Chapter 3

Standard Model Scenario of Mixing and Rare Processes

3.1 Introduction

The neutral meson mixing, such as \( K^0 - \bar{K}^0 \), \( B^0_d - \bar{B}^0_d \), \( B^0_s - \bar{B}^0_s \) and \( D^0 - \bar{D}^0 \) play a very crucial role to test the SM. In 1955 Gell-Mann and Pais proposed the neutral kaon mixing for the first time [45]. Later in 1964, CP violation, one of the most pioneering discovery in particle physics was observed for the first time in the neutral kaon system [1]. This triumph continued through the observation of the \( B^0_d - \bar{B}^0_d \) mixing in 1986 [46], \( B^0_s - \bar{B}^0_s \) mixing in 2006 [47] and \( D^0 - \bar{D}^0 \) mixing in 2007 [48]. Thanks to the two B factories and CDF, D0 collaborations, many important constraints on the CKM parameters are now known.

On the other hand, the neutral meson mixing provides an ideal place to explore new physics beyond the SM. The mixing is caused by flavor-changing neutral current (FCNC) transitions and only occurs via loops in the framework of the SM. The dominant contribution to the mixing comes from the box diagrams. That is why these box diagrams can be very sensitive to the new physics effects.

In the next chapter, we discuss a NP model related to leptoquarks. In that chapter, an analysis of the neutral \( K \) and \( B \) mesons mixing is done in the presence of this NP model. In chapter 4 constraints on the NP are given for the mixing correlated leptonic and semileptonic decays of neutral \( K \) and \( B \).
mesons. Before presenting the NP model scenario, in this chapter we briefly summarize the status of the these neutral mesons mixing and their leptonic and semileptonic decays within the SM. The standard reference for these analysis is [49]. For ready references, in this chapter we have reproduced the SM scenario of the relevant mixing and decay analysis mainly from [49].

3.2 Neutral-Meson Mixing

We denote the neutral meson by \( P^0 \) and its anti-meson by \( \bar{P}^0 \). \( P^0 \) represents any one of the \( K^0, D^0, B^0_d, B^0_s \). \( P^0 \) and \( \bar{P}^0 \) can oscillate between themselves before decaying. In the Wigner-Weisskopf approximation, the two component wave function of an oscillating and decaying beam, in its rest frame can be written as

\[
|\psi(t)\rangle = \psi_1(t)|P^0\rangle + \psi_2(t)|\bar{P}^0\rangle,
\]

(3.1)

where \( t \) is the proper time. The Schrodinger equation for this wave function can be written as

\[
\frac{id}{dt} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}
\]

(3.2)

As the neutral-mesons both oscillate and decay, the matrix \( R \) is not Hermitian. It can be written as

\[
R = M - \frac{i}{2} \Gamma,
\]

(3.3)

with

\[
M = M^\dagger, \quad \Gamma = \Gamma^\dagger.
\]

(3.4)

(3.5)

\( M \) and \( \Gamma \) are associated with \((P^0, \bar{P}^0) \leftrightarrow (\bar{P}^0, P^0)\) transitions via off-shell (dispersive), and on-shell (absorptive) intermediate states, respectively. Diagonal elements of \( M \) and \( \Gamma \) are associated with the flavor-conserving transitions \( P^0 \rightarrow P^0 \) and \( \bar{P}^0 \rightarrow \bar{P}^0 \), while off-diagonal elements are associated with the flavor-changing transitions \( P^0 \rightarrow \bar{P}^0 \). If the two eigenstates of \( R \) are denoted be \( P_H \) and \( P_L \), the mixing parameters can be defined as,

\[
\Delta m \equiv m_H - m_L,
\]

(3.6)

\[
\Delta \Gamma \equiv \Gamma_H - \Gamma_L.
\]

(3.7)
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where \( m_H \) and \( m_L \) correspond to the masses of \( P_H \) and \( P_L \) respectively. \( \Gamma_H, \Gamma_L \) correspond to the decay widths of \( P_H \) and \( P_L \) respectively. The convention is that the mass difference \( \Delta m \) is always positive. The sign of \( \Delta \Gamma \) is fixed through experiments. The mixing in the \( K^0 - \overline{K^0}, B_s^0 - \overline{B_s^0} \) and \( B_s^0 - \overline{B_s^0} \) sectors is discussed below.

