Chapter 2

Parametrization Independent Studies of Neutral B Meson Decays

2.1 Introduction

The two B Factories PEP-II and KEK-B were designed to have peak luminosities $3 \times 10^{33}$ cm$^{-2}$s$^{-1}$ and $1 \times 10^{34}$ cm$^{-2}$s$^{-1}$ respectively. PEP-II, however, reached design luminosity in a remarkably short time, and before shutting down, it exceeded its design performance by a factor of three. KEK-B, with a more ambitious design objective, has also exceeded its design performance, and currently operates at even higher luminosity. The accumulation of all these events allow precision measurements of exclusive B meson decays. These measurements indicate subtle discrepancies between some experimental data and theoretical predictions within the standard model, though at present error bars are still large to come to any concrete conclusion. These discrepancies are quite puzzling and it is difficult to ignore them.

One of such puzzles involve the weak phase $\beta \equiv \arg(-V_{cd}^*V_{cb}/V_{td}^*V_{tb})$ which is defined via the CKM matrix element $V_{td} = |V_{td}|e^{i\beta}$. This phase can be extracted either from the tree-dominated $b \rightarrow c\ell\nu$, e.g. $B \rightarrow J/\psi K_S$ or penguin-dominated $b \rightarrow sq\bar{q}$, e.g. $B \rightarrow \phi K_S$ modes. The two determinations should be same in the SM, but would differ, if new physics contributions modify the penguin dominated decay amplitudes. For several years a large deviation $\Delta S \equiv S_{sq\bar{q}} - S_{c\ell\nu}$ has been measured where $S_i$ has been defined in
Eq. (2.24) in Section (2.3.1). Several studies have been done to estimate the penguin pollution in the $b \rightarrow c \bar{s}s$ trees and the tree pollution in the $b \rightarrow sq\bar{q}$ penguins using various QCD based models and SU(3) based models. However, these studies [27], [28], [29], [30] are unable to produce the observed effect. For most of the cases, these studies indicate that the sign of the discrepancy within SM is opposite to the observed value. It has become one of the most challenging puzzle in B-physics to provide convincing arguments regarding the nature of this discrepancy and whether it can be regarded as an unambiguous signal of NP. In this chapter we have tried to answer these questions.

First we discuss about the parametrization of the most general amplitude for $b \rightarrow q$ transition modes where $q$ is either $d$ or $s$ quark, then we present a quick review about the progress of the SM estimation of these modes. Finally we discuss in details about our own method to try to solve this puzzle.

### 2.2 Parametrizing the amplitude

The most general amplitude for $\bar{b} \rightarrow \bar{q}$ transition modes where $q$ is either $d$ or $s$ quark may be written as [31],

$$A_{\bar{b} \rightarrow \bar{q}} = A_{uq}e^{i\delta_{uq}}v_{uq} + A_{cq}e^{i\delta_{cq}}v_{cq} + A_{tq}e^{i\delta_{tq}}v_{tq},$$

(2.1)

where $v_{jq} = V_{jq}^*V_{jq}$ with $j = u, c$ or $t$ are the product of CKM matrix elements and $A_{jq}$ and $\delta_{jq}$ are the amplitudes and strong phases associated with the CKM factor $v_{jq}$. The unitarity property of CKM matrix gives us a relation $v_{uq} + v_{cq} + v_{tq} = 0$. Using this property we can eliminate any one of the $v_{jq}$ from Eq. (2.1) and express it in terms of only two independent contributions having different weak phases.

In SM, within the framework of Wolfenstein parametrization [20] the various $v_{jq}$ are expressed up to order $\mathcal{O}(\lambda^6)$ as follows:

$$v_{cs} = A\lambda^2 \left(1 - \frac{1}{2}\lambda^2\right),$$

$$v_{cd} = -A\lambda^3,$$

$$v_{us} = A\lambda^4(\rho + i\eta),$$

(2.2)
2.2 Parametrizing the amplitude

\[ v_{ud} = A\lambda^3(\rho + i\eta) \left( 1 - \frac{\lambda^2}{2} \right), \]  
\[ v_{ts} = -A\lambda^2 \left( 1 - \left( \frac{1}{2} - \rho - i\eta \right)\lambda^2 \right), \]  
\[ v_{td} = A\lambda^3 \left( 1 - \left( 1 - \frac{\lambda^2}{2} \right)(\rho + i\eta) \right). \]  

(2.3)

In the above parametrization it is clear that \( v_{cs} \) and \( v_{cd} \) are real at least to order \( \mathcal{O}(\lambda^6) \). The weak phase arising from \( v_{us} \) and \( v_{ud} \) are represented by the familiar unitary triangle angle \( \gamma \), where

\[ \gamma \equiv \arg \left[ -\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} \right] \approx 60^\circ. \]  

(2.4)

The weak phase of \( v_{td} \) is the well known phase \( \beta \), where

\[ \beta \equiv \arg \left[ -\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*} \right] = (21.1 \pm 0.9)^\circ. \]  

(2.5)

The weak phase of \( v_{ts} \) is represented by \( \beta_s \), where

\[ \beta_s \equiv \arg \left[ -\frac{V_{ts}V_{tb}^*}{V_{cs}V_{cb}^*} \right] = 1.045^\circ + 0.061^\circ - 0.057^\circ. \]  

(2.6)

Since \( v_{cq} \) is almost real, the amplitude \( A^{\bar{b} \rightarrow \bar{q}} \) may be rewritten in terms of only one non-zero weak phase, by eliminating either \( v_{uq} \) or \( v_{tq} \) using the unitarity condition. This results in a choice of two different ways to parametrize the \( \bar{b} \rightarrow \bar{q} \) amplitude.

The elimination of \( v_{tq} \) leads to the amplitude being expressed in terms of the weak phase \( \gamma \) independent of \( q \), since \( v_{us} \) and \( v_{ud} \) have the same weak phase. We can write,

\[ A^{\bar{b} \rightarrow \bar{q}} = (A_{cq} e^{i\delta_{cq}} - A_{tq} e^{i\delta_{tq}})v_{cq} + (A_{uq} e^{i\delta_{uq}} - A_{tq} e^{i\delta_{tq}})v_{uq}. \]  

(2.7)

The amplitude may then be re-expressed as follows:

\[ A^{\bar{b} \rightarrow \bar{q}} = e^{i\Theta} \left[ a_q'^* e^{i\gamma} + b_q'^{*} e^{i\gamma} \right], \]  

(2.8)
\[ a'_q = |v_{cq}| \hat{a}'_q = |v_{cq}| |A_{cq}e^{i\delta_{cq}} - A_{tq}e^{i\delta_{tq}}|, \quad (2.9) \]
\[ b'_q = |v_{uq}| \hat{b}'_q = |v_{uq}| |A_{uq}e^{i\delta_{uq}} - A_{tq}e^{i\delta_{tq}}|, \quad (2.10) \]

Here, \( \Theta'_q \) is an overall strong phase which cannot be detected experimentally; we hence set it to be zero. This fact will become clear once we write down the observables in the next section (Sec. 2.3) and find that no observable depends on \( \Theta'_q \). \( \delta'_q \) is the relative strong phase difference between \( a'_q \) and \( b'_q \). At this stage we emphasize an essential difference between our parametrization for \( q = s \) or \( q = d \): the negative sign of \( v_{cd} \) is absorbed into the definition of \( \delta'_s \) for convenience.

