4.0. Introduction:

In Chapter 3, we developed a method for evaluating a cinema plan in terms of reach, frequency and OTS distribution that the plan generated. The purpose of this chapter is to obtain a method for evaluation of a cinema plan when some of the assumptions that were used in the development of a method for evaluation of a plan are further relaxed.

In the development of model 1 in Chapter 3, it was necessary to make an assumption regarding the manner in which the total visits made by a person in the target audience, during the course of a year, would be spread amongst $52M$ potential theatre week combinations ($M$ denotes the number of candidate theatres). We had assumed that there would be no preference for any particular combination and that the $52M$ potential theatre weeks had an equal chance of being chosen. Many experts in this field have suggested to us that it would be desirable to relax
this assumption. It has been pointed out to us that in most of the situations, when there are only 8 to 10 theatres, this assumption poses no problem. In larger towns where there are a large number of theatres available this assumption may prove to be a little stringent. In a large town if the advertiser were to limit himself only to those theatres which are almost similar in every respect, then the list of the theatres might be too restrictive. In case the list of theatres is enlarged to ensure that almost all the visits of a person in the target audience will be within this group of theatres, then the assumption that all potential theatre weeks have an equal chance of being chosen is not likely to be valid. This is because, when a large number of theatres are under consideration, all of them are not similar in every respect. For example, theatres differ in terms of their seating capacity, the locality where they are situated and in terms of the amenities they offer such as parking facilities, etc. It was therefore thought desirable to relax the assumption that there would be no preference to theatre weeks by the members in the target audience.

During our discussion with experts it emerged that it would be possible to classify the list of candidate theatres in two or more classes according to the various
characteristics mentioned above. This classification would be so done that the theatres would be homogeneous within a class but vary substantially between classes. In the light of this classification it would be possible to say that there would be no preference for theatres within a class but there would be a preference between classes.

Considering the fact that this problem is of practical importance, we think it is necessary to develop a methodology for evaluating the reach, frequency and the OTS distribution of the plan when the assumption of equal preference is relaxed. It will be of practical interest to examine the impact of relaxing this assumption on reach, frequency and OTS distribution.

4.1. Assumptions of Model 1

All the assumptions stated for model 1 in Chapter 3 remain except for assumption 3. For ease of reading, however, we are stating the other assumptions that are made in the development of the model.

1. A person visits a given theatre at the most once a week.

2. It is possible to classify the candidate theatres into two categories. All the theatre week combinations
within each category have an equal chance of being selected.

3. A person's choice of a theatre in a given week will not affect his choice of another theatre in the same week.

In addition, for a person in the target audience, whatever may be the choice of theatre week combinations in the first r visits, each of the remaining theatre-week combinations has the same probability of being selected at (r+1) th visit.

4. The probability of a person in the target audience choosing a theatre week combination in one group is \( p \) and in another is \( q \). Further \( p \) is not equal to \( q \), and that the ratio of \( p \) and \( q \) is known. \( p \) and \( q \) are such that they satisfy the expression (4.1) explained in the next section.

5. The probability distribution of the number of visits to theatres made by a member of the target audience is known.

6. Without loss of generality, we can assume that the planning horizon is a year.
4.2. Notations:

Let there be two classes of theatres:

Let $M_1$ denote the number of candidate theatres in class I,

$M_2$ denote the number of candidate theatres in class II,

$i$ denote the number of visits to a theatre by a person in the target audience. Note that it is a random variable.

$i_1$ denote the total number of visits made by a person to a theatre belonging to class I. Obviously $i_1$ is a random variable.

$i_2 = i - i_1$ denote the number of visits made by a person to a theatre belonging to class II.

$f_i$ denote the probability of the random variable $i$.

Obviously $f_i \geq 0$, $\sum_{i=0}^{n} f_i = 1$, where $n$ is the maximum value the random variable $i$ takes.

$F(i_1, i_2 | i)$ denote the probability distribution of the random variable $i_1$ given $i$.

$T_i$ denote the number of theatres selected for screening the advertisement from class I theatres.
T_2 denote the number of theatres selected for screening the advertisement in class II theatres,

W_(1j) denote the number of weeks for which the advertisement is screened in the j th theatre of class I,

W_(2k) denote the number of weeks for which the advertisement is screened in the k th theatre of class II.

\[ W = \sum_{j=1}^{T_1} W_{1j} + \sum_{k=1}^{T_2} W_{2k}. \]

\[ W_1 = \sum_{j=1}^{T_1} W_{1j} \]

\[ W_2 = \sum_{k=1}^{T_2} W_{2k}. \]

Without loss of generality we can assume that p > q.

Strictly speaking we should have used the symbol 'i' to denote the total number of screening weeks selected by a person in the target audience. Similarly the variables i_1 and i_2 should have been used to denote the screening weeks selected by a person in theatre weeks belonging to class I and class II respectively. However by virtue of the assumption that a person visits a theatre at most once a week, it is possible to use the symbol to denote the total number of visits to theatres made by a person in the target audience.

By the same reasoning i_1 and i_2 can be also used to denote the number of visits made by a person to theatres in
class I and class II respectively. In the subsequent sections we denote $i_1$ and $i_2$ to mean the total number of visits to be made by a person in the target audience to the class I theatre week combinations and class II theatre week combinations.

4.3. Approach to the Problem:

Following the approach adopted in section 3.2.3, our aim will be to develop an expression for $P(t_1, t_2)$ where $P(t_1, t_2)$ denotes the probability that a person visits $t_1$'s and $t_2$'s specified theatre-week combinations in class I and class II respectively. Having obtained this expression, we will then obtain the expression for $S_t$ where

$$S_t = \sum_{t_1 + t_2 = t} P(t_1, t_2) = \text{Sum of all the probabilities that a person visits in } t \text{ specified theatre week combinations.}$$

Having obtained an expression for $S_t$, the expression for reach, frequency and OTS distribution can be obtained by standard formulae in theory of probability.\(^{(13)}\)
The difficulty in developing these expressions stems from the fact that the specified theatre week combinations do not belong to the same class of theatres.

To compute \( P(t_1, t_2) \) it is necessary to obtain the probability distribution of \( i_1 \) given that a person has made \( i \) visits. This distribution is not easy to obtain, since all screening weeks do not have an equal probability of being selected.

In the next section we therefore develop an expression for \( P(i_1, i_2 | i) \). Having obtained this expression, next we develop an expression for \( P(t_1, t_2) \) and then for \( S_t \). This is done in section 9.

4.4. Development of the Probability Distribution

\[
P(i_1, i_2 | i) = \]

The general approach for evolving the expression for obtaining \( P(i_1, i_2 | i) \) is developed with the help of some special cases. This will facilitate the understanding of the logic used in developing \( P(i_1, i_2 | i) \).

