CHAPTER IV

ON THE $F_A$ SUMMABILITY OF THE SEQUENCE OF THE FOURIER COEFFICIENTS

4.1 Given an infinite matrix $A = (a_{n,k})$ of real or complex numbers and let $\{s_k\}$ be any sequence of complex numbers. The sequence $\{a_n\}$ defined by

$$t_n = \sum_{k=0}^{\infty} a_{n,k} s_k;$$

is called an $A$-transform of $\{s_k\}$ whenever the series converges for $n = 0, 1, 2, \ldots$. The sequence $\{s_k\}$ is said to be $A$-summable to $S$ if $\{t_n\}$ converges to $S$.

The matrix method $A$ is regular$^1$ if it satisfies the following conditions:

$$\sum_{k=1}^{\infty} |a_{n,k}| \leq M, \ n = 1, 2, \ldots.$$  

(4.1.3) $$\lim_{n \to \infty} a_{n,k} = 0, \ k = 1, 2, 3, \ldots.$$  

and

$$\lim_{n \to \infty} \sum_{k=1}^{\infty} a_{n,k} = 1.$$  

(4.1.4)  

A bounded sequence $\{s_k\}$ is said to be $F_A$-summable$^2$ to the limit $S$ if,

$$a_{n,p} = \sum_{k=0}^{\infty} a_{n,k} s_{k+p};$$

---

2. Lorentz, G.G. (1).
tends to $S$, as $k \to \infty$, uniformly in $p = 0, 1, 2, \ldots$.

If,

$$a_{n,k} = \frac{1}{n}, \quad k = 0, 1, 2, 3, \ldots n-1$$

$$= 0 \quad k \geq n.$$

$F_A$ summability is the same as what is known as almost convergence of the sequence $\{S_p\}$. Moreover, if we take $p = 0$, the above definition reduces to that of $A$-summability.

A sequence is called almost $A$-summable$^4$. If $A$-transform of $\{S_p\}$ is almost convergent.

It is known$^5$ that $F_A$ summable sequence is almost-convergent if the matrix $A$ is regular.

If we superimpose the method $F_A$ on the Cesaro mean of order one, we obtain another method of summation viz. $F_A(C, 1)$.

Further, if we put,

$$p = 0,$$

$$a_{n,k} = \frac{1}{n}, \quad (k \leq n)$$

$$= 0 \quad (k > n);$$

we get the $(C, 1)$ summability method.

4.2. Let $f(t)$ be a periodic function with period $2\pi$ and integrable in the sense of Lebesgue over $(0, 2\pi)$.

5. Lorentz, G.G., (2).
Let,

\[(4.2.1) \ \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)\]

be the Fourier series of \(f(t)\), then the conjugate series of \((4.2.1)\) is

\[(4.2.2) \ \sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{1}^{\infty} B_n (t).\]

We write,

\[(4.2.3) \ \theta(t) = \left[ f(x+t) - f(x-t) \right] ;\]

\[(4.2.4) \ g_k(t) = \left( \frac{\sin \frac{kt}{t} - \cos \frac{kt}{t}}{kt^2} \right) ;\]

\[(4.2.5) \ g_{k+p}(t) = \left( \frac{\sin(k+p) t}{(k+p)t^2} - \frac{\cos(k+p) t}{t} \right),\]

\[(4.2.6) \ G_{k+p}(t) = \sum_{m=1}^{k+p} g_m(t).\]

In Chapter III we have generalized the result of Sharma\(^6,7\) on the product summability \(A.(C, 1)\) of the sequence. The theorem of Sharma\(^7\) has been generalized in another direction by Kathal\(^9\). In fact he proved:

**Theorem II.** If,

\[(4.2.7) \ \theta'(t) = \int_{0}^{t} |\theta(t)| \ dt = o(t), \ \ t \to 0,\]

and

\[(4.2.8) \ \lim_{n \to \infty} \sum_{k=1}^{n} k |a_n, k - a_{n, k+1}| = o(1),\]

then the sequence \(\{nB_n(x)\}\) is summable \(A.(C, 1)\) to the sum \(1/n\).

---

7. Sharma, P.L. (1).
This theorem is an extension of the well-known result of Szasz\(^9\) for \((C, 2)\) summability of the sequence \(\{nB_n(x)\}\).