3.2.1 Mixing in Neutral Kaon system

\( K^0 \) and \( \overline{K^0} \) are the flavour eigenstates in the SM. Following the convention \((CP|K^0) = |\overline{K^0}\rangle\), the CP eigenstates \( K_1 \) and \( K_2 \) are defined as [49]

\[
K_1 = \frac{1}{\sqrt{2}}(K^0 + \overline{K^0}), \quad CP|K_1\rangle = |K_1\rangle, \quad (3.8)
\]

\[
K_2 = \frac{1}{\sqrt{2}}(K^0 - \overline{K^0}), \quad CP|K_2\rangle = -|K_2\rangle. \quad (3.9)
\]

The physical states \( K_L \) and \( K_S \) are the admixtures of \( K_1 \) and \( K_2 \),

\[
K_S = \frac{K_1 + \bar{\varepsilon}K_2}{\sqrt{1 + |\bar{\varepsilon}|^2}}, \quad K_L = \frac{K_2 + \varepsilon K_1}{\sqrt{1 + |\varepsilon|^2}}. \quad (3.10)
\]

The parameter \( \varepsilon \) is very small. It is not a physical parameter as it depends on the the phase convention chosen for \( K^0 \) and \( \overline{K^0} \).

Two pion final states are CP even state and three pion final states are CP odd state. As \( K_L \) and \( K_S \) both consist of \( K_1 \) and \( K_2 \), they decay to \( 3\pi \) via \( K_2 \) and \( 2\pi \) via \( K_1 \) component. The physical parameter \( \varepsilon_K \) is the measurement of the “indirect CP violation”. It is defined as

\[
\varepsilon_K = \frac{A(K_L \to (\pi\pi)_{I=0})}{A(K_S \to (\pi\pi)_{I=0})}, \quad (3.11)
\]

It can also be written as,

\[
\varepsilon_K = \frac{\exp(i\pi/4)}{\sqrt{2}\Delta M_K} (\text{Im}M_{12} + 2\xi\text{Re}M_{12}), \quad (3.12)
\]

where

\[
\xi = \frac{\text{Im}A_0}{\text{Re}A_0} \quad (3.13)
\]
with $A_0 \equiv A(K \rightarrow (\pi\pi)_{I=0})$ and $\Delta M_K$ denoting the $K_L - K_S$ mass difference. The off-diagonal element $M_{12}$ in the neutral $K$-meson mass matrix represents $K^0(\bar{s}d) - \bar{K}^0(sd)$ mixing. It is given by

$$2m_K M_{12} = \langle \bar{K}^0 | H_{\text{eff}}(\Delta S = 2) | K^0 \rangle,$$  

where $H_{\text{eff}}(\Delta S = 2)$ is the effective Hamiltonian for the $\Delta S = 2$ transitions and $m_K$ is the $K$-meson mass.

To lowest order these transitions are induced through the box diagrams shown in Fig. (3.1). Including QCD corrections, the effective low energy Hamiltonian, to be derived from these diagrams, can be written as [50]:

$$H_{\text{eff}}^{\Delta S=2} = \frac{G_F^2}{16\pi^2} M_W^2 \left[ \lambda_c^2 \eta_1 S_0(x_c) + \lambda_t^2 \eta_2 S_0(x_t) + 2\lambda_c \lambda_t \eta_3 S_0(x_c, x_t) \right] \times$$

$$\times \left[ \alpha_s^{(3)}(\mu) \right]^{-2/9} \left[ 1 + \frac{\alpha_s^{(3)}(\mu)}{4\pi} J_3 \right] Q(\Delta S = 2) + h.c. \quad (3.15)$$

where $\lambda_i = V_{is}^* V_{id}$. Using unitarity relation $\lambda_u + \lambda_c + \lambda_t = 0$, $\lambda_u$ is replaced in terms of $\lambda_c$ and $\lambda_t$. Eq. (3.15) is valid for scales $\mu$ below the charm threshold $\mu_c = \mathcal{O}(m_c)$. In this case $H_{\text{eff}}^{\Delta S=2}$ consists of a single four-quark operator

