CHAPTER VIII

ABSOLUTE SUMMABILITY FACTORS OF FOURIER SERIES

3.1. Let $f(t)$ be a periodic function with period $2\pi$ and integrable in the sense of Lebesgue over $(-\pi, \pi)$, then the Fourier series associated with $f(t)$ be

$$
(8.1.1) \quad f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left( a_n \cos nt + b_n \sin nt \right)
$$

$$
= \sum_{n=0}^{\infty} A_n(t)
$$

We write,

$$
\varnothing(t) = \frac{1}{2} (f(x+t) + f(x-t) - 2f(x)).
$$

**Definition.** Suppose $S_n$ denotes the $n$-th partial sum of the series $\Sigma a_n$. Let $S_n^\alpha$ and $t_n^\alpha$ denote the $n$-th Cesaro means of order $\alpha$, $(\alpha > -1)$ of the sequence $\{S_n\}$ and $\{na_n\}$ respectively.

The infinite series $\sum a_n$ is said to be summable $|C, \alpha|$, $\alpha > -1$, if

$$
(8.1.2) \quad \sum_{n=1}^{\infty} |S_n^\alpha - S_{n-1}^\alpha| < \infty,
$$

where,

$$
\frac{\alpha}{n} = \frac{1}{\Lambda_n} \sum_{k=1}^{n} \frac{\alpha}{n-k} S_k
$$

$$
(8.1.3) \quad A_n^\alpha = \left( \begin{array}{c} n + \alpha \\ n \end{array} \right) = \frac{(n+\alpha+1)}{[(n+1) \cdot (\alpha+1)]}
$$

$$
\sim \frac{n^\alpha}{(n+1)} = 0 \ (n^\alpha).
$$

1. Fekete, M. (1), for fractional and negative orders, see Kogbetliantz, E. (1).
If \( t_n^\alpha \) denotes the \((C,\alpha)\) mean of the sequence \( a_n \), then we know\(^2\) that,

\[
t_n^\alpha = \frac{1}{\Lambda_n^\alpha} \sum_{k=0}^{n-k} \alpha^{n-k} k a_k; \\
= n(S_n^\alpha - S_{n-1}^\alpha).
\]

The \(|C,\alpha|\) summability of the series \( a_n \) reduces to the convergence of the series \( n^{-1}|t_n^\alpha| \).

A sequence \( \{\mu_n\} \) is said to be convex, if,

\[
\Delta^2 \lambda_n > 0.
\]

i.e.,

\[
\Delta \lambda_n - \Delta \lambda_{n+1} > 0.
\]

we write,

\[
\varnothing(t) = \frac{1}{2} (f(x+t) + f(x-t) - 2f(x));
\]

\[
\Delta^0 \varepsilon_n = \varepsilon_n, \quad \Delta^1 \varepsilon_n = \Delta \varepsilon_n = \varepsilon_n - \varepsilon_{n+1}
\]

and

\[
\varepsilon_n^\varnothing = \sum_{k=0}^{\infty} \Lambda_k \varepsilon_{k+n}
\]

provided this series is convergent.

8.2. The first application of Absolute summability factors of Fourier series is due to Prasad\(^3\), who proved:

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2. Kogbetliantz, E. (1).
Theorem A. If \( \{ \lambda_n \} \) is any one of the sequences,

\[
(8.2.1) \quad \{(\log n)^{-1-\delta}\}, \{(\log n)^{-1}(\log \log n)^{-1-\delta}\}, \ldots,
\]
\[
\{(\log n)^{-1}(\log \log n)^{-1}(\log \log \ldots \log_{p-1} n)^{-1} \}
\]
\[
(\log \log \ldots \log_p n)^{-1-\delta} \quad \delta > 0
\]

then the series \( \sum_n A_n(t) \) is summable \(|A|\) at every point \( t = x \),

where,

\[
(8.2.2) \quad \int_0^t |\mathcal{O}(u)| du = o(t), \quad \text{as } t \to 0.
\]

Izumi and Kawata\(^4\) extended the result of Prasad by replacing the sequence \( \{ \lambda_n \} \) defined in (8.2.1) by a more general sequence where \( \{ \lambda_n \} \) is a convex sequence such that the series \( \sum n^{-1} \lambda_n \) is convergent. Since the summability \(|C, r, x|, x > 0\) implies the summability \(|A|\) but the converse is not true, hence generalising theorem A of Prasad, Cheng\(^5\) established the following:

Theorem B. If \( \{ \lambda_n \} \) is any one of the sequences defined in (8.2.1) then the series \( \sum \lambda_n A_n(t) \) is summable \(|C, \alpha|, \alpha > 1\), at a point \( t = x \), where (8.2.2) is satisfied.

All the above results are generalized by Pati\(^6\) by proving:

Theorem C. Let \( \{ \lambda_n \} \) be a convex sequence such that \( \sum n^{-1} \lambda_n \) is convergent, and the condition (8.2.2) is satisfied then the series \( \sum \lambda_n A_n(x) \) is summable \(|C, \alpha|, \alpha > 1\).

All these results have been further generalized by Pati\(^7\), and others\(^8\). Very recently Sharma and Kori\(^9\) have generalized the above theorems of Pati and Cheng by proving:

Theorem D. If,

\[ \int_{t}^{\pi} \frac{|\mathcal{O}(u)|}{u} \, du = o(\log 1/t), \quad t \to 0 \]

then the series \( \sum \lambda_n A_n(x) \) is summable \( |C, \alpha|, \alpha > 1 \), where \( \{\lambda_n\} \) is any sequence as defined in (8.2.1).