Similarly the elimination of \( v_{uq} \) leads us to parametrization, the details of which depend on whether \( q = s \) or \( q = d \). We first consider \( q = s \), this results in the \( \beta_s \) parametrization. Here we can write,

\[ A^{5\to\pi} = (A_{cs}e^{i\delta_{cs}} - A_{us}e^{i\delta_{us}})v_{cs} + (A_{ts}e^{i\delta_{ts}} - A_{us}e^{i\delta_{us}})v_{ts}. \quad (2.11) \]

This amplitude may also be re-expressed as follows:

\[ A^{5\to\pi} = e^{i\Theta'_s} [a'_s + b'_se^{i\delta'_s}e^{i\beta_s}], \quad (2.12) \]
\[ a'_s = |v_{cs}| \hat{a}'_s = |v_{cs}| |A_{cs}e^{i\delta_{cs}} - A_{us}e^{i\delta_{us}}|, \quad (2.13) \]
\[ b'_s = |v_{ts}| \hat{b}'_s = |v_{ts}| |A_{ts}e^{i\delta_{ts}} - A_{us}e^{i\delta_{us}}|. \quad (2.14) \]

We will set \( \Theta'_s = 0 \) for the same reasons we set \( \Theta'_q = 0 \) above. \( \delta'_s \) is the relative strong phase difference between \( a'_s \) and \( b'_s \). Note that a negative sign is originating from \( v_{ts} \) has been absorbed in the definition of \( \delta'_s \) for convenience. Interestingly, one may also note that the magnitudes of \( \hat{b}'_s \) and \( \hat{b}'_q \) are same.

The elimination of \( v_{uq} \) for \( q = d \) results in the amplitude:

\[ A^{5\to\pi} = (A_{cd}e^{i\delta_{cd}} - A_{ud}e^{i\delta_{ud}})v_{cd} + (A_{td}e^{i\delta_{td}} - A_{ud}e^{i\delta_{ud}})v_{td}, \quad (2.15) \]

which is re-expressed as,

\[ A^{5\to\pi} = e^{i\Theta'_d} [a'_d + b'_de^{i\delta'_d}e^{i\beta}], \quad (2.16) \]
\[ a'_d = |v_{cd}| \hat{a}'_d = |v_{cd}| |A_{cd}e^{i\delta_{cd}} - A_{ud}e^{i\delta_{ud}}|, \quad (2.17) \]
\[ b'_d = |v_{td}| \hat{b}'_d = |v_{td}| |A_{td}e^{i\delta_{td}} - A_{ud}e^{i\delta_{ud}}|. \quad (2.18) \]
2.3 Observables and Variables

We have thus demonstrated how the amplitudes for \( b \to s \) or \( b \to d \) may be expressed as a sum of two contributions, one with zero weak phase and the other with a chosen weak phase that is either \( \beta_s \) or \( \gamma \) for \( b \to s \), and \( \beta \) or \( \gamma \) for \( b \to d \).

We consider the decay \( B_d^0 \to f_q \), which results from an underlying \( b \to \bar{q} \) quark level process, where \( q \) could be either \( s \) quark or \( d \) quark. From the above arguments it is easy to conclude that the amplitude \( A_q \) for such a decay, can be expressed in a parametrization independent way, in terms of two contributing amplitudes as follows:

\[
A_q = a_q + b_q e^{i\delta_q} e^{i\phi_q},
\]

(2.19)

where \( a_q \) and \( b_q \) are the magnitudes of the two contributions, \( \delta_q \) is the corresponding strong phase difference, and \( \phi_q \) is the weak phase. The weak phase \( \phi_q \) can be chosen from two different values, which define the choice of parametrization. Once the parametrization is chosen, \( \phi_q \) has the same value for all possible final states which result from the same underlying quark level process \( b \to \bar{s} \) or \( b \to \bar{d} \). The values of \( a_q \), \( b_q \) and \( \delta_q \), however, depend on the decay mode.

\( a_q \) can be either \( a'_q \) or \( a''_q \); \( b_q \) can be either \( b'_q \) or \( b''_q \) depending on the two parametrization and \( \phi_q \) can be \( \gamma \) or \( \beta_s \). Assuming the CPT invariance, the amplitude for the CP conjugate mode can be written as:

\[
\overline{A}_q = a_q + b_q e^{i\delta_q} e^{-i\phi_q}.
\]

(2.20)

For simplification of notation we assume that \( A_q \), \( \overline{A}_q \), \( a_q \) and \( b_q \) are normalized by the total decay width of \( B_d^0 \). This will not change the Physics.

2.3 Observables and Variables

2.3.1 Observables

The time dependent decay rate of \( B_d^0 \) to a mode \( f_i \) (or \( f_q \)) can be written as

\[
\Gamma(B_d^0(t) \to f_i) \propto B_i(1 + C_i \cos(\Delta M t) - S_i \sin(\Delta M t)),
\]

(2.21)
Parametrization Independent Studies of Neutral B Meson Decays

where $B_i$ is the branching ratio, $C_i$ is the direct CP asymmetry arising from
the fact that $|A_i| \neq |\bar{A}_i|$ and $S_i$ is the time dependent CP asymmetry which
arises from the interference between the decay. This three quantities are
observables and can be expressed as

$$B_i = \frac{|A_i|^2 + |\bar{A}_i|^2}{2} = a_i^2 + b_i^2 + 2a_ib_i \cos \phi \cos \delta_i,$$

$$C_i = \frac{|A_i|^2 - |\bar{A}_i|^2}{|A_i|^2 + |\bar{A}_i|^2} = \frac{-2a_ib_i \sin \phi \sin \delta_i}{B_i},$$

$$S_i = \sqrt{1 - C_i^2} \sin 2\beta_{\text{meas}}.$$

with

$$\sin 2\beta_{\text{meas}} = -\frac{\text{Im}(e^{-2i\beta_i}A_i^*\bar{A}_i)}{|A_i||\bar{A}_i|},$$

The phase $\beta$ in $S_i$ comes from $B_0^d - \bar{B}_d^0$ mixing box diagrams.

2.3.2 Extraction of $2\beta_{\text{meas}}$

Within the SM, $\beta$ can not be extracted experimentally due to the pollution
through the phase difference between $A_i$ and $\bar{A}_i$. As a result, $S_i$ provides
a measurement of $\sin 2\beta_{\text{meas}}$. Further extraction of $2\beta_{\text{meas}}$ gives a two fold
ambiguity $(2\beta_{\text{meas}}, \pi - 2\beta_{\text{meas}})$. Therefore we get a four fold ambiguity in the
difference between the two values of $2\beta_{\text{meas}}$ as

$$\pm (2\beta_{\text{meas}}^1 - 2\beta_{\text{meas}}^2), \pm \pi \mp (2\beta_{\text{meas}}^1 + 2\beta_{\text{meas}}^2)$$

which can be measured using two different modes $f_1$ and $f_2$. However, we will only be interested in the principal value $2\beta_{\text{meas}}^1$, obtained from $\sin 2\beta_{\text{meas}}^1$, so as to have a well defined value of the difference. This value
of the difference is denoted by $2\omega$ and is defined as

$$2\omega = 2\beta_{\text{meas}}^1 - 2\beta_{\text{meas}}^2.$$

The two modes $f_1$ and $f_2$ are chosen such that $\beta_{\text{meas}}^1 > \beta_{\text{meas}}^2$. This choice
results in $2\omega$ always being positive.
2.3.3 Variables

The phase difference between $A_i$ and $\overline{A_i}$ is defined as $\eta_i$, i.e.,

$$\eta_i = \arg A_i - \arg \overline{A_i}. \quad (2.27)$$

Hence, $A_i^* \overline{A_i} = |A_i||\overline{A_i}|e^{-i\eta_i}$ and the expression for $\sin 2\beta_i^{\text{meas}}$ from Eq. (2.25) implies that $\eta_i = 2\beta_i^{\text{meas}} - 2\beta$. $\eta_i$ is thus the deviation of $2\beta_i^{\text{meas}}$ from $2\beta$. $\omega$ can now be expressed in terms of $\eta_{1,2}$ as

$$2\omega = \eta_1 - \eta_2. \quad (2.28)$$

We have three observables $B_i, C_i, S_i$ and five variables $a_i, b_i, \delta_i, \eta_i$ and $\phi$. It implies the number of independent variables is only two. The choice of these two independent variables is completely in our hand. From Eqs. (2.19) - (2.20) and Eqs. (2.22) - (2.24), $a_i, b_i$ and $\delta_i$ can be expressed in terms of $B_i, C_i, S_i$ and $\eta_i, \phi$ as

$$a_i^2 = \frac{B_i}{2\sin^2 \phi} \left( 1 - \sqrt{1 - C_i^2 \cos (\eta_i - 2\phi)} \right), \quad (2.29)$$

$$b_i^2 = \frac{B_i}{2\sin^2 \phi} \left( 1 - \sqrt{1 - C_i^2 \cos (\eta_i)} \right), \quad (2.30)$$

$$\tan \delta_i = \frac{C_i \sin \phi}{\cos \phi - \sqrt{1 - C_i^2 \cos (\eta_i - \phi)}}. \quad (2.31)$$

Before we discuss about our analysis in Sec. (2.5), we present a brief summary of the bounds obtained from SM analysis in the tree and penguin sector in the next section.