4.4.1. Special Case \( i = 1 \):

In this special case \( i = 1 \), we wish to compute \( P(i_1 = 1, i_2 = 0 | i = 1) \) and \( P(i_1 = 0, i_2 = 1 | i = 1) \).
In order to do this we will first compute 
\[ P'(i_1 = 1, i_2 = 0 \mid i = 1) \] and \[ P'(i_1 = 0, i_2 = 1 \mid i = 1) \]
where \( P'(i_1 = 1, i_2 = 0 \mid i = 1) \) denotes the probability 
that a person selects a specified theatre week combination 
belonging to class I given that he has made one visit.
\[ P'(i_1 = 0, i_2 = 1 \mid i = 1) \]
has the similar meaning.

By our assumption
\[ P'(i_1 = 1, i_2 = 0 \mid i = 1) = p \]
and \[ P'(i_1 = 0, i_2 = 1 \mid i = 1) = q. \]

Since one theatre week combination can be chosen in \( 52M_1 \) 
ways, we have
\[ P(i_1=1, i_2 = 0 \mid i = 1) = (52M_1).'(i_1=1, i_2 = 0 \mid i = 1) = 52M_1 p. \]

Similarly
\[ P(i_1=0, i_2 = 1 \mid i = 1) = (52M_2).P'(i_1=0, i_2 = 1 \mid i = 1) = 52M_2 q. \]

It can be verified that
\[ \sum_{i_1 + i_2 = 1} P(i_1, i_2 \mid i) = 52M_1 p + 52M_2 q = 1 \ldots (4.1) \]
4.4.2. Consider the Case $i = 2$:

We now wish to evaluate the following probabilities namely:

$$P(i_1 = 2, i_2 = 0 \mid i = 2)$$

$$P(i_1 = 1, i_2 = 1 \mid i = 2)$$

$$P(i_1 = 0, i_2 = 2 \mid i = 2)$$

**Evaluation of $P(i_1 = 2, i_2 = 0 \mid i = 2)$:**

We will first evaluate the probability that a person visits two specified theatre week combinations in class I say $T_{k1}$ and $T_{k2}$, given that he has made two visits. Let this event be denoted by $E_2$. Let $E_{21}$ denote the event that he first visits $T_{k1}$ and then visits $T_{k2}$ and $E_{22}$ denote the event that he first visits $T_{k2}$ and then visits $T_{k1}$. Clearly $E_2 = E_{21} \cup E_{22}$. Hence $P(E_2) = P(E_{21}) + P(E_{22})$. Since $E_{21} \cap E_{22} = \emptyset$,

$$P(E_{21}) \text{ by assumption (3) and (4) is } p \left( \frac{p}{1-p} \right).$$

Similarly $P(E_{22}) = p \left( \frac{p}{1-p} \right)$.
The above probability is for two specified visits. The two visits can be chosen from \(52M_1\) choices in \(\binom{52M_1}{2}\) ways.

Therefore \(P(i_1=2, i_2=0 | i=2) = 2! \cdot \binom{52M_1}{2} \frac{p^2}{1-p}\)

............ (4.2)

Evaluation of \(P(i_1 = 1, i_2 = 1 | i = 2):\)

We will first evaluate the probability that a person visits one specified theatre week combination in class I and one specified theatre week combination in class II say \(T_{k1}\) and \(T_{k2}\) given that he has made 2 visits. Let this event be denoted by \(E_{11}\).

Let \(E_{111}\) denote the event that he first visits \(T_{k1}\) and then visits \(T_{k2}\).

Let \(E_{112}\) denote the event that he first visits \(T_{k2}\) and then visits \(T_{k1}\). Then clearly

\[E_{11} = E_{111} \cup E_{112}\]

\[P(E_{11}) = P(E_{111}) + P(E_{112}) \quad \text{(Since } E_{112} \cap E_{111} = \emptyset).\]

By assumption (3) and (4)
\[ P(E_{111}) = p \frac{q}{1 - p} \quad \text{and} \quad P(E_{112}) = q \frac{p}{1 - q} \]

Therefore \( P(E_{11}) = \frac{p - q}{1 - p} + \frac{pq}{1 - q} \)

\( P(E_{11}) \) is the probability for a given specified theatre week combinations. One visit in class I (theatre week combinations) can be made in \( \binom{52M_1}{1} \) ways. Similarly one visit in class II (theatre week combinations) can be made in \( \binom{52M_2}{1} \) ways.

Therefore \( P(i_1 = 1, i_2 = 1, i = 2) = \)

\[ \binom{52M_1}{1} \binom{52M_2}{1} \frac{pq}{1 - p} + \binom{52M_2}{1} \binom{52M_1}{1} \frac{pq}{1 - q} \quad \ldots (6.3) \]

Evaluation of \( P(i_1 = 0, i_2 = 2 \mid i = 2) \):

Using the logic of \( P(i_1 = 2, i_2 = 0 \mid i = 2) \), this can be easily computed as

\[ 2 \cdot \binom{52M_2}{2} \frac{q^2}{1 - q} \]
4.5. Case of a General 11

In all the above illustrations the derivation of the expression (4.3) shows that to obtain an expression for \( P(i_1, i_2 \mid i) \) we need to obtain all the possible configurations in which \( i_1 \) and \( i_2 \) visits to the respective classes can be made. This is so because, as the expression (4.3) shows, the probability of a configuration depends on the order in which the visits have taken place.

The first step in our procedure to obtain \( P(i_1, i_2 \mid i) \) will be to generate all possible configurations in which \( i_1 \) visits to specified theatre week combinations in class I and \( i_2 \) visits to specified theatre week combinations in class II can take place. The sum of the probabilities of these individual configurations will be denoted by \( P'(i_1, i_2 \mid i) \). Then

\[
P(i_1, i_2 \mid i) = \binom{52M_1}{i_1} \binom{52M_2}{i_2} P'(i_1, i_2 \mid i).
\]

We now proceed to evaluate \( P'(i_1, i_2 \mid i) \). Let \( C(i_1, i_2 \mid i) \) denote the set of all possible configurations in which \( i_1 \) visits to the theatre week combinations in class I and \( i_2 \) visits to the theatre week combinations in
class II can take place. In these configurations we ignore the fact that the visits in each of the class I and class II are distinguishable. The configurations in $C(i_1, i_2 | i_1^*)$ will fall in any one of four mutually exclusive and collectively exhaustive categories listed below:

**Category I:**

$r_{11} \ r_{21} \ r_{12} \ r_{22} \ \ldots \ldots \ r_{2k-1} \ r_{1k}$

where $r_{1i}$ denotes the number of visits to theatres belonging to class I.

$r_{1i} > 0$ and $\sum_{i=1}^{k} r_{1i} = i_1$.

$r_{2j}$ denotes the number of visits to theatres belonging to class II.

$r_{2j} > 0$ and $\sum_{j=1}^{k} r_{2j} = i_2$.

$k$ is an integer such that $1 \leq k \leq i_1$, $1 \leq k \leq i_2$.