$$Q(\Delta S = 2) = (\bar{s}d)_{V-A}(\bar{d}s)_{V-A}, \quad (3.16)$$

Functions like $S_0(x_i)$ where $i = u, c, t$ and $S_0(x_c, x_t)$ are the basic loop contributions from the box diagrams without QCD correction. The expressions
3.2 Neutral-Meson Mixing

for these are as follows:

\[
S_0(x_i) = \frac{4x_i - 11x_i^2 + x_i^3}{4(1 - x_i)^2} - \frac{3x_i^2 \ln x_i}{2(1 - x_i)^3}, \quad x_i = \frac{m_i^2}{M_W^2},
\]

(3.17)

\[
S_0(x_c, x_t) = x_c \left[ \ln \frac{x_c}{x_t} - \frac{3x_t}{4(1 - x_t)} - \frac{3x_t^2 \ln x_t}{4(1 - x_t)^2} \right],
\]

(3.18)

Short-distance QCD effects are included in the correction factors \(\eta_1, \eta_2, \eta_3\) and in the explicitly \(\alpha_s\)-dependent terms in Eq. (3.15). The scale dependence and renormalization scheme dependence of \(\alpha_s(\mu)\) and \(J_3\) should cancel with the scale dependence and renormalization scheme dependence of the hadronic matrix element. In the NDR scheme \(J_3 = 1.895\). The NLO values of the QCD factors \(\eta_1, \eta_2\) and \(\eta_3\) are given as follows [51], [52]:

\[
\eta_1 = 1.38 \pm 0.20, \quad \eta_2 = 0.57 \pm 0.01, \quad \eta_3 = 0.47 \pm 0.04.
\]

(3.19)

The renormalization group invariant parameter \(B_K\) can be defined as

\[
B_K = B_K(\mu) \left[ \frac{\alpha_s^{(3)}(\mu)}{4\pi} \right]^{-2/9} \left[ 1 + \frac{\alpha_s^{(3)}(\mu)}{4\pi} J_3 \right]
\]

(3.20)

\[
(\bar{K}^0|(\bar{s}d)_{V-A} (\bar{s}d)_{V-A}|K^0) \equiv \frac{8}{3} B_K(\mu) F_K^2 m_K^2
\]

(3.21)

We have used the value of \(B_K = 0.86 \pm 0.14 \pm 0.06\) given in [15] in our calculation. Using Eq. (3.15) one finds

\[
M_{12} = \frac{G_F^2}{12\pi^2} F_K^2 B_K m_K M_W^2 \left[ \lambda_c^2 \eta_1 S_0(x_c) + \lambda_t^2 \eta_2 S_0(x_t) + 2\lambda_c^* \lambda_t \eta_3 S_0(x_c, x_t) \right],
\]

(3.22)

where \(F_K = 159.8 \pm 1.4 \pm 0.44\) [54] is the \(K\)-meson decay constant.

The last term in Eq. (3.12) can be neglected as compared to other uncertainties for example \(B_K\), as it constitutes at most a 2% correction to \(\varepsilon_K\). Substituting Eq. (3.22) into Eq. (3.12), it can be written as

\[
\varepsilon_K = C_v B_K \text{Im} \lambda_t \{ \text{Re} \lambda_c [\eta_1 S_0(x_c) - \eta_3 S_0(x_c, x_t)] - \text{Re} \lambda_t \eta_2 S_0(x_t) \} \exp(i\pi/4),
\]

(3.23)

where the unitarity relation \(\text{Im} \lambda_c^* = \text{Im} \lambda_t\) is used and \(\text{Re} \lambda_t/\text{Re} \lambda_c = \mathcal{O}(\lambda^4)\) is
Figure 3.2: Box diagram of neutral B-meson mixing.

neglected in the evaluation of $\text{Im}(\lambda^*_c \lambda^*_t)$. The numerical constant $C_\varepsilon$ is given by

$$C_\varepsilon = \frac{G_F^2 F_K^2 m_K M_W^2}{6\sqrt{2}\pi^2 \Delta M_K} = 3.78 \times 10^4.$$  \hspace{1cm} (3.24)

The value of measured $\Delta M_K$ is $(5.31 \pm 0.01) \times 10^{-3} \text{ps}^{-1}$ [54].