2.4 Standard Model analysis

The decay mode $b \to c \bar{s}s$ e.g. $B_d^0 \to J/\psi K_S$ has been regarded as the golden mode for extracting the standard-model parameter $\sin 2\beta$ [33]. The penguin pollution in this mode is only of the order of 5% and hence almost negligible. On the other hand, the $b \to s \bar{q}q$ e.g. $B_d^0 \to \phi K_S$ is penguin dominated and hence there is a large possibility that it can receive a contribution from beyond SM physics. The theoretical estimation of these penguin dominated modes have been progressed mainly in two different directions. Firstly, sev-
eral studies have been done to include QCD corrections based on different hadronic assumptions e.g QCD factorization(QCDF), Soft Collinear Effective Theory(SCET) and Perturbative QCD(PQCD). Secondly SU(3) flavour symmetry between $s$ and $d$ quarks has been used to constrain the penguin dominated modes. Here we present a very brief summary of all these constraints.

<table>
<thead>
<tr>
<th>Mode</th>
<th>QCDF bound</th>
<th>SCET bound</th>
<th>PQCD bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi K_S$</td>
<td>$0.02 \pm 0.01$</td>
<td>-</td>
<td>$0.02 \pm 0.01$</td>
</tr>
<tr>
<td></td>
<td>[35]</td>
<td></td>
<td>[36]</td>
</tr>
<tr>
<td>$\eta' K_S$</td>
<td>$0.01 \pm 0.01$</td>
<td>$-0.019 \pm 0.008$, Sol-I</td>
<td>$0.053_{-0.03}^{+0.02}$</td>
</tr>
<tr>
<td></td>
<td>[37]</td>
<td>$-0.010 \pm 0.010$, Sol-II</td>
<td>[38], [39]</td>
</tr>
<tr>
<td>$\pi^0 K_S$</td>
<td>$0.07_{-0.04}^{+0.05}$</td>
<td>$0.077 \pm 0.030$</td>
<td>[28]</td>
</tr>
<tr>
<td></td>
<td>[29]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho^0 K_S$</td>
<td>$-0.08_{-0.12}^{+0.08}$</td>
<td>-</td>
<td>$0.187_{-0.06}^{+0.10}$</td>
</tr>
<tr>
<td></td>
<td>[29]</td>
<td></td>
<td>[38], [39]</td>
</tr>
<tr>
<td>$\omega K_S$</td>
<td>$0.13 \pm 0.08$</td>
<td>-</td>
<td>$0.153_{-0.07}^{+0.03}$</td>
</tr>
<tr>
<td></td>
<td>[29]</td>
<td></td>
<td>[38], [39]</td>
</tr>
</tbody>
</table>

Table 2.1: Constraints on $\Delta S_i$ from QCDF, SCET, PQCD.

For $B_d^0 \to f_i$ decay, a parameter $r_i$ can be defined from Eq. (2.7) as [34],

$$r_i e^{i\delta_i} = \left| \frac{v_{uq}}{v_{eq}} \right| \left( A_{uq} e^{i\delta_{us}} - A_{tq} e^{i\delta_{ts}} \right) \approx 0.02 \frac{A_u}{A_i^c},$$

(2.32)

where $\delta_i$ is the strong phase. Expanding Eq. (2.25) in terms of the small
2.4 Standard Model analysis

ratio $r_i$,

$$\sin 2\beta_{\text{meas}} = \sin 2\beta + 2r_i \cos \delta_i \cos 2\beta \sin \gamma, \quad (2.33)$$

$$C_i = -2r_i \sin \delta_i \sin \gamma, \quad (2.34)$$

In the limit of negligible $r_i$, $\sin 2\beta_{\text{meas}} = \sin 2\beta$ and $C_i = 0$. If the direct CP asymmetry $C_i$ is found to be non-zero experimentally, it would establish the fact that $r_i$ dependent terms are important. The quantity $\Delta S_i \equiv S_{sq} - S_{cs}$ can be written as

$$\Delta S_i = \sin 2\beta_{\text{meas}} \approx \sin 2\beta_{\text{meas}} = \sin 2\beta_{\text{meas}} - \sin 2\beta_{\text{meas}}. \quad (2.35)$$

Without going into the detail discussion of different hadronic assumptions, we present a summary of the predicted $\Delta S_i$ by different QCD based models like QCDF, SCET and PQCD for different penguin dominated modes in Table. (2.1).

$\Delta S = 0$ modes are related to $\Delta S = 1$ modes by SU(3) flavour symmetry, and using this symmetry the bounds on $r_i$ can be obtained as [40],

$$r_i \leq \frac{\mathcal{R} + \lambda^2}{1 - \mathcal{R}}, \quad \mathcal{R} \leq \sum_{f'} |n_{f'}| \sqrt{\frac{B_{f'}(\Delta S = 0)}{B_{f'}(\Delta S = 1)}}, \quad (2.36)$$

where $n_{f'}$ are numerical coefficients,

$$\mathcal{R}^2 \equiv \frac{\lambda^2 \left( |\sum_f n_f A(f)|^2 + |\sum_f n_f A(f)|^2 \right)}{\left( |A(B^0 \rightarrow \eta' K^0)|^2 + |A(\bar{B}^0 \rightarrow \eta' K^0)|^2 \right)} \quad (2.37)$$

and

$$\lambda = -\frac{V_{cb}^* V_{td}}{V_{cb}^* V_{cs}} \approx 0.225. \quad (2.38)$$

The sum over $f$ in Eq. (2.37) is a sum over all the amplitudes of $\pi^0\pi^0, \pi^0\eta, \pi^0\eta', \eta\eta, \eta'\eta', \eta\eta'$ modes. In the limit in which small amplitudes involving the spectator quarks may be neglected, $\pi^0\eta, \pi^0\eta'$ and $\eta\eta'$ amplitudes can be ignored . The bound on $\mathcal{R}$ is thus in general better, if the sum is over a smaller set of modes $f'$. Furthermore, all the branching ratios $f'$ in the bound need to be measured to have the best bound. Using branching ratios of $\pi^0\eta, \eta'\eta', \pi^0\pi^0, \pi^0\eta, \eta\eta', \mathcal{R}_{\eta'K_s} < 0.116 \ [40]$. Using QCDF predicted
branching ratios, \( \mathcal{R}_{\eta'K_s} < 0.045 \) [37]. The bound on \( \mathcal{R}_{\eta'K_s} < 0.088 \) using SCET predicted branching ratios [28]. Bounds on \( \phi K_S, KK K \) modes are not good [41], [42], [43]. The bounds on \( r_{K^+K^-K^0} < 1.02, r_{K^0K^0K^0} < 0.31 \) [42], [43]. Performing a global fit to experimental data, SU(3) prediction for \( \sin 2\beta^{\text{meas}}_{\pi^0K_s} = -0.81 \pm 0.03 \) which is far away from the measured experimental value [42], [44].