In this category the configurations begin with a run of visits to class I and end with a run of visit to class I.

---

Note that by we mean "In any ordered sequence of elements of two kinds each maximal subsequence of elements of like kind is called a run." (13)
The configurations of this category start with a run of visits to class I and end with a run of visits to class II.

The configurations of this category start with a run of visits to class II and end with a run of visits to class I.

The configurations of this category start and end with a run of visits to class II theatres.

It can be easily verified that the total number of configurations in categories 1, 2, 3, 4 are:

\[
\binom{i_1 - 1}{k - 1} \binom{i_2 - 1}{k - 2}, \quad \binom{i_1 - 1}{k - 1} \binom{i_2 - 1}{k - 1}, \quad \binom{i_1 - 1}{k - 1} \binom{i_2 - 1}{k - 1}
\]

and

\[
\binom{i_1 - 1}{k - 1} \binom{i_2 - 1}{k}
\]

respectively.
Therefore, the total number of elements in the set $C(i_1, i_2 | i)$ is $(\binom{i}{i_1})$. This can be easily seen by the following argument.

The total number of configurations for a given $i_1, i_2$ is

$$
\sum_{k=1}^{i_1} \left[ \binom{i_1-1}{k-1} \binom{i_2-1}{k-2} + 2 \binom{i_1-1}{k-1} \binom{i_2-1}{k-1} + \binom{i_2-1}{k} \right]
$$

$$
= \sum_{k=1}^{i_1} \binom{i_1-1}{k-1} \binom{i_2+1}{k}
$$

$$
= \sum_{k=0}^{i_1} \binom{i_1-1}{i_1-k} \binom{i_2+1}{k}. \text{ Since } \binom{i_1-1}{i_1-k} = 0 \text{ for } k = 0
$$

$$
= \binom{i_1+i_2}{i_1} = \binom{i}{i_1}
$$

Generation of Probability of a Typical Member in Different Categories:

1. Let $P(r_{11} r_{21} r_{12} r_{22} \ldots r_{2k-1} r_{1k})$ denote the
probability of a typical configuration in category 1.

Let \( P_1(\mathbf{i}_1, \mathbf{i}_2 | \mathbf{i}) = \sum_{k=1}^{\infty} \sum_{j \in A} \mathbf{P}(r_{11}, r_{21}, \ldots, r_{2k-1}, r_{1k}) \)

\[
A = \{ j \mid \sum_{j=1}^{k} r_{1j} = i_1, \quad r_{1j} > 0 \} 
\]

\[
B = \{ j \mid \sum_{j=1}^{k-1} r_{2j} = i_2, \quad r_{2j} > 0 \} 
\]

Thus \( P_1(\mathbf{i}_1, \mathbf{i}_2 | \mathbf{i}) \) is the sum of the probabilities of all the configurations of category 1.

Similarly let

\( P(r_{11}, r_{21}, r_{12}, r_{22}, \ldots, r_{2k-1}, r_{1k}, r_{2k}) \) denote the probability of a typical configuration in category 2.

Let \( P_2(\mathbf{i}_1, \mathbf{i}_2 | \mathbf{i}) = \sum_{k=1}^{\infty} \sum_{j \in A} \mathbf{P}(r_{11}, r_{21}, r_{12}, \ldots, r_{2k-1}, r_{1k}, r_{2k}) \)

\[
A = \{ j \mid \sum_{j=1}^{k} r_{1j} = i_1, \quad r_{1j} > 0 \} 
\]

\[
B = \{ j \mid \sum_{j=1}^{k} r_{2j} = i_2, \quad r_{2j} > 0 \} 
\]

Thus \( P_2(\mathbf{i}_1, \mathbf{i}_2 | \mathbf{i}) \) is the sum of all probabilities of the configurations falling in category 2.
Let $P(r_{21} r_{11} r_{22} r_{12} \ldots \ r_{2k-1} r_{1k-1})$ denote the probability of a typical configuration in category 3.

$$P_3(i_1, i_2 | i) = \sum_{k=1}^{i_1} \sum_{j \in A} \sum_{j \in B} P(r_{21} r_{11} \ldots r_{2k-1} r_{1k-1}).$$

$$A = \{ j \mid \sum_{j=1}^{k-1} r_{1j} = i_1, r_{1j} > 0 \}$$

$$B = \{ j \mid \sum_{j=1}^{k-1} r_{2j} = i_2, r_{2j} > 0 \}$$

Thus $P_3(i_1, i_2 | i)$ is the sum of all probabilities falling in the category 3.

Finally let $P(r_{21} r_{11} \ldots r_{1k} r_{2k+1})$ denote the probability of a typical configuration in category 4.

$$P_4(i_1, i_2 | i) = \sum_{k=1}^{i_1} \sum_{j \in A} \sum_{j \in B} P(r_{21} r_{11} \ldots r_{1k} r_{2k+1}).$$

$$A = \{ j \mid \sum_{j=1}^{k} r_{1j} = i_1, r_{1j} > 0 \}$$

$$B = \{ j \mid \sum_{j=1}^{k+1} r_{2j} = i_2, r_{2j} > 0 \}$$

Now $P'(i_1, i_2 | i) = i_1! i_2! \left[ P_1(i_1, i_2 | i) + P_2(i_1, i_2 | i) + P_3(i_1, i_2 | i) + P_4(i_1, i_2 | i) \right]$
and \( P(i_1, i_2 | i) = \binom{52M_1}{i_1} \binom{52M_2}{i_2} p'(i_1, i_2 | i) \).

To illustrate the above logic we now wish to evaluate the following probabilities, namely:

\[
\begin{align*}
P(i_1 = 3, i_2 = 0 | i = 3) \\
P(i_1 = 1, i_2 = 2 | i = 3) \\
P(i_1 = 2, i_2 = 1 | i = 3) \\
\text{and } P(i_1 = 0, i_2 = 3 | i = 3).
\end{align*}
\]

Evaluation of \( P(i_1 = 3, i_2 = 0 | i = 3) \):

This configuration falls in category 1. There is only one run of visits to class I. Therefore there is only one configuration.

That is \( P_1(i_1, i_2 | i) = p \left( \frac{p}{1-p} \right) \left( \frac{p}{1-2p} \right) \)

\[
P'(i_1, i_2 | i) = 3! \ p \left( \frac{p}{1-p} \right) \left( \frac{p}{1-2p} \right)
\]

and \( P(i_1, i_2 | i) = \binom{52M_1}{3} 3! p^3 \).
Evaluation of $P(i_1 = 1, i_2 = 2 \mid i = 3)$:

In this situation there will be configurations in category 2, 3, and 4. Since there is only one visit to class 1, there will not be any configuration in category 1.