3.2.2 Mixing in Neutral B-meson

The strength of $B^0_q - \overline{B}^0_q$ mixing, where $q = d, s$ is described by

$$\Delta M_q = 2|M_{12}^{(q)}|,$$  \hspace{1cm} (3.25)

the mass difference between the mass eigenstates in the $B^0_d - \overline{B}^0_d$ system and the $B^0_s - \overline{B}^0_s$ system, respectively. In this case the off-diagonal term $M_{12}$ of the neutral B-meson mass matrix is given by

$$2m_{B_q} |M_{12}^{(q)}| = |\langle B^0_q |H_{\text{eff}}(\Delta B = 2)|B^0_q \rangle|.$$  \hspace{1cm} (3.26)

These mixings are induced by the box diagrams shown in fig. (3.2). The effective Hamiltonian, valid for the scales $\mu_b = \mathcal{O}(m_b)$, can be written in the case of $B^0_d - \overline{B}^0_d$ mixing as

$$H_{\text{eff}}^{\Delta B = 2} = \frac{G_F^2}{16\pi^2} M_W^2 (V_{tb}^* V_{td})^2 \eta_{B} S_0(x_t) \left[ \alpha_s^{(5)}(\mu_b) \right]^{-6/23} \left[ 1 + \frac{\alpha_s^{(5)}(\mu_b)}{4\pi} J_5 \right] \times$$

$$\times Q(\Delta B = 2) + h.c.$$  \hspace{1cm} (3.27)
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Here

\[ Q(\Delta B = 2) = (\bar{b}d)_{V-A}(\bar{b}d)_{V-A} \]  

(3.28)

and \( \eta_B = 0.55 \pm 0.01 \) [52] is the short distance QCD correction factor. \( J_5 = 1.627 \) in the NDR scheme. In the case of \( B_s^0 - \bar{B}_s^0 \) mixing one should simply replace \( d \rightarrow s \) in Eq. (3.27) and Eq. (3.28) with all other quantities unchanged. Due to CKM suppression, for neutral B-meson box diagrams, the charm quark and charm-top quark contribution terms are negligible compared to the top quark contribution. The \( B_{B_q} \) term is defined as,

\[ B_{B_q} = B_{B_q}(\mu) \left[ \alpha_s^{(5)}(\mu) \right]^{-6/23} \left[ 1 + \frac{\alpha_s^{(5)}(\mu)}{4\pi} J_5 \right] \]  

(3.29)

and

\[ \langle \bar{B}_q^0 |(\bar{b}q)_{V-A}(\bar{b}q)_{V-A}|B_q^0 \rangle = \frac{8}{3} B_{B_q}(\mu) F_{B_q}^2 m_{B_q}^2, \]  

(3.30)

where \( F_{B_q} \) is the \( B_q \)-meson decay constant. Using Eq. (3.27) one finds

\[ \Delta M_q = \frac{G_F^2}{6\pi^2} \eta_B m_{B_q} (B_{B_q} F_{B_q}^2) M_W^2 S_0(x_t) |V_{tq}|^2, \]  

(3.31)

which implies

\[ \Delta M_d = 0.50/\text{ps} \times \left[ \sqrt{\frac{B_{B_d} F_{B_d}}{200 \text{ MeV}}} \right]^2 \left[ \frac{m_t(m_t)}{170 \text{ GeV}} \right]^{1.52} \left[ \frac{|V_{td}|}{8.8 \times 10^{-3}} \right] \left[ \frac{\eta_B}{0.55} \right], \]  

(3.32)

and

\[ \Delta M_s = 15.1/\text{ps} \times \left[ \sqrt{\frac{B_{B_s} F_{B_s}}{240 \text{ MeV}}} \right]^2 \left[ \frac{m_t(m_t)}{170 \text{ GeV}} \right]^{1.52} \left[ \frac{|V_{ts}|}{0.040} \right] \left[ \frac{\eta_B}{0.55} \right]. \]  