Our object is to extend the above theorem of Khalil by proving:

**Theorem 1.** If (4.2.7) holds and

\[
(4.2.9) \quad \lim_{n \to \infty} \sum_{k=1}^{n} (k+p) |a_{n,k} - a_{n,k+1}| = o(1),
\]

uniformly with respect to \(p\), then the sequence \(\{nB_n(x)\}\) is summable \(F_A(C, 1)\) to the sum \(1/\pi\).

Further, we prove:

**Theorem 2.** If (4.2.9) holds and

\[
(4.2.10) \quad \theta'(t) = \int_{0}^{t} \theta(u) \, du = o\left(\frac{t}{\log 1/t}\right), \quad t \to o
\]

then the sequence \(\{nB_n(x)\}\) is summable \(F_A(C, 1)\) to the sum \(1/\pi\).

4.5 Proof of Theorem 1:

If we denote the \((C, 1)\) transform of the sequence \(\{nB_n(x)\}\) by \(a_n\), we have, after Mohanty and Nanda\(^10\)

\[
(4.3.1) \quad a_n - 1/\pi = \frac{1}{\pi} \int_{0}^{\delta} \theta(t) \left( \frac{\sin nt}{nt^2} - \frac{\cos nt}{t} \right) \, dt + o(1),
\]

\(\delta\) being constant greater than zero.

On account of the regularity of the \(A\)-method of summation, we need only to prove that

\[
(4.3.2) \quad I = \frac{1}{\pi} \sum_{k=1}^{n} a_{n,k} \int_{0}^{\delta} \theta(t) \, \frac{e^{k+p}(t)}{\infty} \, dt = o(1),
\]
as \(n \to \infty\), uniformly with \(p\).

---

Where,

\[ g_{k+p} = \left( \frac{\sin (k+p) t}{(k+p) t^2} - \frac{\cos (k+p) t}{t} \right) \]

As in the Chapter III, we have the following estimates which can be easily be obtained by expanding sine and cosine in power of \( k \) and \( t \):

\[
(4.3.3) \quad g_{k+p} (t) = O((k+p)^2 t) \\
= 0 \ (k+p) \\
(4.3.4) \quad g_{k+p} (t) = O(t^{-1}).
\]

Also it is known\(^1\) that for all values of \( n \) and \( t \),

\[
(4.3.5), \quad \sum_{m=1}^{n} \frac{\sin mt}{m} = O(1).
\]

We write,

\[
I = \frac{1}{\pi} \sum_{k=1}^{n} a_{n,k} \left( \int_{-\pi}^{\pi} + \int_{(k+p)-1}^{\delta} \right) \theta(t) \cdot g_{k+p} (t) \ dt
\]

\[
(4.3.6) = \frac{1}{\pi} \sum_{k=1}^{n} a_{n,k} (P+\delta), \ \text{say}.
\]

Using (4.2.7) and (4.3.3), we get,

\[
|P| = \int_{0}^{(k+p)-1} \theta(t) \cdot O(k+p) \ dt \\
= 0 \ (k+p) \ o(k+p)^{-1} \\
= o (1) \text{ as } k \to \infty, \text{ uniformly w.r.t. } p.
\]

Hence the first term in (4.3.6) can be made as small as we please by choosing \( n \) sufficiently large.

\(^1\) Titchmarsh, E.C. (1), page 440.
It is easy to see that \( \mathbb{E} |a_{n,k}| < \infty \) and assumption (4.2.9) imply that \((k+p) a_{n,k} \to 0\). Hence using Abel's transformation, we write,

\[
\frac{1}{\pi} \sum_{k=1}^{n} a_{n,k} \zeta_k = \left| \frac{1}{\pi} \sum_{k=1}^{n} a_{n,k} \int_{(k+p)-1}^{\delta} \theta(t) \left[ G_{k+p}(t) - G_{k+p-1}(t) \right] dt \right|
\]

\[
\leq \left| \frac{1}{\pi} \sum_{k=1}^{n-1} (a_{n,k-a,n,k+1}) \int_{(k+p)-1}^{\delta} \hat{\omega}(t) G_{k+p}(t) dt \right|
\]

\[
+ \left| \frac{1}{\pi} \sum_{k=1}^{n} a_{n,k} \int_{(k+p)-1}^{(k+p)-1} \hat{\Theta}(t) G_{k+p-1}(t) dt \right| + o(1)
\]

\[= \xi_1 + \xi_2 + o(1), \text{ say.}\]

With the help (4.2.4) and (4.3.5), we write,

\[
C_{k+p} = g_1(t) + g_2(t) + \ldots + g_{k+p}(t)
\]

\[
= \frac{1}{t^2} \sum_{k=1}^{k+p} \frac{\sin k t}{k} - 1/t \sum_{k=1}^{k+p} \cos k t
\]

\[= O(1/t^2) - 1/t \nu_k(t).\]

(4.3.7) \[= O(1/t^2).\]

where \( \nu_k(t) \) is the Dirichlet's kernel for convergence of Fourier series, and it is well known that \( \nu_k(t) = O(1/t) \).

