6.1 In a paper S. Gahler[1] investigated the notion of a 2-Metric, a real valued function of a point triples on a set X,

Let \((X, d)\) be a complete metric space, and let \(T:X \rightarrow X\) satisfy.

\[
(6.1.1) \quad d(Tx, Ty) \leq a \cdot d(x, y)
\]

Where \(0 < a < 1\) and \(x, y \in X\).

By Banach's fixed point theorem[2]

\(T\) has a unique fixed point.

In 1968 Kannan[3], investigated mainly under what conditions two mappings each mapping a complete metric space \(X\) into itself have a unique fixed point.

Further Rus[4] has obtain a general result which is the generalization of the above result of Kannan.

[1] Gahler, S.
[2] Banach, S.
[3] Kannan, R.
[4] Rus, I.A.
On the other hand Demarr[5] obtain a common fixed point theorem for a family of mapping. Wong[6] has proved a very general theorem on common fixed point of two mapping which generalizes the Banach contraction theorem, result of Hardy and Rogers[7], Kannan[8], Rakotch[9], Reich[10], Shrivastava and Gupta[11], also gives a different proof of the special cases. Recently Iseki[12] in his three papers generalized the above mentioned result and also an interesting theorem of Maia[13].

In 1974, Sehgal[14] has also proved a common fixed point theorem which includes as special cases the result of Rakotch[15]Boyd and

[12] Iseki, K.  [1,5]
Wong[16], Fukushima[17], Kannan[18], Maki[19], Sehgal[21], Singh[22] and Shrivastava and Gupta[23].

Further some results also given by Singh[24], Singh[25] and Sharma and Yuel[26].

Recently in 1986 Paliwal[27] study a fixed point theorem in 2-metric space.

The object of this chapter is to extend the well known theorem on a contraction mapping by Banach[28].

[16] Boyd, D.W. and Wong, J.S.W.
[17] Fukushima, H.
[18] Kannan, R.
[19] Maki, H.
[20] Reich, S.
[21] Sehgal, V.M.
[22] Singh, S.P.
[23] Shrivastava, P. and Gupta, V.K.
[25] Singh, S.L.
[26] Sharma, P.L. and Yuel, A.K.
[27] Paliwal, Y.C.
[28] Banach, S.
Before proving the result here we state the following definitions:

**DEFINITION 1.**

A. 2-metric space is a space \( X \) in which for each triple points \( a, b, c \) there exists a real function \( d(a, b, c) \) such that to each pair of points \( a, b \) (\( a \neq b \)) from \( X \), there is \( c \in X \) satisfying.

\[(6.1.2a)\]
\[d(a, b, c) \neq 0\]

\[(6.1.2b)\]
\[d(a, b, c) = 0 \text{ only when at least two of the three points are equal,}\]

\[(6.1.3)\]
\[d(a, b, c) = d(a, c, b) = d(b, c, a).\]

\[(6.1.4)\]
\[d(a, b, c) \leq d(a, b, e) + d(a, e, c) + d(e, b, c).\]

A 2-metric space is called bounded, if there exists a constant \( k \) such that \( d(a, b, c) \leq k \) for all \( a, b, c \in X \).

**DEFINITION 2.**

A sequence \( \{x_n\} \) in 2-metric space \( X \) is called a convergent sequence if there an \( x \in X \). Such that \( \lim d(x_n, x, y) = 0 \) for all \( y \in X \). Here \( x \) is called the limit of \( \{x_n\} \).

**DEFINITION 3.**

A sequence \( \{x_n\} \) in a 2- metric space \( X \) is called a cauchy
sequence, if \( \lim d(x_m, x_n, y) = 0 \) for all \( y \in X \).

**DEFINITION 4.**

A 2-metric space in which every Cauchy sequence converges is called a complete 2-metric space.