(3.33)

For our calculation, we have used [15], [53],

\[ F_{B_d} \sqrt{B_{B_d}} = (0.228 \pm 0.033) \text{ GeV}, \]  

(3.34)

\[ F_{B_s} \sqrt{B_{B_s}}_{\text{LQCD}} = (0.245 \pm 0.021^{+0.003}_{-0.002}) \text{ GeV}. \]  

(3.35)
3.3 Neutral Meson Mixing Correlated decay

Various leptonic and semileptonic decay couplings of neutral $K$ and $B$ mesons are related to the couplings of the $K^0 - \bar{K}^0$, $B^0_q - \bar{B}^0_q$ mixing respectively. Bounds can be obtained for the same CKM matrix elements from both mixing and correlated decays. In this section we discuss about the SM scenario of those correlated leptonic and semileptonic decays.

3.3.1 Neutral Kaon decay

- $K_L \to l^+l^-$

The decay $K_L \to l^+l^-$, where $l = e, \mu$, proceeds through loop diagrams. In the SM, the dominant contributions to this decay come from the $W$ box and $Z$ penguin diagrams. In addition, it receives long distance contributions from the two-photon intermediate states, which are difficult to calculate reliably. But the SM predicted terms are one order off from the NP terms. That is why we have neglected these terms in our calculation. At next-to-leading order, the effective Hamiltonian for $K_L \to l^+l^-$ can be written as

$$H_{\text{eff}} = -\frac{G_F}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2 \Theta_W} (V^*_{cs} V_{cd} Y_{NL} + V^*_{ts} V_{td} Y(x_t))(5d)_{V-A}(\bar{l}l)_{V-A} + h.c.$$  \hspace{1cm} (3.36)

The function $Y(x_t)$ is given by $Y(x_t) \approx 1.03Y_0(x)$, where

$$Y_0(x) = \frac{x}{8} \left[ \frac{x - 4}{x - 1} + \frac{3x}{(x - 1)^2} \log(x) \right]$$  \hspace{1cm} (3.37)

The renormalized group (RG) expression $Y_{NL}$ represents the charm contribution. It has two parts, one coming from the $Z$ penguin and the other coming from the box diagrams. The detail expressions are given in [55].

- $K_L \to \pi^0 e^+e^-$

The effective Hamiltonian for $K_L \to \pi^0 e^+e^-$ at scales $\mu < m_c$ is given
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as follows:

\[ \mathcal{H}_{\text{eff}}(K_L \to \pi^0e^+e^-) = \frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} \left[ \sum_{i=1}^{6,7} \left[ z_i(\mu) + \tau y_i(\mu) \right] Q_i + \tau y_{7A}(M_W) Q_{7A} \right], \]  

where the operators \( Q_i \) are given explicitly as follows:

**Current–Current :**

\[ Q_1 = (\bar{s}_\alpha u_\beta)V_{-A} (\bar{u}_\beta d_\alpha)V_{-A}, \]  
\[ Q_2 = (\bar{s}u)V_{-A} (\bar{u}d)V_{-A}. \]

**QCD–Penguins :**

\[ Q_3 = (\bar{s}d)V_{-A} \sum_{q=u,d,s} (\bar{q}q)V_{-A}, \]  
\[ Q_4 = (\bar{s}_\alpha d_\beta)V_{-A} \sum_{q=u,d,s} (\bar{q}_\beta g_\alpha)V_{-A}, \]  
\[ Q_5 = (\bar{s}d)V_{-A} \sum_{q=u,d,s} (\bar{q}q)V_{+A}, \]  
\[ Q_6 = (\bar{s}_\alpha d_\beta)V_{-A} \sum_{q=u,d,s} (\bar{q}_\beta g_\alpha)V_{+A}. \]