Thus from the estimate (4.3.7), we get

\[
(1) = O(1) \cdot \sum_{k=1}^{n-1} |a_{n,k-a,n,k+1}| \int_{(k+p)-1}^{\delta} \hat{\Theta}(t) \cdot O(1/t^2) \, dt
\]

\[
= O(1) \cdot \sum_{k=1}^{n-1} |a_{n,k-a,n,k+1}| \left( \frac{\hat{\Theta}(t)}{t^2} \right)_{(k+p)-1} + o(1) \int_{(k+p)-1}^{\delta} \frac{dt}{t^2}
\]

\[
= O(1) \cdot \sum_{k=1}^{n-1} |a_{n,k-a,n,k+1}| \left( (k+p) - \delta - 1^2 \right)
\]

\[
= O(1) \cdot \sum_{k=1}^{n-1} (k+p) |a_{n,k-a,n,k+1}|
\]
\[ = o(1), \text{ as } n \to \infty, \]

uniformly with \( p \), by (4.2.9).

Further,
\[
B_2 = o(1) \sum_{k=1}^{n} |a_{n,k}| \frac{(k+p-1)^{-1} |\theta(t)|}{(k+p)^{-1}} \frac{dt}{t^2}
\]
\[ = o(1) \sum_{k=1}^{n} |a_{n,k}| \frac{(\theta'(t))(k+p-1)^{-1}}{(k+p)^{-1}} + o(1) \int (k+p-1)^{-1} \frac{dt}{t^2}
\]
\[ = o(1), \text{ as } n \to \infty, \text{ uniformly with } p, \text{ by (4.1.2).}
\]

This completes the proof of Theorem 1.

4.4 **Proof of Theorem 2**

Proceeding as in the proof of Theorem 1, we write,
\[
I = \frac{1}{\pi} \sum_{k=1}^{n} a_{n,k} \left\{ \int_0^{(k+p)^{-1}} + \int_{(k+p)^{-1} + \delta}^{(k+p)^{-1} + \frac{1}{k+p}} \right\} \theta(t) \cdot g_{k+p}(t) \frac{dt}{t^2}
\]
\[ (4.4.1) = \frac{1}{\pi} \sum_{k=1}^{n} a_{n,k} \left( I_1 + I_2 + I_3 \right), \text{ say,}
\]

where \( 0 < r < 1/2 \).

The following estimates can easily be obtained by expanding sine and cosine in the powers of \( k \) and \( t \).

(4.4.2) \[ \frac{d}{dt} (g_{k+p}(t)) = O((k+p)^2), \quad 0 < t < \frac{1}{k+p}, \]

(4.4.3) \[ \frac{d}{dt} (g_{k+p}(t)) = O(1/t^2), \quad 1/k+p < t < 5, \]

and

(4.4.4) \[ \frac{d}{dt} (g_{k+p}(t)) = O( \frac{1}{t^2} + \frac{(k+p)}{t^2} ) \]
From the estimates (4.3.3), (4.4.2) and the hypothesis (4.2.10), we get

\[
I_1 = \int_0^{(k+p)-1} \theta(t) g_{k+p}(t) \, dt
\]

\[
= \left(\theta_1(t) g_{k+p}(t)\right)_0^{(k+p)-1} - \int_0^{(k+p)-1} \theta_1(t) \frac{d}{dt} g_{k+p}(t) \, dt
\]

\[
= \left[o(t)\cdot o((k+p)^2)\right]_0^{(k+p)-1} + o(1) \int_0^{(k+p)-1} t \cdot o((k+p)^2) \, dt
\]

\[
= o(1) + o((k+p)^2) \cdot (t^2)_0^{(k+p)-1}
\]

\[
= o(1).
\]

Similarly, using (4.4.3), (4.3.4) and (4.2.10, we have,

\[
I_2 = \left[\left(\theta_1(t) \cdot g_{k+p}(t)\right)\right]_0^{(k+p)-1} - \int_0^{(k+p)-1} \theta_1(t) \frac{d}{dt} g_{k+p}(t) \, dt
\]

\[
= \left[o(t)\cdot o(1/t)\right]_0^{(k+p)-1} + o(1) \int_0^{(k+p)-1} \left(\frac{t}{\log 1/t}\right) \cdot o(1/t) \, dt
\]

\[
= o(1) + o(1) \int_0^{(k+p)-1} \frac{d}{dt} t \log 1/t
\]

\[
= o(1).
\]

Thus the first two terms in (4.4.1) can be made as small as we please by choosing \(n\) sufficiently large.