In category 2, there will be only one configuration. Therefore

$$P_2(i_1, i_2 \mid i) = p \left( \frac{q}{1 - p} \right) \left( \frac{q}{1 - p - q} \right).$$

In category 3, there will be only one configuration

$$P_3(i_1, i_2 \mid i) = q \left( \frac{q}{1 - q} \right) \left( \frac{p}{1 - 2q} \right).$$

In category 4 also there will be only one configuration

$$P_4(i_1, i_2 \mid i) = q \left( \frac{p}{1 - q} \right) \left( \frac{q}{1 - p - q} \right).$$

Therefore

$$P'(i_1, i_2 \mid i) = 2 \cdot 1 \cdot [P_2(i_1, i_2 \mid i) + P_3(i_1, i_2 \mid i) + P_4(i_1, i_2 \mid i)].$$

and

$$P(i_1, i_2 \mid i) = \binom{52M_1}{1} \binom{52M_2}{2} P'(i_1, i_2 \mid i).$$
By using the same logic we can evaluate $P(i_1=2, i_2=1 \mid i = 3)$ and $P(i_1 = 0, i_2 = 3 \mid i = 3)$.

4.6. Computational Efforts in Computing $P(i_1, i_2 \mid i)$:

In principle the exact expression for $P(i_1, i_2 \mid i)$ can be obtained. It will be however impractical to use the exact expression. The reasons for this assertion can be easily demonstrated.

The conditional mass function $P(i_1, i_2 \mid i)$ has $2^i$ elements. This is because the number of elements is equal to $\sum_{i_1=0}^{i} \binom{i}{i_1} = 2^i$. Each element of $P(i_1, i_2 \mid i)$ contains a sum of $\binom{i}{i_1}$ expressions.

Even for moderately large values of $i$ the computational time required will not be negligible.

Assuming that each element of $P(i_1, i_2 \mid i)$ requires only one nanosecond ($10^{-9}$ seconds), which is negligible for fast computers, since there are $2^i$ such terms the total time required will be $(2^i) \binom{10^{-9}}{\text{nanosecond}}$. 

For a town frequently encountered in media planning, \( i \) is of the order of 100. Even assuming \( i = 50 \), the total computer time required will be of the order of 227 hours.\(^{19}\) This clearly shows that it will be highly uneconomical to spend so much amount on computers for a given value of \( i \). We have therefore developed an approximation for \( P(i_1, i_2 | i) \). The approximation that we have tried to develop is such that it substantially cuts down the computer time and yet gives a close approximation to the exact value of \( P(i_1, i_2 | i) \).

Fortunately this is possible because values of \( p \) and \( q \) are fairly small. From expression (4.1), we note that \( 52M_1p + 52M_2q = 1 \) and even for moderately small values of \( M_1 \) and \( M_2 \), \( p \) and \( q \) will be small in magnitude. Therefore we now develop the approximations.

\(^{19}\) \( 2^{50} \) is approximately equal to \( 10^{15} \).

\[ 10^{15} / 10^9 = 10^6 / 3600 = 227 \text{ hours}. \]
4.7. Preliminaries to the Approximations

4.7.1. The most probable and the least probable configurations in which $i_1$ visits and $i_2$ visits can take place:

To obtain this approximation we now state and prove a lemma regarding the most probable and the least probable configurations in the set $C(i_1, i_2 | i)$.

**Lemma 1:** The least probable configuration belongs to category 3 with $k = 1$, $r_{21} = i_2$ and $r_{11} = i_1$. The most probable configuration belongs to category 2 with $k = 1$, $p_{11} = i_1$ and $r_{21} = i_2$.

**Proof:** The least probable configuration is $r_{21} = i_2$, $r_{11} = i_1$ and $k = 1$. We denote the configuration by $C_L(i_2, i_1)$. To prove this we need to prove that the probability of any configuration belonging to $C(i_1, i_2 | i)$ is greater than or equal to the probability of this configuration. ($C_L(i_2, i_1)$).

Let $P_L = P[C_L(i_2, i_1)] = \frac{i_1 \cdot i_2}{D_1}$ ..... (4.4)

where $D_1 = (1-q)(1-2q) .... (1-i_2 q)(1-i_2 q-p)$;  $...(1-i_2 q - \sqrt{i_1 -1p})$. ........ (4.4a).
We now show that the probability of any configuration is greater than or equal to $P_{\infty}$. Let us consider a configuration from category 1.

$$P(r_{11} r_{21} \cdots r_{2k-1} r_{1k}) = \frac{i_1 i_2}{D_2} \quad \ldots \quad (4.5)$$

where $D_2 = (1-p) \cdots (1-r_{11} p)(1-r_{11} p-q) \cdots$

$$(1 - r_{11} p - r_{21} q) \quad \ldots \quad (4.5 \text{ a})$$

We note that the numerators of $(4.4)$ and $(4.5)$ are the same. The denominators of both the expressions contain $i_1 + i_2 - 1$ terms. To show that $P(r_{11} r_{21} \cdots r_{2k-1} r_{1k})$ is greater than or equal to $P_{\infty}$, it is sufficient to show that $D_1 \geq D_2$, since both the numerators are the same.

To show that $D_1 \geq D_2$, we will show that each term of $D_1$ (as arranged in expression $(4.4 \text{ a})$) is greater than or equal to corresponding term in $D_2$ (as arranged in expression $(4.5 \text{ a})$).
Before we prove this assertion we make a few observations on the nature of the terms in $D_1$ and $D_2$. Let us consider the $s$th term of both the expressions.

$s$th term in $D_1$: By examination of these terms, we see that $s$th term is of the form $1 - n_1p - n_2q - \delta_1p - \delta_2q$, where $n_1$ and $n_2$ are the number of visits to theatres in class I and class II that immediately precedes the $s$th visit.

$\delta_1 = 1$ if the visit immediately precedes the $s$th visit to class I

$\delta_1 = 0$ Otherwise, and

$\delta_2 = 1$ if the visit immediately precedes the $s$th visit to class II

$\delta_2 = 0$ Otherwise.

Thus if $s < i_2$, then

$n_1 = 0, \quad n_2 = s - 1 \quad \delta_1 = 0 \quad \text{and} \quad \delta_2 = 1$

Hence the $s$th term is $1 - (s - 1)q - q = 1 - sq \quad \ldots \ldots \ldots \ldots \ldots (4.6)$.

If $s > i_2$, then $n_2 = i_2, \quad n_1 = s - i_2 - 1$

$\delta_1 = 1, \quad \delta_2 = 0$

hence the $s$th term is $1 - (s - i_2 - 1)p - i_2q - p = 1 - (s - i_2)p - i_2q \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (4.7)$
s th Term in D2: The nature of the terms in D2 follows a similar pattern. By examination of the terms in D2, we see that the s th term has the same pattern

\[ 1 - n_1p - n_2q - \delta_1p - \delta_2q, \]

where \( n_1, n_2, \delta_1, \delta_2 \) notations have the same meaning.

However, the determination of \( \delta_1 \) and \( \delta_2 \), is not so simple as it was in the case of D1.