We shall prove the following theorem which is a generalization of Paliwal[29]

**THEOREM 1.**

If \( T \) is a mapping of complete 2-metric space \( X \) into itself, satisfying the following condition.

\[
d(Tx, Ty, a) \leq a_1 \left[ \frac{d(x, Tx, a) \cdot d(y, Ty, a)}{d(x, y, a)} \right] \\
+ a_2 \left[ \frac{d(x, Ty, a) \cdot d(y, Tx, a)}{d(x, y, a)} \right] \\
+ a_3 \left[ d(x, Tx, a) + d(y, Ty, a) \right] \\
+ a_4 \left[ d(x, Ty, a) + d(y, Tx, a) \right] \\
+ a_5 \left[ d(x, y, a) \right]
\]

\[ ..............(1) \]

For all \( x, y, a \) in \( X \), where \( 0 \leq \frac{a_3 + a_4 + a_5}{1 - a_1 - a_3 - a_4} < 1 \)

\( a_3 + a_4 < 1 \), and \( a_2 + 2a_4 + a_5 < 1 \).

[29] Y.C. Paliwal
Then $T$ has a unique fixed point.

PROOF.

Let $x_0$ be an arbitrary point of $X$. We define a sequence $\{x_n\}$ by

$$x_n = Tx_{n-1} = T^n x_0, \ n = 1, 2, 3 \ldots$$

If $x_n = x_{n+1}$ for some $n$ then the result is immediate. So let $x_n \neq x_{n+1}$ for each $n = 0, 1, 2 \ldots$ then by (1) for $x = x_{n-1}$ and $y = x_n$ we have

$$d(x_n, x_{n+1}, a) = d(Tx_{n-1}, Tx_n, a)$$

$$\leq a_1 \left[ \frac{d(x_{n-1}, Tx_{n-1}, a) \cdot d(x_n, Tx_n, a)}{d(x_{n-1}, x_n, a)} \right]$$

$$+ a_2 \left[ \frac{d(x_{n-1}, Tx_n, a) \cdot d(x_n, Tx_{n-1}, a)}{d(x_{n-1}, x_n, a)} \right]$$

$$+ a_3 [d(x_{n-1}, Tx_{n-1}, a) + d(x_n, Tx_n, a)]$$

$$+ a_4 [d(x_{n-1}, Tx_n, a) + d(x_n, Tx_{n-1}, a)]$$

$$+ a_5 [d(x_{n-1}, x_n, a)]$$

$$= a_1 \left[ \frac{d(x_{n-1}, x_n, a) \cdot d(x_n, x_{n+1}, a)}{d(x_{n-1}, x_n, a)} \right]$$
\[+ a_2 \left[ \frac{d(x_{n-1}, x_{n+1}, a) \cdot d(x_n, x_n, a)}{d(x_{n-1}, x_n, a)} \right]\]

\[+ a_3 \left[ d(x_{n-1}, x_n, a) + d(x_n, x_{n+1}, a) \right]\]

\[+ a_4 \left[ d(x_{n-1}, x_{n+1}, a) + d(x_n, x_n, a) \right]\]

\[+ a_5 \left[ d(x_{n-1}, x_n, a) \right]\]

Which implies that

\[d(x_n, x_{n+1}, a) \leq \frac{a_3 + a_4 + a_5}{(1 - a_1 - a_3 - a_4)} \cdot d(x_{n-1}, x_n, a)\]

\[= q \cdot d(x_{n-1}, x_n, a)\]

where

\[0 \leq \frac{a_3 + a_4 + a_5}{1 - a_1 - a_3 - a_4} = q < 1.\]

\[\leq q^2 \cdot d(x_{n-2}, x_{n-1}, a)\]

\[\ldots\]

\[\ldots\]

\[\leq q^n \cdot d(x_0, x_1, a)\]

For \(m > n\) we have
\[ d(x_n, x_m, a) \leq d(x_n, x_{n+1}, a) + d(x_{n+1}, x_{n+2}, a) + \ldots . \]

\[ + d(x_{m-1}, x_m, a) \]

\[ \leq (q^n + q^{n+1} + \ldots + q^{m-1}) d(x_0, x_1, a) \]

\[ \leq \frac{q^n}{1 - q} d(x_0, x_1, a) \]

\[ \to 0 \text{ as } m, n \to \infty \]

Since \( q \) is less then \( 1 \) i.e. \( q < 1 \), it follows that \( \{x_n\} \) is cauchy sequence. Since \( X \) is complete there exists a point \( u \in X \) such that \( x_n \to u \).