As in theorem 1, \((k+p) a_{n,k} - o\), since \(\varepsilon |a_{n,k}| < \infty\) and by (4.2.9), hence by Abel-transformation, we write,

\[
\left| \frac{1}{\pi} \sum_{1}^{n} a_{n,k} I_2 \right| = \left| \frac{1}{\pi} \sum_{1}^{n} a_{n,k} \int_0^{(k+p)-1} \theta(t) \left[G_{k+p}(t) - G_{k+p-1}(t)\right] dt \right|
\]
\[= \left\lfloor \frac{1}{\pi} \sum_{n=1}^{n-1} (a_{n,k} - a_{n,k+1}) \int_{(k+p)-r}^{5} \theta(t) \cdot G_{k+p}(t) \, dt \right\rfloor + \frac{1}{\pi} \sum_{n=1}^{n} a_{n,k} \int_{(k+p)-r}^{(k+p)-r} \theta(t) \cdot G_{k+p-1}(t) \, dt \right\rfloor + o(1).\]

\[= I_{3.1} + I_{3.2} + o(1), \text{ say.}\]

From the estimate (4.3.7), second mean value theorem and the continuity part of the integral, \(\int |\theta(t)| \, dt\), we have

\[I_{3.1} = 0(1) \cdot \sum_{n=1}^{n-1} |a_{n,k} - a_{n,k+1}| \int_{(k+p)-r}^{5} |\theta(t)| \cdot O(1/t^2) \, dt\]

\[= 0(1) \cdot \sum_{n=1}^{n-1} |a_{n,k} - a_{n,k+1}| \cdot (k+p)^{2r} \int_{(k+p)-r}^{5} |\theta(t)| \, dt\]

\[= 0(1) \cdot \sum_{n=1}^{n-1} |a_{n,k} - a_{n,k+1}| \cdot (k+p) \left( \frac{1}{(k+p)^{1-2r}} \right) \cdot O(1)\]

\[= 0(1) \cdot \sum_{n=1}^{n-1} (k+p) |a_{n,k} - a_{n,k+1}| \cdot o(1) \quad \text{for} \quad 0 < r < 1/2\]

\[= o(1), \quad \text{as} \quad n \to \infty, \quad \text{uniformly with} \quad p, \quad \text{by (4.2.9)}.\]

Again, using (4.3.7), (4.4.4) and the hypothesis (4.2.10), we get

\[I_{3.2} = \left\lfloor \frac{1}{\pi} \sum_{n=2}^{n} a_{n,k} \int_{(k+p)-r}^{(k+p)-r} \theta(t) \cdot G_{k+p-1}(t) \, dt \right\rfloor\]

\[= 0(1) \sum_{n=2}^{n} |a_{n,k}| \cdot \left[ \Theta_1(t) \cdot G_{k+p-1}(t) \right]_{(k+p)-r}^{(k+p)-r}\]

\[+ o(1) \int_{(k+p)-r}^{(k+p)-r} \frac{1}{t^2} + \frac{(k+p)}{t^2} \, dt\]
\[
= o(1) \left( \frac{n}{2} \left| a_{n,k} \right| (o(1)((k+p-1)^r-(k+p)^r)) + o(k+p) \int (k+p)^r \frac{dt}{t} \right)
\]

\[
= o(1) \left( \frac{n}{2} \left| a_{n,k} \right| ((k+p-1)^r-(k+p)^r) + o(1) \frac{n}{2} \left| a_{n,k} \right| (k+p) \log(1- \frac{1}{k+p}) \right)
\]

\[
= o(1), \quad \text{for } 0 < r < 1/2.
\]

since \((a_{n,k})\) is regular; \(\{(k+p-1)^r-(k+p)^r\}\) and \(\{(k+p) \log(1- \frac{1}{k+p})\}\)

are null sequences.

This complete the proof of theorem 2.