixed point of \( T \), and the proof of the theorem is complete.

**THEOREM 2:**

Let \( X \) be a metric space with two metrics \( d \) and \( \partial \). If \( x \) satisfies the following conditions:

1. \((8.2.1)\) \( d(x, y) \leq \partial(x, y) \) for every \( x, y \) in \( X \),
2. \((8.2.2)\) \( X \) is complete with respect to \( d \),
3. \((8.2.3)\) the mapping \( T : X \to X \) is continuous with respect to the metric \( d \), and
\( (8.2.4) \)

\[ \vartheta(Tx, Ty) \leq q \max \left\{ \vartheta(x, y), \frac{\vartheta(y, Ty)[1+\vartheta(x, Tx)]}{1+\vartheta(x, y)} \right\} \]

\[ \leq \frac{\vartheta(x, Ty)[1+\vartheta(x, Tx)+\vartheta(y, Tx)]}{1+\vartheta(x, y)} \]

\[ \frac{\vartheta(x, Tx) \vartheta(y, Ty)}{\vartheta(x, y)} \]

\[ \frac{\vartheta(x, Ty) \vartheta(y, Tx)}{\vartheta(x, y)} \]

\[ \vartheta(x, y) \]

\( \ldots \ldots \ldots (8) \)

for every \( x \neq y \in X \) and \( q \in (0, 1) \), then \( T \) has a unique fixed point in \( X \).

**PROOF:**

Let \( x_0 \in X \) be arbitrary and define

\[ x_1 = Tx_0, \ x_{n+1} = Tx_n, \ n = 1, 2, \ldots \]

\[ \vartheta(x_n, x_{n+1}) = \vartheta(Tx_{n-1}, Tx_n) \]

\[ \leq q \max \left\{ \vartheta(x_{n-1}, x_n), \frac{\vartheta(x_{n-1}, Ty_n)[1+\vartheta(x_{n-1}, Tx_n)]}{1+\vartheta(x_{n-1}, x_n)} \right\} \]

\[ \leq \frac{\vartheta(x_{n-1}, Ty_n)[1+\vartheta(x_{n-1}, Tx_n)+\vartheta(x_n, Tx_n)]}{1+\vartheta(x_{n-1}, x_n)} \]

\[ \frac{\vartheta(x_{n-1}, Tx_{n-1}) \vartheta(x_n, Tx_n)}{\vartheta(x_{n-1}, x_n)} \]

\[ \vartheta(x_{n-1}, x_n) \]
\[ \delta(x_{n-1}, Tx_n) \leq \delta(x_n, T x_{n-1}) \]
\[ = q \max \{ \delta(x_{n-1}, x_n), \delta(x_n, x_{n+1}), \frac{1}{2} \delta(x_{n-1}, x_{n+1}) \} \]
\[ \delta(x_n, x_{n+1}) \]

Now by following the similar argument as in the proof of Theorem 1, we obtained
\[ \delta(x_n, x_{n+1}) \leq q^n \delta(x_0, x_1), \]
and hence
\[ \delta(x_n, x_{n+1}) \leq \frac{q^n}{1-q} \delta(x_0, x_1) \]

Where \( p \) is any positive integer.

Therefore, by \( d \leq \delta \);
\[ d(x_n, x_{n+p}) \leq \frac{q^n}{1-q} \delta(x_0, x_1) \]

This shows that the sequence \( \{ x_n \} \) is a Cauchy sequence with respect to \( d \).

Since \( X \) is complete with respect to \( d \), \( \{ x_n \} \) has a limit point \( u \) in \( X \),
i.e. \[ \lim_{n \to \infty} x_n = u \] Hence, by the continuity of \( T \) with respect to the metric \( d \),
\[
\begin{align*}
  u &= \lim_{n \to \infty} x_{2n+1} = \lim_{n \to \infty} T x_{2n} \\
  &= T \left( \lim_{n \to \infty} x_{2n} \right) = Tu.
\end{align*}
\]

Therefore \( u \) is a fixed point of \( T \). Uniqueness of \( u \) follows by the similar argument as in Theorem 1.

Hence \( T \) has a unique fixed point in \( X \), which complete the proof.