Now

\[ d(u, Tu, a) \leq d(u, x_n, a) + (x_n, Tu, a) \]

\[ \leq 1 \ d(u, x_n, a) + \alpha_1 \frac{d(x_{n-1}, x_n, a) d(u, Tu, a)}{d(x_{n-1}, u, a)} \]

\[ + \alpha_2 \frac{d(x_{n-1}, Tu, a) d(u, x_n, a)}{d(x_{n-1}, u, a)} \]

\[ + \alpha_3 [d(x_{n-1}, x_n, a) + d(u, Tu, a)] \]

\[ + \alpha_4 [d(x_{n-1}, Tu, a) + d(u, x_n, a)] \]

\[ + \alpha_5 d(x_{n-1}, u, a) \]
Taking the limit $n\to\infty$, we have

$$d(u, Tu, a) \leq \left( a_3 + a_4 \right) d(u, Tu, a)$$

it follows that $Tu = u$ as $a_3 + a_4 < 1$.

Now we prove uniqueness

Suppose $z$ and $u$ are two fixed point of $T$ in $X$ that is $Tu = u$ and $Tz = z$.

$$d(z, u, a) = d(Tz, Tu, a)$$

$$\leq a_1 \frac{d(z, Tz, a) d(u, Tu, a)}{d(z, u, a)}$$

$$+ a_2 \frac{d(z, Tu, a) d(u, Tz, a)}{d(z, u, a)}$$

$$+ a_3 \left[ d(z, Tz, a) + d(u, Tu, a) \right]$$

$$+ a_4 \left[ d(z, Tu, a) + d(u, Tz, a) \right]$$

$$+ a_5 \left[ d(z, u, a) \right]$$

Hence,

$$d(z, u, a) \leq \left( a_2 + 2a_4 + a_5 \right) d(z, u, a)$$

Since $a_2 + 2a_4 + a_5 < 1$, therefore it follows that $u = z$. So $u$ is the unique fixed point of $T$.

REMARK:

1. If we put $a_2 = 0$ then, we get the result of Paliwal.
In 1982 Iseki, Rajput and Sharma[30] has given "An Extension of Banach contraction principle through Rational Expression" in complete 1-metric space. Our object of theorem is to prove in complete 2-metric space in the following form.

In fact we prove

THEOREM 2.

Let T is mapping of complete 2-metric space into itself, satisfying the following condition

\[ d(Tx, Ty, a) \leq \alpha_1 \frac{d(y, Ty, a)}{[1 + d(x, Tx, a)]} + \alpha_2 \frac{d(x, Ty, a)}{d(x, y, a)} + \alpha_3 [d(x, Tx, a) + d(y, Ty, a)] + \alpha_4 [d(x, Ty, a) + d(y, Tx, a)] + \alpha_5 [d(x, y, a)] \]

For all \( x, y, a \) in \( X \), where

\[ \alpha_1 + 2 \alpha_3 + 2 \alpha_4 + \alpha_5 < 1 \] and
\[ \alpha_2 + 2 \alpha_4 + \alpha_5 < 1. \]

Then T has a unique fixed point.

PROOF:

Let \( x_0 \) be an arbitrary point of \( X \).

We define a sequence \( \{x_n\} \) by

\[
x_n = T^n x_0, \quad n = 1, 2, 3 \ldots
\]

If \( x_n = x_{n+1} \) for some \( n \) then the result is immediate. So let \( x_n \neq x_{n+1} \) for each \( n = 0, 1, 2 \ldots \).