**Electroweak–Penguins :**

\[ Q_7 = \frac{3}{2} (\bar{s}d)V_{-A} \sum_{q=u,d,s} e_q (\bar{q}q)V_{+A}, \]  
\[ Q_8 = \frac{3}{2} (\bar{s}_\alpha d_\beta)V_{-A} \sum_{q=u,d,s} e_{q_\beta}(\bar{q}_\beta g_\alpha)V_{+A}, \]  
\[ Q_9 = \frac{3}{2} (\bar{s}d)V_{-A} \sum_{q=u,d,s} e_q (\bar{q}q)V_{-A}, \]  
\[ Q_{10} = \frac{3}{2} (\bar{s}_\alpha d_\beta)V_{-A} \sum_{q=u,d,s} e_{q_\beta}(\bar{q}_\beta g_\alpha)V_{-A}. \]
and

\[ Q_{7V} = (\bar{s} d)_{V-A} (\bar{\tau} e)_V, \]
\[ Q_{7A} = (\bar{s} d)_{V-A} (\bar{\tau} e)_A. \]

Here, \( e_q \) denotes the electrical quark charges reflecting the electroweak origin of \( Q_7, \ldots, Q_{10} \) and \( \alpha, \beta \) are the colour indices. \( V(A) \) stands for the Lorentz structure \( \gamma^\mu (\gamma^\mu \gamma^5) \).

The Wilson coefficient (WC) functions \( z_i(\mu) \) and \( y_i(\mu) \) were calculated including the complete next-to-leading order (NLO) corrections in [56, 57, 58]. The details of these calculations can be found there and in the review [50]. These WCs describe the strength with which a given operator enters the Hamiltonian. The WCs are controlled by the renormalization group equations, and their values at a high energy scale (typically \( M_W \)) is supplied. They include all the perturbative corrections to the operators in question. The nonperturbative part comes in evaluating the matrix elements of the operators \( Q_i \) between initial and final states. The regularization scale \( \mu \) is an arbitrary point (of the order \( m_c \)) that separates the high-energy perturbative corrections and the low-energy nonperturbative contributions. The final result, theoretically, should not depend on \( \mu \).

Three different type of contributions: CP conserving, indirectly CP violating and directly CP violating type can contribute in \( K_L^0 \to \pi^0 l^+ l^- \). The estimation of the CP conserving part is very difficult as it can only be done outside the perturbative framework. The SM estimations give:

\[ Br(K_L \to \pi^0 e^+ e^-)_{cons} \approx \begin{cases} 
(0.3 - 1.8) \times 10^{-12} & [59] \\
4.0 \times 10^{-12} & [60] \\
(5 \pm 5) \times 10^{-12} & [61]. 
\end{cases} \]

The SM estimation of indirectly CP violating branching ratio [62], [63] is

\[ Br(K_L \to \pi^0 e^+ e^-)_{indir} \leq 1.6 \times 10^{-12}, \]
and the directly CP violating branching ratio [49] is

\[
Br(K_L \rightarrow \pi^0 e^+ e^-)_{\text{dir}} = \left\{ \begin{array}{ll}
(4.5 \pm 2.6) \times 10^{-12} & \text{Scanning} \\
(4.2 \pm 1.4) \times 10^{-12} & \text{Gaussian},
\end{array} \right.
\] (3.53)

3.3.2 \( B^0_q \) MESON DECAY

The B meson decay is controlled by an effective Hamiltonian of the form

\[
\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} \sum_i V_{CKM}^i C_i(\mu) O_i(\mu),
\] (3.54)

where \( O_i \) are the relevant local operators which govern the decays in question. \( V_{CKM}^i \) represents the CKM factors. Below we show six classes of operators which play the dominant role in the phenomenology of weak decays and mixing. We assume the charged current decay \( b \rightarrow c \) of a \( B \) meson. The subscripts 1 and 8 denote whether the currents are in singlet-singlet or octet-octet combination of colour SU(3).

**Current-Current:**

\[
\begin{align*}
O_1 &= (\bar{\tau}b)_{8,V-A} (\bar{s}c)_{8,V-A}, \\
O_2 &= (\bar{\tau}b)_{1,V-A} (\bar{s}c)_{1,V-A}.
\end{align*}
\] (3.55) (3.56)

Only a typical combination \( \bar{s}c \) is shown; there may be other combinations.