The experimental values of \( \Delta S_i = \) are found negative in most of the cases as can be seen from Fig. (2.1). These values are less than the theoretically predicted values also. Presently the error in these experimental values are in such a regime that the tree pollutions in different \( b \rightarrow sqq \) modes with \( q \rightarrow d, s \) can not be neglected any more. Even with the SU(3) approach, at present, only upper limits are available for many of the branching ratios that enter into Eq. (2.36). That is why these bounds obtained from the SU(3) analysis are probably a significant overestimate and will improve with further data. Hence, in both approaches, the theoretical bounds will be more robust with further availability of the experimental data.

In the next section we present a completely different approach based on geometrical interpretation to argue whether the discrepancies in the measured values of \( \Delta S_i = \) can be an indication of NP or not.

### 2.5 Relation between \( \omega \) and \( \phi \)

We want to find a relation between \( \omega \) and \( \phi \). Eq. (2.28) depicts that \( \omega \) and \( \eta \)'s are related to each other. Using this fact we first try to obtain a relation between \( \eta \) and \( \phi \) which finally leads to a relation between \( \omega \) and \( \phi \). The first thing which can be noticed is how sign of \( \eta \) depends on sign of \( \phi \) and then how amplitude of \( \eta \) depends on amplitude of \( \phi \). In this section we present a geometric approach, though we have verified all the results numerically. The simplicity of the arguments is the real beauty of this approach.

#### 2.5.1 Relation between sign of \( \eta \) and \( \phi \)

\( A_i \) and \( \overline{A_i} \) are represented geometrically. Given values of \( a_i, b_i, \delta_i \) and \( \phi, |A_i| \) and \( |\overline{A_i}| \) are as shown in the Fig. 2.2. For the purpose of illustration first we choose \( \delta_i > 0 \) and \( \phi > 0 \). \( \overline{a_i} \) is represented by \( QV \), and \( \overline{b_i} \) is represented by \( SV \) or \( PV \) depending on the phase \( \delta_i + \phi \) or \( \delta_i - \phi \), resulting into the
\[ \sin(2\beta_{\text{eff}}) \equiv \sin(2\phi_{1\text{eff}}) \]

<table>
<thead>
<tr>
<th>Process</th>
<th>World Average</th>
<th>BaBar</th>
<th>Belle</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ K^+ \to \pi^0 ]</td>
<td>[0.67 \pm 0.02]</td>
<td>[0.26 \pm 0.06 \pm 0.03]</td>
<td>[0.67 \pm 0.22]</td>
</tr>
<tr>
<td>[ \eta K^0 ]</td>
<td>[0.57 \pm 0.08 \pm 0.02]</td>
<td>[0.64 \pm 0.10 \pm 0.04]</td>
<td>[0.68 \pm 0.07]</td>
</tr>
<tr>
<td>[ n^0 K^0 ]</td>
<td>[0.90 \pm 0.18 \pm 0.08]</td>
<td>[0.30 \pm 0.32 \pm 0.08]</td>
<td>[0.44 \pm 0.18]</td>
</tr>
<tr>
<td>[ K_S^0 K_L^0 ]</td>
<td>[0.55 \pm 0.20 \pm 0.03]</td>
<td>[0.67 \pm 0.31 \pm 0.08]</td>
<td>[0.74 \pm 0.17]</td>
</tr>
<tr>
<td>[ \phi K^0 ]</td>
<td>[0.35 \pm 0.06 \pm 0.03]</td>
<td>[0.64 \pm 0.25 \pm 0.09]</td>
<td>[0.57 \pm 0.17]</td>
</tr>
<tr>
<td>[ K_S^0 K^0 ]</td>
<td>[0.59 \pm 0.20 \pm 0.04]</td>
<td>[0.55 \pm 0.20 \pm 0.03]</td>
<td>[0.67 \pm 0.31 \pm 0.08]</td>
</tr>
<tr>
<td>[ \pi^0 K^0 ]</td>
<td>[0.64 \pm 0.25 \pm 0.09]</td>
<td>[0.64 \pm 0.25 \pm 0.09]</td>
<td>[0.57 \pm 0.17]</td>
</tr>
<tr>
<td>[ \omega K_S^0 ]</td>
<td>[0.55 \pm 0.20 \pm 0.03]</td>
<td>[0.67 \pm 0.31 \pm 0.08]</td>
<td>[0.74 \pm 0.17]</td>
</tr>
<tr>
<td>[ K_{12} K^0 ]</td>
<td>[0.43 \pm 0.22 \pm 0.06 \pm 0.10]</td>
<td>[0.48 \pm 0.50]</td>
<td>[0.48 \pm 0.53]</td>
</tr>
<tr>
<td>[ f_0 K^0 ]</td>
<td>[0.20 \pm 0.07 \pm 0.07]</td>
<td>[0.20 \pm 0.07 \pm 0.07]</td>
<td>[0.20 \pm 0.07 \pm 0.07]</td>
</tr>
<tr>
<td>[ \pi^0 K_{12} K^0 ]</td>
<td>[0.72 \pm 0.77 \pm 0.08]</td>
<td>[0.72 \pm 0.77 \pm 0.08]</td>
<td>[0.72 \pm 0.77 \pm 0.08]</td>
</tr>
<tr>
<td>[ \pi^0 K^0 ]</td>
<td>[0.43 \pm 0.22 \pm 0.06 \pm 0.10]</td>
<td>[0.48 \pm 0.50]</td>
<td>[0.48 \pm 0.53]</td>
</tr>
<tr>
<td>[ K_{12} K_S^0 ]</td>
<td>[0.20 \pm 0.07 \pm 0.07]</td>
<td>[0.20 \pm 0.07 \pm 0.07]</td>
<td>[0.20 \pm 0.07 \pm 0.07]</td>
</tr>
<tr>
<td>[ f_0 K_S^0 ]</td>
<td>[0.72 \pm 0.77 \pm 0.08]</td>
<td>[0.72 \pm 0.77 \pm 0.08]</td>
<td>[0.72 \pm 0.77 \pm 0.08]</td>
</tr>
</tbody>
</table>

Figure 2.1: \( \sin 2\beta^\text{eff} \equiv \sin 2\beta^\text{meas} \) from the HFAG collaboration [32].
Figure 2.2: The amplitudes $A_i$ and $\overline{A}_i$ in terms of $a_i$ and $b_i$ for the case $\phi > 0$ and $\delta_i > 0$.

amplitude $A_i$ and $\overline{A}_i$ respectively. It may be noted that the same values of $|A_i|$, $|\overline{A}_i|$ and $\eta_i$ can be obtained using different values of $a_i$, $b_i$, $\delta_i$ and $\phi$. The set of points for which this is possible is obtained by moving the point $V$ along the bisector to $SP$, since $SV$ and $PV$ are both $b_i$, they must always be equal. It is hence essential to express all quantities in terms of irreducible variables.

In Fig. 2.2(a) we choose $\delta_i$ to lie in the range between 0 and $\pi/2$. Clearly $\eta_i$ is always positive (if $\phi > 0$) irrespective of the values of the amplitudes $a_i$ and $b_i$. If $\delta_i$ is increased beyond $\pi/2$, at some critical value of $\delta_i = \delta_i^c$, $\eta_i$ becomes 0. This is expressed in Fig. 2.2(b). This critical value $\delta_i^c$ can be easily derived from Eq. (2.31) substituting $\eta_i = 0$ and the expression for it is

$$\tan \delta_i^c = \frac{C_i \sin \phi}{\cos \phi \left(1 - \sqrt{1 - C_i^2}\right)}.$$  \hspace{1cm} (2.39)

If $\delta_i$ is increased further beyond $\delta_i^c$ the sign of $\eta_i$ depends on the relative
2.5 Relation between $\omega$ and $\phi$

magnitudes of $a_i$ and $b_i$; $\eta < 0$ if $b_i < a_i$ (Fig. 2.2(c) ) and $\eta > 0$ if $b_i > a_i$ (Fig. 2.2(d) ). It is easy to generalize to the cases where both $\delta_i$ and $\phi$ can be positive or negative. Note that from Eq. (2.23), if $\delta_i$ and $\phi$ have the same sign then $C_i < 0$, else $C_i > 0$.