Then by (2) for \( x = x_{n-1} \) only \( y = x_n \) we have

\[
d(x_n, x_{n+1}, a) = \frac{(T x_{n-1}, T x_n, a)}{[1 + d(x_{n-1}, x_n, a)]}
\]

\[
\leq a_1 \frac{d(x_n, T x_n, a) [1 + d(x_{n-1}, T x_{n-1}, a)]}{[1 + d(x_{n-1}, x_n, a)]}
\]

\[
+ a_2 \frac{d(x_{n-1}, T x_n, a) d(x_n, T x_{n-1}, a)}{d(x_{n-1}, x_n, a)}
\]

\[
+ a_3 [d(x_{n-1}, T x_{n-1}, a) d(x_n, T x_n, a)]
\]

\[
+ a_4 [d(x_{n-1}, T x_n, a) + d(x_n, T x_{n-1}, a)]
\]

\[
+ a_5 [d(x_{n-1}, x_n, a)]
\]

\[
= a_1 \frac{d(x_n, x_{n+1}, a) [1 + d(x_{n-1}, x_n, a)]}{[1 + d(x_{n-1}, x_n, a)]}
\]

\[
+ a_2 \frac{d(x_{n-1}, x_{n+1}, a) d(x_n, x_n a)}{d(x_{n-1}, x_n, a)}
\]
\[ + \alpha_3 [d(x_{n-1}, x_n, a) + d(x_n, x_{n+1}, a) \]
\[ + \alpha_4 [d(x_{n-1}, x_{n+1}, a) + d(x_n, x_n, a)] \]
\[ + \alpha_5 [d(x_{n-1}, x_n, a)] \]

Which implies that
\[ d(x_n, x_{n+1}, a) \leq \frac{\alpha_3 + \alpha_4 + \alpha_5}{1 - \alpha_1 - \alpha_3 - \alpha_4} d(x_{n-1}, x_n, a) \]
\[ \leq q \cdot d(x_{n-1}, x_n, a) \]

Where
\[ 0 \leq \frac{\alpha_3 + \alpha_4 + \alpha_5}{1 - \alpha_1 - \alpha_3 - \alpha_4} = q < 1 \]
\[ \leq q^2 d(x_{n-2}, x_{n-1}, a) \]
\[ \ldots \]
\[ \ldots \]
\[ \ldots \]
\[ \leq q^n d(x_0, x_1, a) \]

for \( m > n \) we have
\[ d(x_n, x_m, a) \leq d(x_n, x_{n+1}, a) + d(x_{n+1}, x_{n+2}, a) + \ldots \]
\[ + d(x_{m-1}, x_m, a) \]
\[ \leq \left( q^n + q^{n+1} + \ldots + q^{m-1} \right) d(x_0, x_1, a) \]

\[ \leq \frac{q^n}{1 - q} \quad d(x_0, x_1, a) \]

\[ \rightarrow 0 \quad \text{as} \quad m, n \rightarrow \infty \]

Since \( q \) is less than 1 i.e. \( q < 1 \).

It follows that \( \{x_n\} \) is a Cauchy sequence since \( X \) is complete, there exists a point \( u \in X \) such that \( x_n \rightarrow u \).

Now

\[ d(u, Tu, a) \leq d(u, x_n, a) + d(x_n, Tu, a) \]

\[ \leq d(u, x_n, a) + \alpha_1 \frac{d(u, Tu, a)[1 + d(x_{n-1}, x_n, a)]}{[1 + d(x_{n-1}, u, a)]} \]

\[ + \alpha_2 \frac{(x_{n-1}, Tu, a) d(u, x_n, a)}{d(x_{n-1}, u, a)} \]

\[ + \alpha_3 [d(x_{n-1}, x_n, a) + d(u, Tu, a)] \]

\[ + \alpha_4 [d(x_{n-1}, Tu, a) + d(u, x_n, a)] \]

\[ + \alpha_5 d(x_{n-1}, u, a) \]

Taking the limit \( n \rightarrow \infty \), we have

\[ d(u, Tu, a) \leq (\alpha_3 + \alpha_4) d(u, Tu, a) \]

It follows that \( Tu = u \) as
Now for uniqueness, suppose \( z \) and \( u \) are two fixed point of \( T \) in \( X \) that is \( Tu = u \) and \( Tz = z \)

\[
\begin{align*}
    d(z, u, a) &= d(Tz, Tu, a) \\
    &\leq \alpha_1 \frac{d(u, Tu, a) [1+d(z,Tz,a)]}{[1+d(z,u,a)]} \\
    &+ \alpha_2 \frac{d(z, Tu, a) d(u, Tz, a)}{d(z, u, a)} \\
    &+ \alpha_3 [d(z,Tz, a) + d(u, Tu, a)] \\
    &+ \alpha_4 [d(z, Tu, a) + d(u, Tz, a)] \\
    &+ \alpha_5 d(z, u, a)
\end{align*}
\]

Hence

\[
d(z, u, a) \leq (\alpha_2 + 2 \alpha_4 + \alpha_5) d(z, u, a)
\]

Since \( \alpha_2 + 2 \alpha_4 + \alpha_5 < 1 \), therefore it follows that \( u = z \).