To determine the values of \( \delta_1 \) and \( \delta_2 \) it is necessary to know the pattern of runs of visits to class I and class II theatres before the s th visit. Here two cases can arise:

1. The first case is when the s th falls in the \((2\alpha + 1)\) th run. In other words 'a' runs of visits to class I and 'a' run of visits to class II are over and s th visit is in the \((2\alpha + 1)\) th run. Since in category 1, the run opens with a run of visits to class I, the \((2 \alpha + 1)\) th visit will consist of a visit to class I.

Now assume that 't-1' visits in \((2\alpha + 1)\) th run are over before the s th visit. In this situation,
The second case is the other possibility of the $s^{th}$ visit falling in the $(2a)$ th run. In this case, a run of visits to class I and $a-1$ run of visits to class II are over and we are in the $(2a)$ th run. Here again assume $t-1$ visits precede the $s^{th}$ visit. Since in this category the $(2a)$ th run will correspond to visits in class II, we have

$$n_1 = \sum_{j=1}^{a} r_{1j}$$

$$n_2 = \sum_{j=1}^{a-1} r_{2j} + t - 1,$$

$$\delta_1 = 0, \quad \text{and} \quad \delta_2 = 1.$$

Hence the $s^{th}$ term in $D_2$ is $(1-p \sum_{j=1}^{a} r_{1j} - q \sum_{j=1}^{a-1} r_{2j} - tq)$. 

\[\ldots\ldots(4.9)\]
Now we will show that in both of these circumstances the $s^{th}$ term of $D_1$ is greater than or equal to $s^{th}$ term in $D_2$.

Case I: $s > i_2$:

(a) Let $s = \sum_{j=1}^{a} r_{1j} + \sum_{j=1}^{a} r_{2j} + t$.

From (4.8) it follows that the $s^{th}$ term of $D_2$ is

$$1 - p \left( \sum_{j=1}^{a} r_{1j} + t \right) - q \left( \sum_{j=1}^{a} r_{2j} \right).$$

$$= 1 - p(s - \sum_{j=1}^{a} r_{2j}) - q \left( \sum_{j=1}^{a} r_{2j} \right)$$

$$= 1 - p[(i_2 - \sum_{j=1}^{a} r_{2j}) + (s - i_2)] - q \left( \sum_{j=1}^{a} r_{2j} \right)$$

$$= 1 - p[(i_2 - \sum_{j=1}^{a} r_{2j})] - (s - i_2)p - q \left( \sum_{j=1}^{a} r_{2j} \right)$$

$$< 1 - q[(i_2 - \sum_{j=1}^{a} r_{2j})] - (s - i_2)p - q \left( \sum_{j=1}^{a} r_{2j} \right)$$

$$= 1 - q[(i_2 - \sum_{j=1}^{a} r_{2j} + \sum_{j=1}^{a} r_{2j}) - (s - i_2)p$$
\[ l - q \, i_2 - (s - i_2)p = \text{th term of } D_1. \]

In this situation \( s \) th term of \( D_1 \) is larger than or equal to \( s \) th term of \( D_2 \).

(b) Let \( s = \alpha \sum_{j=1}^{\alpha-1} r_{1j} + \sum_{j=1}^{\alpha-1} r_{2j} + t \)

From expression (4.9) we know that \( s \) th term of \( D_2 \) is

\[ l - p \sum_{j=1}^{\alpha-1} r_{1j} - q \sum_{j=1}^{\alpha-1} r_{2j} - tq. \]

\[ = 1 - p\left[ s - \sum_{j=1}^{\alpha-1} r_{2j} - t \right] - q \left[ \sum_{j=1}^{\alpha-1} r_{2j} - t \right] - tq. \]

\[ = 1 - p\left[ s - i_2 + i_2 - \sum_{j=1}^{\alpha-1} r_{2j} - t \right] - q \left[ \sum_{j=1}^{\alpha-1} r_{2j} - t \right] - tq. \]

\[ = 1 - p(s - i_2) - p\left[ i_2 - \sum_{j=1}^{\alpha-1} r_{2j} - t \right] - q \left[ \sum_{j=1}^{\alpha-1} r_{2j} - t \right] - tq. \]

\[ \leq 1 - p(s - i_2) - q \left[ i_2 - \sum_{j=1}^{\alpha-1} r_{2j} - t \right] - q \left[ \sum_{j=1}^{\alpha-1} r_{2j} - t \right] - tq \]

\[ = 1 - p(s - i_2) - q \, i_2. \]
Here we see that the $s^{th}$ term of $D_1$ is greater than or equal to the $s^{th}$ term in $D_2$.

Case III: $s < i_2$

(a) Let $s = \sum_{j=1}^{\alpha} r_{1j} + \sum_{j=1}^{\alpha} r_{2j} + t$

From (4.8) we have

\[
l - p \sum_{j=1}^{\alpha} r_{1j} - q \sum_{j=1}^{\alpha} r_{2j} - tp
\]

\[
= l - p(\sum_{j=1}^{\alpha} r_{1j} + t) - q \sum_{j=1}^{\alpha} r_{2j}
\]

\[
\leq l - q(\sum_{j=1}^{\alpha} r_{1j} + t) - q(\sum_{j=1}^{\alpha} r_{2j})
\]

\[
= l - q \left[ \sum_{j=1}^{\alpha} r_{1j} + \sum_{j=1}^{\alpha} r_{2j} + t \right]
\]

\[
= l - q s \text{ which is the } s^{th} \text{ term of } D_1 \text{ when } s < i_2.
\]

(b) Let $s = \sum_{j=1}^{\alpha} r_{1j} + \sum_{j=1}^{\alpha-1} r_{2j} + t$.

From (4.9) we have

\[
l - p \sum_{j=1}^{\alpha} r_{1j} - q \sum_{j=1}^{\alpha-1} r_{2j} - tq
\]

\[
= l - p \sum_{j=1}^{\alpha} r_{1j} - q \left[ \sum_{j=1}^{\alpha-1} r_{2j} + t \right]
\]
\[ = 1 - p \sum_{j=1}^{s} r_{1j} - q \left[ s - \sum_{j=1}^{s} r_{1j} \right] \]

\[ \leq 1 - q \sum_{j=1}^{s} r_{1j} - qs + q \sum_{j=1}^{s} r_{1j} \]

\[ = 1 - sq, \text{ which is the } s^{th} \text{ term of } D_1. \text{ Thus in all cases } s^{th} \text{ term of } D_1 \text{ is greater or equal to the corresponding term in } D_2. \]

Hence \( P_{\text{L}} \) is the least probable configuration.

We do not have to give separate proofs for configurations belonging to other categories. This is because the \( s^{th} \) term is determined by the number of visits to class I and class II and the visit which immediately precedes the \( s^{th} \) visit. The nature of the \( s^{th} \) term therefore will remain the same in each of the categories. By a similar reasoning used for a configuration belonging to category 1, the result can be proved for all the configurations in other categories.