$\delta_i^c$ lies in the range $\pi/2$ to $\pi$ for $C_i < 0$ and $-\pi$ to $-\pi/2$ for $C_i > 0$. Here we want to mention that $\delta_i^c$ is strictly not defined for $C_i = 0$, but may be taken to have any value between $\pi/2$ and $\pi$. Hence, we conclude that $\eta_i$ always has the same sign as $\phi$ if $|\delta_i| \leq |\delta_i^c|$. The weak phase $\phi$ is fixed by the parametrization chosen within SM and is same for all modes. Hence as long as $|\delta_i| \leq |\delta_i^c|$ for each mode $f_i$, the sign of $\phi$ and $\eta_i$ must be same for all modes.

2.5.2 Relation between magnitude of $\eta$ and $\phi$

Next, we want to see how the magnitude of $\eta_i$ depends on magnitude of $\phi$, when the magnitude of the strong phase is constrained to be less than $\delta_i^c$, i.e. $|\delta_i| < \delta_i^c$.

Cases for $\phi > 0$: To start with, let us consider $\phi > 0$. The magnitude of $\eta$ depends on both $\delta_i$ and $\phi$. $|\delta_i|$ itself can be either less than $\phi$ or larger than $\phi$. Further, $\delta_i$ can be both positive and negative. We thus require a case by case study depending on the value of $\delta_i$. The three possible cases that need individual consideration are shown in Fig 2.3.

Fig. 2.3(a) represents the case with positive $\delta_i$ greater than $\phi$, i.e. $0 \leq \phi \leq \delta_i \leq \delta_i^c$. Using simple geometry it is easy to deduce that

$$2\phi = \eta + \zeta_i - \overline{\zeta_i}.$$  \hspace{1cm} (2.40)

In Eq. (2.23), since the amplitudes $a_i, b_i$ and the branching ratio $B_i$ are all positive quantities, it is clear that for this case ($0 \leq \phi \leq \delta_i \leq \delta_i^c$), $C_i < 0$. Eq. (2.23) also in turn implies that $|A_i| < |\overline{A_i}|$, if $C_i < 0$. It is then easy to prove that if $|A_i| < |\overline{A_i}|$, $|\zeta_i| < |\overline{\zeta_i}|$ must hold. Before we present the proof we focus on Fig. 2.4(a) and 2.4(b). Fig. 2.4(a) is a repetition of Fig. 2.3(a) with only the essential labels retained. In Fig. 2.4(b) the triangle $\triangle QVP$ of Fig. 2.4(a) is flipped to triangle $\triangle QVP'$. In $\triangle QSP'$,
Figure 2.3: Case $\phi > 0$ and $-\delta_i^c < \delta_i < \delta_i^c$. The amplitudes $A_i$ and $\overline{A}_i$ in terms of $a_i$ and $b_i$. The three possible cases need individual consideration. In cases (a) and (b) we consider $\phi \leq |\delta_i|$ where as in case (c) we consider $|\delta_i| < \phi$. The two cases requiring different treatment for $\phi \leq |\delta_i|$ i.e. $0 \leq \delta_i \leq \delta_i^c$ and $0 \leq \delta_i \leq \delta_i^c$ are considered in (a) and (b) respectively.

It is proved that $|\zeta_i| < |\eta_i|$ for $0 \leq \phi \leq \delta_i \leq \delta_i^c$. From Eq. (2.40), hence it can be concluded that for the case under consideration $\eta_i \leq 2\phi$.

The next case, $-\delta_i^c \leq \delta_i \leq 0$ is depicted in Fig. 2.3(b). For this case $\eta_i$ and $\phi$ are related by

$$2\phi = \eta_i - \zeta_i + \overline{\zeta}_i.$$  

Using Eq. (2.23) and logic similar to the case above, it is easy to see that $0 < C_i$ is implying that $|\overline{A}_i| < |A_i|$. Thus, $|\zeta_i| < |\overline{\zeta}_i|$ must hold, as can be seen by a proof analogous to the above. Hence, it can be deduced that
2.5 Relation between $\omega$ and $\phi$

Figure 2.4: For $|\overline{A_i}| > |A_i|$, (a) before flipping $\Delta QVP$, (b) after flipping $\Delta QVP$ to $\Delta QVP'$.

$\eta_i \leq 2\phi$ even when $-\delta_i^c \leq \delta_i \leq 0$ as long as $\phi$ is positive.

The case of Fig. 2.3(c), when $|\delta_i| < \phi$ is simpler to deal with as it does not depend on the sign of $\delta_i$. It is easy to see that for this case

$$2\phi = \eta_i + \zeta_i + \overline{\zeta_i}. \quad (2.42)$$

Hence, $\eta_i \leq 2\phi$ for this case as well. Having considered all the three possible cases for $0 < \phi$ and $|\delta_i| \leq \delta_i^c$ we can conclude that $0 \leq \eta_i \leq 2\phi$.

Cases for $\phi < 0$:

The different cases for $\phi < 0$ can be treated in a way that is essentially similar to those for $0 < \phi$. However, we discuss these cases in some detail for establishing the completeness of our conclusion. Moreover, due to the negative value of $\phi$ complications arise, that warrant a detailed consideration. To begin with, since $\phi$ is negative, it is easy to see from Fig. 2.5 that $\eta_i < 0$ as well. Hence we need to consider $|\phi|$ and $|\eta_i|$ to follow an approach that is analogous to the one used for $0 < \phi$. Further, the direct CP-asymmetry $C_i$ has opposite sign when compared to the corresponding cases for $0 < \phi$. Also, Eqs. (2.19) and (2.20) imply that $|A_i|$ and $|\overline{A_i}|$ switch. The flip in the positions of $|A_i|$ and $|\overline{A_i}|$ can be seen when comparing Fig. 2.5 with Fig. 2.3.

Fig. 2.5(a) represents the case $\phi \leq 0 \leq \delta_i \leq \delta_i^c$. It is easy to conclude that for
Figure 2.5: Case $\phi < 0$ and $-\delta_i^c < \delta_i < \delta_i^c$. The amplitudes $A_i$ and $\bar{A}_i$ in terms of $a_i$ and $b_i$.

this case, $2 |\phi| = |\eta_i| + \bar{\zeta}_i - \zeta_i$. Since, $\phi < 0$ and $0 < \delta_i$, $C_i$ must be positive, i.e. $0 < C_i$, implying that $|\bar{A}_i| < |A_i|$. Following an approach identical to the one introduced for the case of Fig. 2.3(a) in Fig. 2.4, we conclude that $|\zeta_i| < |\bar{\zeta}_i|$. Hence, $|\eta_i| \leq 2 |\phi|$ for $\phi \leq 0 < \delta_i \leq \delta_i^c$. We next consider Fig. 2.5(b), where $\delta_i^c \leq \delta_i \leq \phi \leq 0$. For this case, $2 |\phi| = |\eta_i| + \bar{\zeta}_i - \zeta_i$. Here, $C_i < 0$, as both $\phi < 0$ and $\delta_i < 0$ imply that $|A_i| \leq |\bar{A}_i|$ and $|\bar{\zeta}_i| < |\zeta_i|$. We hence conclude that $|\eta_i| \leq |\phi|$. We finally consider the case when $|\delta_i| \leq |\phi|$ but $\phi$ itself is negative. This case is straightforward; since, $2 |\phi| = |\eta_i| + \bar{\zeta}_i + \zeta_i$, we easily conclude that $|\eta_i| \leq 2 |\phi|$. It can be concluded that even for each of the $\phi < 0$ sub-cases, $|\eta_i| \leq 2 |\phi|$, though within the SM, the weak phase $\phi$ is always positive as mentioned earlier.