So \( u \) is the unique fixed point of \( T \).

**THEOREM 3.**

Let \( T \) is mapping of complete 2-metric space into itself satisfying the following condition:

\[
d(Tx,Ty,a) \leq \alpha_1 \frac{d(y, Ty, a) [1+d(x, Tx, a)]}{[1+d(x, y, a)]}
\]
\[ + \alpha_2 \quad [d(x, Tx, a) \cdot d(y, Ty, a)] \\
\quad d(x, y, a) \]

\[ + \alpha_3 \quad d(x, Ty, a) \cdot d(y, Tx, a) \\
\quad d(x, y, a) \]

\[ + \alpha_4 \quad [d(x, Tx, a) + d(y, Ty, a)] \]

\[ + \alpha_5 \quad [d(x, Ty, a) + d(y, Tx, a)] \]

\[ + \alpha_6 \quad d(x, y, a) \quad ........... \quad (3) \]

For all \( x, y, a \) in \( X \).

Where \( \alpha_3 + 2\alpha_5 + \alpha_6 < 1 \) and

\[ \alpha_1 + \alpha_2 + 2\alpha_4 + 2\alpha_5 + \alpha_6 < 1 \]

Then \( T \) has a unique fixed point.

**PROOF:**

Let \( x_0 \) can be an arbitrary point in \( X \) we define sequence

\[ \{ x_n \} \]

by

\[ x_n = T x_{n-1} = T x_0^n, \quad n = 1, 2, 3, \quad ........ \]

If \( x_n = x_{n+1} \) for some \( n \) then the result is immediate. So let

\[ x_n \neq x_{n+1} \quad \text{for each} \quad n = 0, 1, 2, \ldots \]

Then by (3) for \( x = x_{n-1} \) and \( y = x_n \).
We have

\[ d(x_n, x_{n+1}, a) = d(Tx_{n-1}, Tx_n, a) \]

\[ \leq a_1 \frac{d(x_n, Tx_n, a) \cdot [1 + d(x_{n-1}, Tx_{n-1}, a)]}{1 + d(x_{n-1}, x_n, a)} + a_2 \frac{d(x_{n-1}, Tx_{n-1}, a) \cdot d(x_n, Tx_n, a)}{d(x_{n-1}, x_n, a)} + a_3 \frac{d(x_{n-1}, Tx_n, a) \cdot d(x_n, Tx_{n-1}, a)}{d(x_{n-1}, x_n, a)} + a_4 [d(x_{n-1}, Tx_n, a) + d(x_n, Tx_{n-1}, a)] + a_5 [d(x_{n-1}, Tx_n, a) + d(x_n, Tx_{n-1}, a)] + a_6 d(x_{n-1}, x_n a) \]

\[ = a_1 \frac{d(x_n, x_{n+1}, a) \cdot [1 + d(x_{n-1}, x_n, a)]}{1 + d(x_{n-1}, x_n, a)} + a_2 \frac{d(x_{n-1}, x_n, a) \cdot d(x_n, x_{n+1}, a)}{d(x_{n-1}, x_n, a)} + a_3 \frac{d(x_{n-1}, x_{n+1}, a) \cdot d(x_n, x_n, a)}{d(x_{n-1}, x_n, a)} + a_4 [d(x_{n-1}, x_{n+1}, a) + d(x_n, x_n, a)] \]
\[ + \frac{a_5}{\alpha} \left[ d(x_{n-1}, x_{n+1}, a) + d(x_n, x, a) \right] \]
\[ + \frac{a_6}{\alpha} \left[ d(x_{n-1}, x_n, a) \right] \]