4.7.2. Most Probable Configuration:

To prove the second part of the lemma we follow a similar logic which is outlined below:

The most probable configuration is \( r_{11} = i_1 \), \( r_{21} = i_2 \) and \( k = 1 \).
To prove this we need to prove that the probability of any configuration belonging to $C(i_1, i_2 | i)$ is less than or equal to the probability of this configuration namely $[ C_u (i_1, i_2) ]$.

$$P_u = P[C_u(i_1, i_2)] = \frac{p_{i_1} q_{i_2}}{D_3} \quad \text{...... (4.10)}$$

where $D_3 = (1-p)(1-2p) \ldots (1-i_1p)(1-i_2p-q)$

$$\ldots \ldots (1-i_1p - i_2 - i_1q) \quad \ldots \quad (4.10 \text{ a})$$

We now show that the probability of any configuration is less than or equal to $P_u$. Let us consider the configuration from category 1, which is taken to prove the first part of the lemma.

Once again we note that numerators of (4.5) and (4.10) are the same. Therefore we have to show that each term of $D_3$ (as arranged in 4.10 a) is less than or equal to the corresponding terms (as arranged in 4.5a) of $D_2$.

$s$ th Term of $D_3$:

Using the same logic as done in the case of $D_1$,

when $s < i_1$ the $s$ th term of $D_3$ is $(1 - sp)$ and when $s > i_1$ the $s$ th term of $D_3$ is of the form $1-pi_1 - (s - i_1)^q$. 
As done in the earlier ease, we show that under all the conditions the $s$-th term of $D_3$ is less than or equal to the corresponding $s$-th term in $D_2$.

Case I: $s > i_1$:

(a) Let $s = \sum_{j=1}^{a} r_{1j} + \sum_{j=1}^{a} r_{2j} + t$

From expression (4.8) we know that $s$-th term in $D_2$ is

$$1 - p \sum_{j=1}^{a} r_{1j} - q \sum_{j=1}^{a} r_{2j} - tp$$

$$= 1 - p \sum_{j=1}^{a} r_{1j} - q \left[ s - \sum_{j=1}^{a} r_{1j} - t \right] - tp$$

$$= 1 - p \sum_{j=1}^{a} r_{1j} - q \left[ s - i_1 + i_1 - \sum_{j=1}^{a} r_{1j} - t \right] - tp$$

$$= 1 - p \sum_{j=1}^{a} r_{1j} - q(s - i_1) - q[i_1 - \sum_{j=1}^{a} r_{1j} - t] - tp$$

$$\geq 1 - p \sum_{j=1}^{a} r_{1j} - q(s - i_1) - p[i_1 - \sum_{j=1}^{a} r_{1j} - t] - tp$$

$$= 1 - q(s - i_1) - p i_1.$$
Similarly when \( s = \sum_{j=1}^{a} r_{1j} + \sum_{j=1}^{a-1} r_{2j} + t \)

\( s \) th term of \( D_{2} \gtrsim 1 - p_{1\perp} - (s - i_{1\perp}) q_{1} \).

This is shown below:

Let \( s = \sum_{j=1}^{a} r_{1j} + \sum_{j=1}^{a-1} r_{2j} + t \)

From expression (4.9) the \( s \) th term of \( D_{2} \) is

\[
1 - p \sum_{j=1}^{a} r_{1j} - q \sum_{j=1}^{a-1} r_{2j} + t q
\]

\[
= 1 - p \sum_{j=1}^{a} r_{1j} - q (s - \sum_{j=1}^{a} r_{1j} - t) - t q
\]

\[
= 1 - p \sum_{j=1}^{a} r_{1j} - q (s - i_{1\perp} + i_{1\perp} - \sum_{j=1}^{a} r_{1j} - t) - t q
\]

\[
\gtrsim 1 - p \sum_{j=1}^{a} r_{1j} - q(s - i_{1\perp}) - p_{1\perp} - p \sum_{j=1}^{a} r_{1j}
\]

\[
= 1 - q(s - i_{1\perp}) - p i_{1\perp}.
\]
Case II: $s < i_1$

(a) Let $s = \sum_{j=1}^{a} r_{1j} + \sum_{j=1}^{c} r_{2j} + t$

From expression (4.8), the $s$th term of $D_2$ is

$$l - p \sum_{j=1}^{a} r_{1j} - q \sum_{j=1}^{c} r_{2j} - tp.$$  

$$= l - p \left[ \sum_{j=1}^{a} r_{1j} + t \right] - q \sum_{j=1}^{c} r_{2j}$$

$$\geq l - p \left[ \sum_{j=1}^{a} r_{1j} + t \right] - p \sum_{j=1}^{a} r_{2j}$$

$$= l - p \left[ \sum_{j=1}^{a} r_{1j} + t + \sum_{j=1}^{a} r_{2j} \right]$$

$$= l - sp. \quad \text{This is the $s$th term of the $D_3$.}$$

Similarly when $s = \sum_{j=1}^{a} r_{1j} + \sum_{j=1}^{a-1} r_{2j} + t$

we can show the result. Hence the result.
4.7.3. Corollary

To show that the upper bound (probability of the most probable configuration) and the lower bound (probability of the least probable configuration) are equal if \( p = q \) and the probability of each of the configurations in \( \mathcal{C}(i_1, i_2 | i) \) is equal to \( \frac{1}{\binom{52M}{i}} \).

Symbolically \( P_{\text{L}} = P_{\text{U}} = \frac{1}{\binom{52M}{i}} \) if \( p = q \).

We know

\[
P_{\text{L}} = \frac{i_1 i_2}{(1-q)(1-2q)\ldots(1-i_2q)(1-i_2q-p)\ldots(1-i_2q-1-i_2p)}
\]

and

\[
P_{\text{U}} = \frac{i_1 i_2}{(1-p)(1-2p)\ldots(1-i_1p)(1-i_1p-q)\ldots(1-i_1p-1-i_2q)}
\]

From (4.1) we know that \( 52M_1 p + 52M_2 q = 1 \).