2.5.3 CONCLUSION OF THIS SECTION

As a conclusion of this section it can be stated that,

1. $\eta_i$ always has the same sign as of $\phi$ if $|\delta_i| \leq |\delta_i^c|$.

2. $|\eta_i| \leq 2 |\phi|$.
Combining these two solutions, the constraints are

\[ |\delta_i| < |\delta_i^c| \Rightarrow 0 \leq \eta_i \leq 2\phi, \quad (2.43) \]
\[ |\delta_i| > |\delta_i^c| \Rightarrow \eta_i < 0 \text{ or } \eta_i > 2\phi. \quad (2.44) \]

### 2.6 Constraints on $\eta_i$ and $2\beta$

The present world average value of $\sin 2\beta_{\text{mass}}(b \rightarrow c\bar{s}s) = 0.67 \pm 0.02$. The measured values of $\sin 2\beta_{\text{mass}}(b \rightarrow s\bar{q}q)$ are given in Fig. (2.1). The modes $f_1$ and $f_2$ are chosen in such a way that the estimated values of $2\omega$'s are always positive. These values are listed in Table. (2.2).

$\beta$ can be chosen in three possible ways, it can be either greater than both of $\beta_{1\text{mass}}, \beta_{2\text{mass}}$ or in between them or less than both of them. In Table. (2.3) we consider these three cases with the possible sub cases depending on the value of $\phi = \gamma$ parametrization to obtain bounds on $\eta_1, \eta_2$ and $2\beta$. In Table. (2.4) the bounds are given for $\phi = \beta_s$ parametrization.

### 2.7 $A_j$'s as a Function of $\eta_i$

![Diagram](image)

Figure 2.6: In (a) geometric representation of Eq. (2.8) with $\phi = \gamma$ parametrization and in (b) geometric representation of Eq. (2.12) with $\phi = \beta_s$ parametrization.

Fig. 2.6(a) and Fig. 2.6(b) are the geometrical representation of Eq. (2.8) with $\phi = \gamma$ parametrization and Eq. (2.12) with $\phi = \beta_s$ parametrization.
### Table 2.2: Estimated $2\omega$ and its error $\Delta 2\omega$ values from Fig. (2.1)

<table>
<thead>
<tr>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$2\omega$</th>
<th>$\Delta 2\omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b \to c\bar{c}s$</td>
<td>$\phi K^0$</td>
<td>15.96°</td>
<td>$+11.02°$ $-11.67°$</td>
</tr>
<tr>
<td>$b \to c\bar{c}s$</td>
<td>$\eta'/K^0$</td>
<td>5.91°</td>
<td>±5.21°</td>
</tr>
<tr>
<td>$K_SK_SK_S$</td>
<td>$b \to c\bar{c}s$</td>
<td>5.66°</td>
<td>±14.75°</td>
</tr>
<tr>
<td>$b \to c\bar{c}s$</td>
<td>$\pi^0 K^0$</td>
<td>7.32°</td>
<td>±12.04°</td>
</tr>
<tr>
<td>$b \to c\bar{c}s$</td>
<td>$\rho^0 K_S$</td>
<td>9.38°</td>
<td>$+12.45°$ $-24.53°$</td>
</tr>
<tr>
<td>$b \to c\bar{c}s$</td>
<td>$\omega K_S$</td>
<td>15.32°</td>
<td>±15.67°</td>
</tr>
<tr>
<td>$b \to c\bar{c}s$</td>
<td>$f^0 K_S$</td>
<td>5.20°</td>
<td>$+8.05°$ $-9.48°$</td>
</tr>
<tr>
<td>$b \to c\bar{c}s$</td>
<td>$f_2 K_S$</td>
<td>13.38°</td>
<td>±37.21°</td>
</tr>
<tr>
<td>$b \to c\bar{c}s$</td>
<td>$f_X K_S$</td>
<td>30.53°</td>
<td>±32.79°</td>
</tr>
<tr>
<td>$b \to c\bar{c}s$</td>
<td>$\pi^0\pi^0 K_S$</td>
<td>73.40°</td>
<td>±28.74°</td>
</tr>
<tr>
<td>$b \to c\bar{c}s$</td>
<td>$\pi^+\pi^- K_S$</td>
<td>41.49°</td>
<td>±19.34°</td>
</tr>
<tr>
<td>$K^+K^-K^0$</td>
<td>$b \to c\bar{c}s$</td>
<td>13.02°</td>
<td>±7.19°</td>
</tr>
<tr>
<td>$K^+K^-K^0$</td>
<td>$\pi^+\pi^- K_S$</td>
<td>54.51°</td>
<td>±20.61°</td>
</tr>
<tr>
<td>$K_SK_SK_S$</td>
<td>$\pi^+\pi^- K_S$</td>
<td>47.16°</td>
<td>±24.56°</td>
</tr>
<tr>
<td>$K^+K^-K^0$</td>
<td>$\eta' K^0$</td>
<td>18.93°</td>
<td>±8.62°</td>
</tr>
</tbody>
</table>
2.7 $\mathcal{A}_j$ as a function of $\eta_i$

respectively. The co-ordinates $(x'_1, y'_1)$ and $(x'_2, y'_2)$ and $l'$ can be solvable as a function of $\mathcal{A}_j$, $a'_i, b'_i$ and $\delta'$. Similarly we can solve for the co-ordinates $(x'_1, y'_1), (x'_2, y'_2)$ and $l'$ as a function of $\mathcal{A}_j, a'_i, b'_i$ and $\delta'_i$. Following Eqs. (2.29) - (2.31), $a'^2, b'^2$, tan $\delta'_i$ and $a'^2, b'^2$, tan $\delta'_i$ can be expressed as a function of only $\eta_i$, as $B_i$ and $C_i$ are observables and $\phi$ is known experimentally. This straight-forwardly leads to express $\mathcal{A}_i$ and $\Delta \equiv |\delta_c - \delta_a|$ in terms of $\mathcal{A}_c, \mathcal{A}_u$ and $\eta_i$.

<table>
<thead>
<tr>
<th>$\Delta_{2\text{max}}$</th>
<th>$\Delta_{1\text{max}}$</th>
<th>$\eta_1$ bound</th>
<th>$\eta_2$ bound</th>
<th>$\beta$ bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta_{2\text{max}} \leq \Delta_{1\text{max}} \leq \beta$</td>
<td>I</td>
<td>$\eta_2 \leq \eta_1 \leq 0 \leq 2\gamma$</td>
<td>$\eta_1 \leq -2\omega$</td>
<td>$\eta_1 \leq 0$</td>
</tr>
<tr>
<td>$\beta \leq \Delta_{1\text{max}}$</td>
<td>II</td>
<td>$\eta_2 \leq 0 \leq \eta_1 \leq 0 \leq 2\gamma$</td>
<td>$2\omega \leq \eta_1 \leq 2\gamma$</td>
<td>$2\beta_{\text{meas}} \leq 2\beta \leq 2\beta_{1\text{meas}}$;</td>
</tr>
<tr>
<td>$\Delta_{2\text{max}} \leq \Delta_{1\text{max}} \leq \beta$</td>
<td>III(a)</td>
<td>$0 \leq \eta_2 \leq \eta_1 \leq 0 \leq 2\gamma$</td>
<td>$2\omega \leq \eta_1 \leq 2\gamma$</td>
<td>$2\beta_{\text{meas}} - 2\gamma \leq 2\beta \leq 2\beta_{1\text{meas}}$;</td>
</tr>
<tr>
<td></td>
<td>III(b)</td>
<td>$0 \leq \eta_2 \leq \eta_1 \leq 2\gamma - \eta_2 \leq 2\gamma$</td>
<td>$2\gamma \leq \eta_1 \leq 2\gamma + 2\omega$</td>
<td>$2\beta_{\text{meas}} - 2\gamma - 2\omega \leq 2\beta \leq 2\beta_{1\text{meas}} - 2\gamma$;</td>
</tr>
<tr>
<td></td>
<td>III(c)</td>
<td>$0 \leq \eta_2 \leq \eta_1 \leq 2\gamma - \eta_2 \leq 2\gamma$</td>
<td>$2\omega + 2\gamma \leq \eta_1$</td>
<td>$2\beta \leq 2\beta_{1\text{meas}} - 2\gamma$;</td>
</tr>
</tbody>
</table>

Table 2.3: Constraints on $\eta_i$ and $2\beta$ for the $\gamma$ parametrization.