Which implies that

\[ d(x_n, x_{n+1}, a) \leq \frac{\frac{a_4}{\alpha} + \frac{a_5}{\alpha} + \frac{a_6}{\alpha}}{1 - \frac{a_1}{\alpha} - \frac{a_2}{\alpha} - \frac{a_4}{\alpha} - \frac{a_5}{\alpha}} \cdot d(x_{n-1}, x_n, a) \]
\[ \leq q \cdot d(x_{n-1}, x_n, a) \]

Where
\[ 0 \leq \frac{\frac{a_4}{\alpha} + \frac{a_5}{\alpha} + \frac{a_6}{\alpha}}{1 - \frac{a_1}{\alpha} - \frac{a_2}{\alpha} - \frac{a_4}{\alpha} - \frac{a_5}{\alpha}} = q < 1 \]
\[ \leq q^2 \cdot d(x_{n-1}, x_n, a) \]
\[ \vdots \]
\[ \vdots \]
\[ \vdots \]
\[ \leq q^n \cdot d(x_0, x_1, a) \]

For \( m > n \) we have

\[ d(x_n, x_m, a) \leq d(x_n, x_{n+1}, a) + d(x_{n+1}, x_{n+2}, a) + \ldots + d(x_{m-1}, x_m, a) \]
\[ \leq (q^n + q^{n+1} + \ldots + q^{m-1}) \cdot d(x_0, x_1, a) \]
\[
< \frac{q^n}{1-q} \quad d\left( x_0, x_1, a \right)
\]

\( \to 0 \) as \( m, n \to \infty \)

Since \( q < 1 \)

It follows that \( \{ x_n \} \) is a Cauchy sequence,

Since \( X \) is complete, there exists a point \( u \in X \) such that \( x_n \to u \).

Now

\[
d(u, T u, a) \leq d(u, x_n, a) + d\left( x_n, T u, a \right)
\leq d(u, x_n, a) + \alpha_1 \frac{d(u, T u, a) \left[ 1 + d(x_{n-1}, x_n, a) \right]}{[1 + d(x_{n-1}, x_n, a)]}
+ \alpha_2 \frac{d(x_{n-1}, x_n, a) d(u, T u, a)}{d(x_{n-1}, u, a)}
+ \alpha_3 \frac{d(x_{n-1}, T u, a) d(u, x_n, a)}{d(x_{n-1}, u, a)}
+ \alpha_4 \left[ d(x_{n-1}, x_n, a) + d(u, T u, a) \right]
+ \alpha_5 \left[ d(x_{n-1}, T u, a) + d(u, x_n, a) \right]
+ \alpha_6 d\left( x_{n-1}, u, a \right)
\]

Taking the limit \( n \to \infty \) we have
\[ d(u, Tu, a) \leq (\alpha_4 + \alpha_5) d(u, Tu, a) \]

It follows that \( Tu = u \) as

Now for uniqueness, suppose \( z \) and \( u \) are two fixed point of \( T \) in \( X \) that is \( Tu = u \) and \( Tz = z \)

\[
d(z, u, a) = d(Tz, Tu, a)
\]

\[
\leq \alpha_1 \frac{d(u, Tu, a)}{1+d(z, u, a)} \frac{1+d(z, Tz, a)}{1+d(z, u, a)}
+ \alpha_2 \frac{d(z, Tz, a)}{d(z, u, a)} d(u, Tu, a)
+ \alpha_3 \frac{d(z, Tu, a)}{d(z, u, a)} d(u, Tz, a)
+ \alpha_4 \frac{[d(z, Tz, a) + d(u, Tu, a)]}{d(z, u, a)}
+ \alpha_5 \frac{[d(z, Tu, a) + d(u, Tz, a)]}{d(z, u, a)}
+ \alpha_6 d(z, u, a)
\]

Hence \( d(z, u, a) \leq (\alpha_3 + 2\alpha_5 + \alpha_6) d(z, u, a) \)

Since \( \alpha_3 + 2\alpha_5 + \alpha_6 < 1 \) therefore

It follows that \( u = z \)

So \( u \) is the unique fixed point of \( T \).

This complete the theorem.