**QCD Penguins:**

\[
\begin{align*}
O_{3(4)} &= (\bar{s}b)_{1(8),V-A} \sum_q (\bar{q}q)_{1(8),V-A}, \\
O_{5(6)} &= (\bar{s}b)_{1(8),V-A} \sum_q (\bar{q}q)_{1(8),V+A}.
\end{align*}
\] (3.57) (3.58)

The sum runs over all the lighter flavours \( (u, d, s, c) \).
Electroweak Penguins:

\[ O_{7(8)} = \frac{3}{2} (\pi b)_{1(8),V-A} \sum_q e_q \ (\bar{q}q)_{1(8),V+A}, \quad (3.59) \]

\[ O_{9(10)} = \frac{3}{2} (\pi b)_{1(8),V-A} \sum_q e_q (\bar{q}q)_{1(8),V-A}. \quad (3.60) \]

Magnetic Penguins:

\[ O_{7\gamma} = \frac{e}{8\pi^2} m_b \tilde{\sigma}^{\mu\nu} (1 + \gamma_5) b F_{\mu\nu}, \quad (3.61) \]

\[ O_{8G} = \frac{g}{8\pi^2} m_b \tilde{\sigma}^{\mu\nu} (1 + \gamma_5) T^a_{\alpha\beta} b_\beta G_{\mu\nu}^a. \quad (3.62) \]

Here \( \alpha \) and \( \beta \) are colour indices and \( T^a \) are the SU(3) generators.

Semileptonic Operators:

\[ O_{9V} = (\bar{d}b)_{1,V-A}(\bar{e}e)_V, \quad (3.63) \]

\[ O_{10A} = (\bar{d}b)_{1,V-A}(\bar{e}e)_A \quad (3.64) \]

These operators also contribute to the leptonic decays. Again, this basis is for the SM only.

- **Leptonic decays** \( B^0_q \rightarrow l^+l^- \)

The decay \( B_q \rightarrow l^+l^- \), where \( q = d \) or \( s \) and \( l = e, \mu \) or \( \tau \), proceeds through loop diagrams. In the SM, the dominant contribution to this decay comes from the \( W \) box and \( Z \) penguin diagrams. A significant contribution to this decay is made by the top quark in the loop. At low energies (of order \( m_b \)), the decay can be described by a local \( (\bar{b}q)(\bar{t}l) \) coupling. These kind of couplings can appear through the effective Hamiltonian which is similar to the one given in Eq. (3.54). The branching fraction is given by
\[ Br(B^0_q \rightarrow l^+ l^-) = \frac{G_F^2}{8\pi} f_{B_q}^2 \tau_{B_q} m_{B_q}^3 \sqrt{(1 - \frac{4m^2_l}{m_{B_q}^2})|C_{P}^q|^2 - \frac{2m_l}{m_{B_q}} C_{A}^q|C_{P}^q|^2} \]

+ \left( 1 - \frac{4m^2_l}{m_{B_q}^2} \right) |C_{P}^{q'}|^2 \] (3.65)

In the SM \( C_{P}^{q'} \) and \( C_{P}^q \) arise from penguin diagram with physical and non-physical neutral scalar exchange, and are suppressed by a factor \((m_b/M_W)^2\). The decay rate is controlled by the coefficient

\[ [C_{A}^q]_{SM} = \frac{\alpha V_{tb} V_{tq}^*}{\sqrt{8\pi} \sin^2 \theta_w} Y(x_t) \] (3.66)

where \( \sin^2 \theta_w \) is the weak mixing angle. The expression for \( Y(x_t) \) is given in Eq. (3.37). For different lepton flavour the SM branching fractions are [64]

\[
\begin{align*}
Br(B^0_d \rightarrow e^+ e^-) & \approx \mathcal{O}(10^{-14}), \\
Br(B^0_d \rightarrow \mu^+ \mu^-) & \approx \mathcal{O}(10^{-10}), \\
Br(B^0_d \rightarrow \tau^+ \tau^-) & \approx \mathcal{O}(10^{-8}), \\
Br(B^0_s \rightarrow e^+ e^-) & \approx \mathcal{O}(10^{-13}), \\
Br(B^0_s \rightarrow \mu^+ \mu^-) & \approx \mathcal{O}(10^{-9}), \\
Br(B^0_s \rightarrow \tau^+ \tau^-) & \approx \mathcal{O}(10^{-7}).
\end{align*}
\] (3.67)

These numbers show that purely leptonic decays are too rare to be observed unless they are significantly enhanced by new physics.