Since \( p = q \), we have \( q(52M_1 + 52M_2) = 1 \).

\[
q = \frac{1}{52(M_1 + M_2)} = \frac{1}{52 M} \quad \ldots \quad (4.11)
\]
Since \( p = q \), \( \mathbf{P}_L = \mathbf{P}_U \)

\[
q^i = \frac{q^i}{(1-q)(1-2q)(1-i_2q) \ldots (1-i_2q - 1 - 1 q)} \quad \ldots (4.12)
\]

Replacing 1 by \( 52 M q \) from (4.11) we have

\[
q^i = \frac{q^i}{(52Mq-q)(52Mq-2q) \ldots (52Mq - i_2q - 1 - I q)}
\]

\[
q^i = \frac{q^i}{q^{i-1} (52M-1) (52M-2) \ldots (52M-i_2 - 1 - 1)}
\]

\[
q^i = \frac{q}{(52M-1) (52M - 2) \ldots (52M - i_2 - 1 - 1)}
\]

\[
q^i = \frac{1}{52M (52M-1) \ldots (52M - i_2 - 1 - 1)}
\]

\[
q^i = \frac{1}{(52 \cdot M)^i} = \frac{1}{i! \left( \begin{array}{c} 52M \\ i \end{array} \right)}
\]

From this corollary, we can deduce the result of model 1. Each configuration of the set \( C(i_1, i_2 \mid i) \) has
the probability \( \frac{1}{i!(\binom{52M}{i})} \). Since there are \( \binom{i}{i_1} \)

configuration in the set \( C(i_1, i_2 \mid 1) \)

\[
P'(i_1, i_2 \mid 1) = \binom{i}{i_1} \frac{i_1! \cdot i_2!}{i! (\binom{52M}{i})}
\]

\[
P(i_1, i_2 \mid 1) = \binom{52M_1}{i_1} \binom{52M_2}{i_2} P'(i_1, i_2 \mid 1)
\]

\[
= \frac{\binom{52M_1}{i_1} \binom{52M_2}{i_2}}{i_1! (\binom{52M}{i})} \cdot i_1! \cdot i_2! \binom{i}{i_1}
\]

\[
= \frac{\binom{52M}{i_1} \binom{52M}{i_2}}{(\binom{52M}{i})} \text{ which is the}
\]
probability of $i_1$ visits in class I and $i_2$ visits in class II theatres, given that $i$ visits are made when $p = q$.

4.8. Approximation to $P(i_1, i_2 | i)$:

We now turn to our main task of finding an approximation to $P(i_1, i_2 | i)$. In this task the closeness of bounds developed in section 4.7 is of vital interest to us.

Unfortunately this question cannot be answered analytically. We therefore evaluated these bounds for different values of $p$, $q$, $i_1$ and $i_2$ (given in appendix III). For each of these cases we found that upper and lower bounds did not differ more than $10^{-9}$. As the computational experience shows that these bounds are close to each other, we decided to approximate the probabilities of each of these configurations by $\frac{L + U}{2}$ where

$$L = \frac{i_1}{p} \frac{i_2}{q} \frac{1}{(1-q)(1-2q)\ldots(1-i_2q)(1-i_2q-p)\ldots(1-i_2q-\frac{i_1-1}{p})}$$

$$U = \frac{i_1}{p} \frac{i_2}{q} \frac{1}{(1-p)(1-2p)\ldots(1-i_1p)(1-i_1p-q)\ldots(1-i_1p - \frac{i_2-1}{q})}$$
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Hence \( P'(i_1, i_2 \mid i) = \frac{1}{i_1 + \frac{U}{2}} \left( \frac{l}{i_1} \right) \left( \frac{i}{i_1} \right) \).

This is so because in each of the configurations visits to respective classes are distinguishable and there are totally \( \left( \begin{array}{c} i \\ i_1 \end{array} \right) \) such configurations.

Hence \( P(i_1, i_2 \mid i) = \left( \begin{array}{c} 52M_1 \\ i_1 \end{array} \right) \left( \begin{array}{c} 52M_2 \\ i_2 \end{array} \right) P'(i_1, i_2 \mid i) \).

To approximate \( P(i_1, i_2 \mid i) \) we make use of the fact that \( p \) and \( q \) are small but \( 52M_1 p \) and \( 52M_2 q \) are not small. Therefore we can simplify \( P(i_1, i_2 \mid i) \) further.

This is accomplished by expanding the denominator of \( \frac{L}{i \cdot U} \) by using Binomial Theorem and ignoring second and higher powers of \( p \) and \( q \).

\[
P(i_1, i_2 \mid i) = \left( \begin{array}{c} 52M_1 \\ i_1 \end{array} \right) \left( \begin{array}{c} 52M_2 \\ i_2 \end{array} \right) i_1! \cdot i_2! \cdot p^{i_1} \cdot q^{i_2}
\]

\[
\frac{1}{2} \left\{ \frac{1}{(1-q)(1-2q) \ldots (1-i_2q)(1-i_2q-p) \ldots (1-i_2q - i_1-1p)} + \frac{1}{(1-p)(1-2p) \ldots (1-i_1p)(1-i_1p-q) \ldots (1-i_1p - i_2-1q)} \right\}
\]
Consider

\[
\frac{1}{(1-q)(1-2q)\ldots(1-i_2q)(1-i_2q-p)\ldots(1-i_2q-i_1-1p)}
\]

1\textsuperscript{2}

\[
(1+q)(1+2q)\ldots(1+i_2q)(1+i_2q+p)\ldots(1+i_2q+i_1-1p)
\]

1\textsuperscript{2}

\[
[l+q(l+2q+\ldots+i_2)] + [l+(i_2q+p+\ldots+i_2q+i_1-1p)]
\]

1\textsuperscript{2}

\[
[l+q(l+2q+\ldots+i_2)+(i_2q+p+\ldots+i_2q+i_1-1p)]
\]

\[
= 1 + \frac{q\, i_2\,(i_2+1)}{2} + i_2q\,(i_1-1) + \frac{p\, i_1}{2}\,(i_1-1)
\]

\[
= 1 + q\, i_2\left(\frac{i_2+1+2i_1-2}{2}\right) + p\left(\frac{i_1\,(i_1-1)}{2}\right)
\]

\[
= 1 + q\, i_2\left[\frac{i + i_1-1}{2}\right] + p\, \left(\frac{i_1\,(i_1-1)}{2}\right)
\]

\[
= 1 + q\, \frac{i_2\, i}{2} + \frac{(i_1\!-\!1)}{2} (qi_2 + pi_1).
\]

Similarly consider

\[
\frac{1}{(1-p)(1-2p)\ldots(1-i_1p)(1-i_1p-q)\ldots(1-i_1p-i_2-1q)}
\]
We see that \( P(1, i_2 | i) \) is a legitimate mass function.

Because:
(1) \( P(i_1, i_2 \mid i) \) is a non-negative quantity for all values of \( i, i_1 \) and \( i_2 \).

(2) \[
\sum_{i_1+1_2=i} P(i_1, i_2 \mid i) = \sum_{i_1=0}^{1} P(i_1, i_2 \mid i) = 1.
\]

The multiplier \( K \) in the expression \((4.14)\) is so determined that

\[
\frac{1}{\sum_{i_1=0}^{1} P(i_1, i_2 \mid i)} = K \left( \sum_{i_1=0}^{1} \binom{1}{i_1} \left( \frac{52M_1}{i_1} \right) \left( \frac{52M_2}{i_2} \right) \right)
\]

\[
= K \cdot \frac{1}{K} = 1.
\]

Therefore we have the expression \((4.15)\) for \( K \).
4.9. Development of $S_t$:

As mentioned in section 4.3, we are interested in developing an expression for $S_t$.