2.7.1 SOLUTION OF $\mathcal{A}_i$ AND $\Delta$ IN $\gamma$ PARAMETRIZATION

From Fig. 2.6(a),

\[
\begin{align*}
\mathcal{A}_c^2 & = (x'_2 - l')^2 + y_2^2, \\
\mathcal{A}_u^2 & = (x'_2 + l')^2 + y_2^2, \\
\mathcal{A}_l^2 & = (x'_2 - x'_1)^2 + (y'_2 - y'_1)^2, \\
\tilde{a}^2 & = (x'_1 - l')^2 + y_1^2, \\
\tilde{b}^2 & = (x'_1 + l')^2 + y_1^2, \\
4l'^2 & = \tilde{a}^2 + \tilde{b}^2 - 2\tilde{a}\tilde{b}\cos\delta'.
\end{align*}
\] (2.45)
Table 2.4: Constraints on \( \eta_i \) and \( 2\beta \) for \( \beta_s \) parametrization.

Considering the present experimental data, it is a valid assumption to assume the direct CP asymmetry \( C_i = 0 \) which is used to plot Fig. 2.7. \( C_i = 0 \) implies either \( \delta'_i = 0 \) or \( \delta'_i = \pi \). Eq. (2.46) follows \( \delta'_i = 0 \) has two sub-cases \( l' = \pm \frac{(a' - b')}{2} \). These cases are discussed in details in this subsection.

**Case-I (a):** \( \delta'_i = 0 \) and \( l' = \frac{(a' - b')}{2} \)

Eqns. (2.45) - (2.46) leads to

\[
\begin{align*}
x'_1 &= -\frac{(a' + b')}{2}, \\
y'_1 &= 0, \\
x'_2 &= \frac{A_u^2 - A_c^2}{2(a' - b')}, \\
y'_2 &= \pm \left[ A_c^2 - \left( \frac{A_u^2 - A_c^2}{2(a' - b')} \right)^2 \right]^{\frac{1}{2}}, \\
A_t &= \pm \left[ 2\dot{a}'\dot{b}' + \frac{(\dot{a}'A_{u}^{2} - \dot{b}'A_{c}^{2})}{(a' - b')} \right]^{\frac{1}{2}}, \\
\Delta &= \cos^{-1} \left[ \frac{A_{u}^{2} + A_{c}^{2} - 4l'^{2}}{2A_{u}A_{c}} \right].
\end{align*}
\]

**Case-I (b):** \( \delta'_i = 0 \) and \( l' = \frac{(b' - a')}{2} \)
In this case,

\[
x'_1 = \frac{\hat{a}' + \hat{b}'}{2}, \\
y'_1 = 0, \\
x'_2 = \frac{A^2_u - A^2_c}{2(\hat{b}' - \hat{a}')}, \\
y'_2 = \pm \left[ A^2_c - \left( \frac{A^2_u - A^2_c - (\hat{b}' - \hat{a}')^2}{2(\hat{b}' - \hat{a}')} \right)^{\frac{1}{2}} \right],
\]

(2.48)

\[
A_t = \pm \left[ \frac{-2\hat{a}'\hat{b}' + (\hat{a}'A^2_u + \hat{b}'A^2_c)}{(\hat{a}' + \hat{b}')} \right]^{\frac{1}{2}}, \\
\Delta = \cos^{-1} \left[ \frac{A^2_u + A^2_c - 4l'^2}{2A_uA_c} \right].
\]

Finally in both of the sub-cases for \( \delta'_i = 0 \), the values of \( A_t \) and \( \Delta \) remain same.

**Case-II:** \( \delta'_i = \pi \) and \( l' = \frac{(\hat{a}' + \hat{b}')}{2} \)

In this case,

\[
x'_1 = \frac{\hat{b}' - \hat{a}'}{2}, \\
y'_1 = 0, \\
x'_2 = \frac{A^2_u - A^2_c}{2(\hat{b}' + \hat{a}')}, \\
y'_2 = \pm \left[ A^2_c - \left( \frac{A^2_u - A^2_c - (\hat{a}' + \hat{b}')^2}{2(\hat{a}' + \hat{b}')} \right)^{\frac{1}{2}} \right],
\]

(2.49)

\[
A_t = \pm \left[ \frac{-2\hat{a}'\hat{b}' + (\hat{a}'A^2_u + \hat{b}'A^2_c)}{(\hat{a}' + \hat{b}')} \right]^{\frac{1}{2}}, \\
\Delta = \cos^{-1} \left[ \frac{A^2_u + A^2_c - 4l'^2}{2A_uA_c} \right].
\]

These are the possible cases for \( \gamma \) parametrization. The cases for \( \beta_s \) parametriza-
tion are discussed in the next subsection.

2.7.2 Solution of $\mathcal{A}_t$ and $\Delta$ in $\beta_s$ Parametrization

Fig. 2.6(b) follows:

$$
\begin{align*}
\mathcal{A}_c^2 &= (x'_2 - l')^2 + y'^2_2, \\
\mathcal{A}_u^2 &= (x'_2 - x'_1)^2 + (y'_2 - y'_1)^2, \\
\mathcal{A}_t^2 &= (x'_2 + l')^2 + y'^2_2, \\
\hat{a}^2 &= (x'_2 - l')^2 + y'^2_2, \\
\hat{b}^2 &= (x'_2 + l')^2 + y'^2_2,
\end{align*}
$$

(2.50)

$$
\Delta = \cos^{-1} \left[ \frac{\mathcal{A}_u^2 + \mathcal{A}_c^2 - \hat{a}^2}{2\mathcal{A}_u \mathcal{A}_c} \right].
$$

The $C_i = 0$ cases are discussed below.

Case-IV (a): $\delta'_i = 0$ and $l' = \frac{(\hat{a}' - \hat{b}')}{2}$.

From Eqs.(2.50)-(2.51),

$$
\begin{align*}
x'_1 &= - \frac{(\hat{a'} + \hat{b}')}{2}, \\
y'_1 &= 0, \\
x'_2 &= \frac{\mathcal{A}_u^2 - \mathcal{A}_c^2 - \hat{a}' \hat{b}'}{2\hat{a}'}, \\
y'_2 &= \pm \left[ \mathcal{A}_u^2 - \left(\frac{\mathcal{A}_u^2 - \mathcal{A}_c^2 + \hat{a}^2}{2\hat{a}'}\right)^{\frac{1}{2}} \right], \\
\mathcal{A}_t &= \pm \left[ \mathcal{A}_u^2 + \hat{b}^2 - \frac{\hat{b}'}{\hat{a}'}(\mathcal{A}_u^2 - \mathcal{A}_c^2 + \hat{a}^2) \right]^{\frac{1}{2}},
\end{align*}
$$

(2.52)

Case-IV (b): $\delta'_i = 0$ and $l' = \frac{(\hat{b}' - \hat{a}')}{}$. 