Theorem E. If \( \{\lambda_n\} \) is a convex sequence such that \( \sum \lambda_n \) is convergent and the condition (8.2.3) is satisfied then the series \( \sum \lambda_n A_n(x) \) is summable \( |C, \alpha|, \alpha > 1 \).

It is known\(^{10}\) that if (8.2.2) holds, then,

\[ \int_{t}^{\pi} \frac{|\mathcal{O}(u)|}{u} \, du = o(\log 1/t), \quad t \to 0. \]

On the other hand, if (8.2.4) is true, then,

\[ \int_{t}^{\pi} |\mathcal{O}(u)| \, du = o(t \cdot \log 1/t), \quad t \to 0, \]

and this result is best possible. Thus (8.2.4) is a weaker assertion than (8.2.5).

The object of the present Chapter is to prove the following theorem:

**Theorem.** If,

\[ \int_{t}^{\pi} \frac{|\mathcal{O}(u)|}{u} \, du = O(\log 1/t), \quad t \to 0 \]

then the series \( \sum A_n(t)/n^{1-\alpha} \cdot (\log n)^{1+\varepsilon} \)

is summable \( |C, \alpha|, \alpha < 1 \).

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8. For reference, see Prasad, B.N. (3).  
8.3. We require the following Lemmas:

**Lemma 1.** Let $\theta_n^{\alpha}(t)$ denotes the $(c, \alpha)$ mean of the sequence $\{n \cos nt\}$, and if $0 < \alpha < 1$, then,

$$(8.3.1) \quad \theta_n^{\alpha}(t) = \begin{cases} 0(n^{2-\alpha}), & \text{for } 0 < t \leq 1/n \\ O(n^{1-\alpha} t^{-\alpha}), & \text{for } 1/n < t \leq \pi. \end{cases}$$

This follows from the lemma 4 of Cheng. 11

**Lemma 2.** If, $0 < \alpha < 1$, then,

$$(8.3.2) \quad \frac{1}{k} \sum_{k=1}^{n} |\theta_k^{\alpha}(t)| = 0(n^{2-\alpha}), \text{ for } 0 < t \leq 1/n$$

and,

$$0(t^{\alpha-2}), \text{ for } 1/n < t \leq \pi$$

**Proof.** For $0 < t \leq 1/n$, we have,

$$\frac{1}{k} \sum_{k=1}^{n} |\theta_k^{\alpha}(t)| = O(1) \sum_{k=1}^{n} k^{2-\alpha}$$

$$= O(1) \sum_{k=1}^{n} k^{1-\alpha}$$

$$= 0(n^{2-\alpha}),$$

and for, $1/n < t \leq \pi$, we have,

$$\frac{1}{k} \sum_{k=1}^{n} |\theta_k^{\alpha}(t)| = O(1) \sum_{k=1}^{1/t} k^{2-\alpha}$$

$$+ O(1) \sum_{k=1}^{[1/t]+1} k^{1-\alpha}$$

$$= 0(t^{\alpha-2}).$$

where $[1/t]$ denotes the integral part of $1/t$, hence the proof of the lemma 2.

**Lemma 3.** Let $a > 0$, $r$ being the least integer $\geq r \geq a$, $\{\theta_n\}$ is positive non decreasing, $\epsilon_n \rightarrow 0$, as $n \rightarrow \infty$, and

$$
\sum_{n=1}^{\infty} n^r \cdot \theta_n \left| \frac{n}{\theta_n} \epsilon_n \right| < \infty.
$$

Let $t_{n\alpha}$ and $\tau_{n\alpha}$ denotes the $(C, \alpha)$ means of the sequence $\{n\alpha_n\}$ and $\{na_n \epsilon_n\}$, such that,

$$
\sum_{k=1}^{n} |t_{k\alpha}| \cdot 1/k = O(\theta_n); \quad \text{holds},
$$

then,

$$
\sum_{n=1}^{\infty} \frac{|\tau_{n\alpha}|}{n} < \infty.
$$

This is known by Lemma 7, of Sharma and Kori.\(^{12}\)

**Proof of the theorem.** Let $T_{n\alpha}$ denotes the $(C, \alpha)$ mean of the sequence $\{n\alpha_n(t)\}$, then,

$$
T_{n\alpha} = 2/\pi \int_{0}^{\pi} \phi(t) \cdot \theta_{n\alpha}(t) \cdot dt.
$$

where $\theta_{n\alpha}(t)$ denotes the $(C, \alpha)$ Kernel.

Hence,

$$
\sum_{k=1}^{n} 1/k \cdot |T_{k\alpha}| \leq 2/\pi \int_{0}^{\pi} |\phi(t)| n \sum_{k=1}^{n} 1/k \cdot |\theta_{k\alpha}(t)| \cdot dt
$$

$$
= O(1) \cdot \left( n^{2-\alpha} \int_{0}^{1/n} |\phi(t)| dt \right) + O(1) \left( \int_{1/n}^{\pi} \frac{|\phi(t)|}{t^{2-\alpha}} \cdot dt \right)
$$

$$
= O(1) \cdot \left( n^{2-\alpha} (1/n \log n) \right) + O(1) \left( \int_{1/n}^{\pi} \frac{|\phi(t)|}{t} \cdot t^{\alpha-1} \cdot dt \right)
$$

$$
= O(1) \cdot \left( n^{1-\alpha} \cdot \log n \right) + O(1) \left( n^{1-\alpha} \cdot \log n \right)
$$

\[ = 0(n^{1-\alpha} \cdot \log n) \]

The conditions of lemma 3 are clearly satisfied for \( \theta_n = (n^{1-\alpha}, \log n) \), and,

\[ \epsilon_n = n^{\alpha-1} \cdot (\log n)^{-1-\epsilon}, \]

\[ a_n = A_n(t). \]

Thus the proof of the theorem is complete.