In doing so, we use the same logic as in the case of model 1 in Chapter 3.

Let $A_{j_1, j_2, \ldots, j_t}$ be the event that a person in the target audience visits $j_1, j_2, \ldots, j_t$ specified theatre week combinations. The $t$ specified visits can be made by visiting $t_1$ specified theatre week combinations in class I and $t_2$ specified theatre week combinations in class II. Note that $t = t_1 + t_2$.

$$S_t = \sum_{j_1 < j_2 < \ldots < j_t} P_{j_1, j_2, \ldots, j_t}$$

$$= \sum P(t_1, t_2) \quad (4.16),$$

where $P(t_1, t_2)$ is the probability of $t_1$ specified visits in class I and $t_2$ specified visits in class II. Now we are interested in calculating $P(t_1, t_2)$.

$$P(t_1, t_2) = \sum_{i \geq t} P(t_1, t_2 | i) f_i \quad (4.17).$$
The 'i' visits to theatre week combinations can be made by making \( i_1 \) visits to theatre week combinations in class I and \( i_2 \) visits to theatre week combinations in class II. Obviously \( i = i_1 + i_2 \).

Next our interest is in calculating \( P(t_1, t_2 | i) \).

Let \( A_{i_1} \) denote the event a member in the target audience visits \( t_1 \) specified theatre week combinations in class I and \( B_{i_1} \) denote the event a person in the target audience visits \( i_1 \) theatre week combinations. Clearly we are interested in

\[
P(\bigcup_{i_1} A_{i_1} \cap B_{i_1}) = \sum_{i_1 = t_1}^{i-t+t_1} P(A_{i_1} \cap B_{i_1})
\]

Note that the summation with respect to \( i_1 \) starts from \( t_1 \), when \( t_1 \) specified visits are to be made in class I and \( t_2 = t - t_1 \) specified visits are to be made in class II.

It is imperative that \( i_1 \), the number of visits made to class I, has to be greater than or equal to \( t_1 \) and \( i_2 \) (\( = i - i_1 \)) has to be greater than or equal to \( t_2 \) (\( = t - t_1 \)).

That is \( i_1 \geq t_1 \) and \( i-t+t_1 \geq i_1 \).
We note that

\[
P'(i_1, i_2 | i) = P'((t_1, t_2, i_1, i_2 | i) = \binom{i_1}{i_1} \binom{i_2}{i_2} \left[ 1 + (p_{i_1} + q_{i_2}) \left( \frac{i - 1}{2} \right) \right] \ast K
\]

\[\ldots\ldots(4.18)\]

where

\[
K = \frac{1}{\sum_{i_1=0}^{1} [1+(i_1 p+i_2 q) \left( \frac{i-1}{2} \right)] \binom{52M1}{i_1} \binom{52M2}{i_2} p_{i_1} q_{i_2} \binom{i}{i_1} \binom{i}{i_2} i_1 i_2 !}
\]

\[\ldots\ldots(4.19)\]

Given that a member of the target audience makes \( i \) visits,

\( t_1 \) visits can be made in \( \binom{52M - t_1}{i_1 - t_1} \) ways, and similarly

\( t_2 \) visits can be made in \( \binom{52M - t_2}{i_2 - t_2} \) ways. These are

mutually exclusive and each of these ways has a probability of \( P'(t_1, t_2, i_1, i_2 | i) \). Therefore
\[ P(A_t \cap B_{t_1}) = \binom{52M_1-t_1}{i_1-t_1} \binom{52M_2-t_2}{i_2-t_2} \cdot p_{i_1} q_{i_2} \]

\[ \cdot \frac{i_1! \cdot i_2!}{i_1!} \left[ 1 + (p_{i_1} + q_{i_2}) \left( \frac{i_1 - 1}{2} \right) \right] \cdot K \]

Therefore

\[ P(t_1 \cdot t_2 | i) = \sum_{i_1 = t_1}^{i-t+t_1} P(i_1 \cdot i_2 \cdot t_1 \cdot t_2 | i). \]

\[ = \sum_{i_1 = t_1}^{i-t+t_1} \binom{52M_1-t_1}{i_1-t_1} \binom{52M_2-t_2}{i_2-t_2} \cdot p_{i_1} q_{i_2} \]

\[ \cdot \frac{i_1! \cdot i_2!}{i_1!} \left[ 1 + (p_{i_1} + q_{i_2}) \left( \frac{i_1 - 1}{2} \right) \right] \cdot K \]

where \( K \) is given by the expression (4.19).

Hence

\[ P(t_1 \cdot t_2) = \sum_{i \geq t} \sum_{i_1 = t_1}^{i-t+t_1} \binom{52M_1-t_1}{i_1-t_1} \binom{52M_2-t_2}{i_2-t_2} \cdot p_{i_1} q_{i_2} \]

\[ \cdot \frac{i_1! \cdot i_2!}{i_1!} \left[ 1 + (p_{i_1} + q_{i_2}) \left( \frac{i_1 - 1}{2} \right) \right] \cdot K \cdot f_1 \]

\[ \ldots (4.20) \]
The advertiser has chosen $W_1$ screening weeks from class I and $W_2$ screening weeks from class II.

The $t_1$ specified theatre weeks can be chosen in $\binom{W_1}{t_1}$ ways and $t_2$ specified theatre week can be chosen in $\binom{W_2}{t_2}$ ways. The number of terms in the expression \((4.20)\) will be $\sum_{t=t_1+t_2}^{t_1+t_2} (\binom{W_1}{t_1})(\binom{W_2}{t_2})$ where $\binom{W_1}{t_1}(\binom{W_2}{t_2})$ terms will have the same probability of $P(t_1, t_2)$. Therefore

$$S_t = \sum_{t=t_1+t_2}^{t_1+t_2} P(t_1, t_2) \binom{W_1}{t_1}\binom{W_2}{t_2}.$$  

Hence

$$S_t = \sum_{t_1+t_2=t} \sum_{i_1+t_1} P_{i_1, i_2} \binom{52M_1-t_1}{i_1-t_1} \binom{52M_2-t_2}{i_2-t_2}$$

$$= \frac{1}{i_2} \binom{i_1}{i} \binom{1}{\frac{i-1}{2}} [1 + (p_{1i} + q_{1i})] \binom{W_1}{t_1}\binom{W_2}{t_2} \ast K.$$
Having generated $S_t$, using the logic used in Chapter 3, we have

$$\text{Reach} = S_1 - S_2 + \ldots + S_N$$

and

$$P[m] = S_m - \binom{m+1}{m} S_{m+1} + \binom{m+2}{m} S_{m+2} + \ldots + \binom{N}{m} S_N,$$

where $P[m]$ is the probability of exactly $m$ exposure.

Using this logic we illustrate some problems in Chapter 6.