In this case,

\[
\begin{align*}
x_1' &= \frac{(\dot{a}' + \dot{b}')}{2}, \\
y_1' &= 0, \\
x_2' &= -\frac{A_u^2 - A_c^2 - \dot{a}' \dot{b}'}{2\ddot{a}'}, \\
y_2' &= \pm \left[ A_u^2 - \left( \frac{A_u^2 - A_c^2 + \dot{a}^2}{2\ddot{a}'} \right)^2 \right]^{\frac{1}{2}}, \\
A_t &= \pm \left[ A_u^2 + \dot{b}^2 - \frac{\dot{b}'}{\ddot{a}'} (A_u^2 - A_c^2 + \dot{a}^2)^2 \right]^{\frac{1}{2}}, \\
\Delta &= \cos^{-1} \left[ \frac{A_u^2 + A_c^2 - \dot{a}^2}{2A_uA_c} \right].
\end{align*}
\] (2.53)

Finally in both of the sub-cases for \(\delta_i' = 0\), the value of \(A_t\) and \(\Delta\) remain same.

Case-V \(\colon\) \(\delta_i' = \pi\)

For this case,

\[
\begin{align*}
l' &= \frac{(\dot{a}' + \dot{b}')}{2}, \\
x_1' &= \frac{(\dot{b}' - \dot{a}')}{2}, \\
y_1' &= 0, \\
x_2' &= -\frac{A_u^2 - A_c^2 + \dot{a}' \dot{b}'}{2\ddot{a}'}, \\
y_2' &= \pm \left[ A_u^2 - \left( \frac{A_u^2 - A_c^2 + \dot{a}^2}{2\ddot{a}'} \right)^2 \right]^{\frac{1}{2}}, \\
A_t &= \pm \left[ A_u^2 + \dot{b}^2 + \frac{\dot{b}'}{\ddot{a}'} (A_u^2 - A_c^2 + \dot{a}^2)^2 \right]^{\frac{1}{2}}, \\
\Delta &= \cos^{-1} \left[ \frac{A_u^2 + A_c^2 - \dot{a}^2}{2A_uA_c} \right].
\end{align*}
\] (2.54)

In Fig. 2.7, the values of \(A_t\) are plotted as a function of \(A_c\) and \(A_u\) for six
Figure 2.7: Values of \( \mathcal{A}_t \) and \( \Delta \equiv |\delta_u - \delta_c| \) as a function of \( \mathcal{A}_u \) and \( \mathcal{A}_c \) for \( C_i = 0 \). \( \mathcal{A}_j \) are normalized such that, if \( \mathcal{A}_u = 0 \) and \( \mathcal{A}_t = 0 \), \( \mathcal{A}_c \) would be unity. The allowed values are bounded by the curves for \( \Delta = 0, \pi \). The unlabelled parabolic curves represent \( \Delta = \frac{\pi}{2}, \frac{\pi}{3} \) and \( \frac{\pi}{6} \).

fixed values of \( \eta_i \) for \( C_i = 0 \).

2.8 Analysis of the bounds

2.8.1 \( f_1 \to (b \to c\bar{c}s), f_2 \to (b \to s\bar{s}q) \)

In \( b \to c\bar{c}s \) channel, tree contributions dominate over the penguin, which leads to the constraint \( \mathcal{A}_c > \mathcal{A}_{u,t} \). In \( b \to s\bar{s}q \) channel only penguin diagram contributes, it does not lead to any such constraint. From Fig. 2.7, it is clear that large \( \eta > 0 \) (of the order of \( 5^\circ \)) is easily obtained by having only \( \mathcal{A}_u \) sizeable but \( \mathcal{A}_c > \mathcal{A}_{u,t} \) only for negative \( \eta \) cases. This immediately rules out all the cases except Case I of Table. (2.3) and Table. (2.4). In
these cases, \( \eta_1 \) is negative, but \( \eta_2 \leq -2\omega \). The estimated values of \( 2\omega \)'s are given in Table. (2.2), though \( \Delta 2\omega \)'s are very large, still for most of the cases, the central value of \( 2\omega > 5^\circ \). Following Fig. 2.7, it can be seen that for Case I, this large negative value of \( \eta_2 \) requires the square of quark level amplitudes \( |A_c|^2 \) and \( |A_t|^2 \) which are at least 10 times larger than the observed branching ratio. Eq. (2.7) or Eq. (2.11) implies that these large quark level amplitudes have to be fine tuned such a way that they produce \( A_{b\to s} \) of the order one. It becomes extremely difficult to fine tune in such a fashion to explain all the channels within SM. Hence, none of the cases can be accommodated within the SM, unless one requires that the observed branching ratios result from considerable fine tuned cancellations of quark level amplitudes.

### 2.8.2 \( f_1 \to (b \to s q_1 \bar{q}_1), f_2 \to (b \to c \bar{c} s) \)

According to the previous logic, in this case \( A_c > A_u,t \) demands negative \( \eta_2 \). This automatically rules out the Case III of Table. (2.3) and Table. (2.4). Case I is also not relevant as \( \eta_2 \leq -2\omega \). Hence, this case is also ruled out according to the same logic discussed in the previous case. Case II of Table. (2.3) and Case II(a) and II(b) of Table. (2.4) are allowed. The constraints obtained from Case II(a) and II(b) of Table. (2.4) are tighter than their counterparts of Table. (2.3).

### 2.8.3 \( f_1 \to (b \to s q_1 \bar{q}_1), f_2 \to (b \to s q_2 \bar{q}_2) \)

In this case none of the \( \eta \) belongs to \( b \to c \bar{c} s \). \( q_1 \) and \( q_2 \) can be either same quarks or different quarks. Case III(b) and III(c) of Table. (2.3) are naturally ruled out as it is expected that \( \gamma \sim 60^\circ \). Case I of both Table. (2.3) and Table. (2.4) and Case III(a) and III(b) of Table. (2.4) are ruled out due to \( \eta_2 \leq -2\omega, 2\omega \leq \eta_1 \) and \( 2\omega + 2\beta_s \leq \eta_1 \) constraints respectively. Case II, III(a) of Table. (2.3) and Case II(a), II(b) of Table. (2.4) are allowed by the present experimental data. In this case also, constraints obtained from \( \beta_s \) parametrization is better than the ones from \( \gamma \) parametrization.

The above discussion clearly indicates that in all cases a large value of \( 2\omega \) must correspond to a large \( \eta \) for at least one of the modes being compared. The values of the amplitudes \( A_u, A_c \) and \( A_t \) and their relative strong phases are depicted in Fig. 2.7. It is easy to conclude from Fig. 2.7 that in all cases,
the amplitudes $A_u$, $A_c$ and $A_t$ must be large and destructively interfere in order to obtain negative $\eta$ larger than a few degrees. A scenario of large destructive interferences among the different contributions seems unnatural. It may be noted that within the same mode it is not possible to change the relative strengths of the amplitudes $A_u$, $A_c$ and $A_t$ by rescattering, since by definition they are distinct amplitudes corresponding to $v_u$, $v_c$ and $v_t$ respectively. Given our relatively successful understanding of $B_d$ decay amplitudes, it seems unlikely that our estimates for the contributing amplitudes of an individual mode (excluding coupled channel final state interactions) are incorrect by a factor of 3 or more; it is even more unlikely that large enough strong phases are generated so as to result in fine-tuned cancellations of these large amplitudes. One may finally consider the possibility of large coupled channel rescattering effects. In such a case, one can fine tune the individual contributions for each mode. However, note that large $\eta$ necessitates a large contribution from $A_t$ and at least one of $A_u$ or $A_c$. In modes that can be coupled by rescattering, a large deviation $\eta$ must be compensated by a mode that has reduced contribution from $A_t$ and an enhanced contribution to $A_u$ and $A_c$. Significantly larger data samples would shed light on whether such a scenario is plausible. Large coupled channel effects would require unnatural fine-tuning of amplitudes, and large coupled channel effects are not expected theoretically.

To conclude, without making any hadronic model assumptions, we have shown that it would be impossible to explain within SM a large discrepancy in the $B^0_d - \bar{B}^0_d$ mixing phase measured using various modes. The only possibility to forgo this conclusion is to accept that the observed branching ratios result from rather fine-tuned cancellations of significantly larger amplitudes.