- **Semileptonic decays**

The semileptonic inclusive decays \( B \rightarrow X_{s,d} l^+ l^- \), originating from the parton level process \( b \rightarrow s(d) l^+ l^- \), can be calculated using the effective Hamiltonian formalism. The amplitude reads
\[ A(B \to X_{s}l^{+}l^{-}) = \frac{\sqrt{2}G_F}{\pi} V_{tb}V_{ts}^{*} [C_{7}^{\text{eff}} s_{L}^{\gamma} \gamma^{\mu} b_{L} \not{l} \gamma_{\mu} + C_{9}^{\text{eff}} s_{L}^{\gamma} \gamma^{\mu} b_{L} \not{l} \gamma_{5} \gamma_{\mu} l \nonumber \\
-2C_{7}^{\text{eff}} m_{b}s_{L}^{\gamma} \gamma^{\mu} q_{\mu} q_{2}^{\gamma} b_{R} \not{l} \gamma_{\mu} l], \quad (3.68) \]

where \( q^{2} \) is the momentum transferred to the lepton pair. In addition to the RG evolutions of \( C_{7}^{\text{eff}} \) and \( C_{9}^{\text{eff}} \) at the weak scale, the WCs \( C_{7}^{\text{eff}} \) and \( C_{9}^{\text{eff}} \) contain, the mixing effects with operators \( O_{1-6} \) (for \( C_{9}^{\text{eff}} \)) and \( O_{2} \) and \( O_{8} \) (for \( C_{7}^{\text{eff}} \)); hence the superscript. There is also a sizeable long-distance contribution coming from \( B \to K(\ast) \psi \) and \( \psi \to l^{+}l^{-} \), where \( \psi \) is a generic vector \( c\bar{c} \) state.

For the semileptonic decays, we use

\[ \langle N(p_{2}) | \not{s} \gamma^\mu b | M(p_{1}) \rangle = P^\mu F_{1}(q^{2}) + q^\mu \frac{m_{B}^{2} - m_{K}^{2}}{q^{2}} (F_{0}(q^{2}) - F_{1}(q^{2})) , \]

\[ \langle \phi(p_{2}, \epsilon) | V_{\mu} = A_{\mu} | B_{s}(p_{1}) \rangle = \frac{1}{m_{B_{s}} + m_{\phi}} [-iV(q^{2}) \delta_{\mu\alpha} \beta \epsilon^{\star \nu} P^{\alpha} q^{\beta} \nonumber \\
\pm A_{0}(q^{2})(P \cdot q) \epsilon_{\mu}^* \pm A_{\pm}(q^{2})(\epsilon^{\star} \cdot p_{1}) P_{\mu} \nonumber \\
\pm A_{-}(q^{2})(\epsilon^{\star} \cdot p_{1}) q_{\mu}] \quad (3.69) \]

where \( M \) may be \( B_{d} \) or \( B_{s} \) and \( N \) may be \( \pi^{0} \) or \( K^{0} \). The \( m_{B_{s}} \) and \( m_{\phi} \) are the meson masses, \( p_{1}(p_{2}) \) is the momentum of the initial (final) meson, \( \epsilon \) is the polarization vector of the vector meson \( \phi \), \( P = p_{1} + p_{2} \), \( q = p_{1} - p_{2} \), \( V_{\mu} = \overline{q}_{2} \gamma_{\mu} q_{1} \), \( A_{\mu} = \overline{q}_{2} \gamma_{\mu} \gamma_{5} q_{1} \). \( V, A_{0, \pm} \) and \( F_{0,1}(q^{2}) \) are the form factors. The values of these form factors are taken from [65], [66].