CHAPTER II

2-METRIC SPACE AND TOPOLOGICAL SEMIFIELDS

2.1 Gähler \(^1\) in his paper has discussed the concept of 2-metric space over reals, a function of point triples on a set \(X\) whose abstract properties were suggested by the area function for a triple determined by a triangle in a Euclidean space, associated with a given 2-metric. It is proved that every metric space is 2-metrizable.

The object of this chapter is to generalize the concept of Gähler's 2-metric space over reals to 2-metric space over topological semifield \(^2\).

DEFINITION 1. Let \(F\) be a semifield and \(X\) be the set of all its positive elements. The set \(X\) is called a 2-metric space over the semifield \(F\) if there exists a mapping (called 2-metric) \(f : X \times X \times X \rightarrow F\) for each point triple \(x, y, z \in X\) such that, to each pair of points \(x\) and \(y\), \(x \neq y\) in \(X\), there exists a \(w \in X\) satisfying:

\[f(x, y, z) = f(x, w, z) = f(w, y, z) = f(x, x, z)\]

1) Gähler, S. (1)

2) Antonovskii, N. Ya; Boltyanskii, V. G. and Sarymaskov, T. A. (1)
(2.1.1) \[ p(x,y,z) \neq 0, \]

(2.1.2) \[ p(x,y,z) = 0 \text{ if and only if at least two of the three elements are equal,} \]

(2.1.3) \[ p(x,y,z) = p(x,z,y) = p(y,z,x), \]

(2.1.4) \[ p(x,y,z) \leq p(x,y,w) + p(x,w,z) + p(w,y,z). \]

**Definition 2:** A set \( S \) in a 2-metric space \( X \) over the topological semifield \( \mathbb{F} \) is bounded if the set \( S \) consisting of all elements \( p(x,y,z) \) where \( x, y, z \) are in \( S \), is bounded above in \( \mathbb{F} \).

**Definition 3:** Let \( U \) be an arbitrary neighbourhood of zero in \( \mathbb{F} \). For \( x, y \in X \),

\[ \bigcap (x,y; U) = \{ z \in X \mid p(x, y, z) \in U \}. \]

**Remarks:-**

1. If \( \mathbb{F} = \mathbb{R} \), the field of real numbers, we arrive at the definition of a 2-metric space 3).

3) Gähler, I. (1)
2. If \( X \) consists of only two points, then we set the definition of the metric space over \( \mathbb{R} \).

3. We shall denote the 2-metric space \( X \) over \( \mathbb{R} \) by \( (X, \rho, E) \).

We shall prove:

**Theorem 1:** The system of all \( \bigcap (x, y; U) \) builds a neighbourhood base for the topology in \( X \).

**Proof:** It is enough if we prove that:

i) \( \bigcap (x, y; U) \cap \bigcap (x, y; V) = \bigcap (x, y; U \cap V) \);

ii) If \( z \in \bigcap (x, y; U) \) then there exists a neighbourhood \( V \) of zero in \( E \) such that

\[
\bigcap (x, z; V) \subset \bigcap (x, y; U).
\]

i) is obvious; to prove ii),

since \( z \in \bigcap (x, y; U) \), we have \( \rho(x, y, z) \in U \), where \( U \) is an arbitrary neighbourhood of zero in \( E \).

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4) Antonovskii, V.V., Boltyanski, V.I., and Sarymsakov, T.A. (1), (2).
Let \( \mathcal{N} \) be a neighbourhood of zero in \( \mathbb{F} \) such that

\[
p(x,y,z) + x + y + z + s \subseteq \mathbb{F}.
\]

Such a neighbourhood exists because

\[
p(x,y,z) + 0 + 0 + 0 + 0 \subseteq \mathbb{F} \quad \text{and} \quad 0 \text{ is symmetric, i.e., satisfying the condition } -0 = 0.
\]

Let \( \mathcal{V} \) be a saturated neighbourhood of zero, \( \mathcal{V} \subseteq \mathcal{N} \).

For, let \( p \in \neg\neg (x, z ; \mathcal{V}) \).

By (2.1.4) we get

\[
p(x,y,z) - p(x,z,p) - p(z,y,p)
\]

\[
\ll p(x,y,p)
\]

\[
\ll p(x,y,z) + p(x,z,p)
\]

\[
+ p(z,y,p).
\]

Adding \( p(x,y,z) + p(x,z,p) + p(z,y,p) \) to this inequality, we get,

\[
0 \ll p(x,y,p) + p(x,z,p) + p(z,y,p) - p(x,y,z)
\]

5) Antonovskii, M.Ya; Boltyanskii, V.G. and Tymoshkov, T.A.(1)
\[ << 2 \left[ \rho(x, z, p) + \rho(z, y, p) \right]. \]

Thus, setting \( v' = \frac{1}{2} \left[ \rho(x, y, p) + \rho(x, z, p) + \rho(z, y, p) - \rho(x, y, z) \right] \),

\[ v = \rho(x, z, p) + \rho(z, y, p), \]

we get,

\[ \mathcal{O} \ll v' \ll v. \]

Now \( \rho(x, y, z) \in \mathcal{U}, \ \rho(x, z, p) \in \mathcal{V}, \text{ and hence } \rho(z, y, p) \in \mathcal{U} \cap \mathcal{V} \subset \mathcal{W}. \)

Therefore, by virtue of the choice of neighbourhood, we have \( v' \in \mathcal{V}. \) Hence \( 2v' \in \mathcal{V} + \mathcal{V}, \) which implies

\[ \rho(x, y, p) + \rho(x, z, p) + \rho(z, y, p) - \rho(x, y, z) \in \mathcal{V} + \mathcal{V}. \]

Since \( \mathcal{V} \subset \mathcal{V} \) and \( v \in \mathcal{V} \), we get

\[ (x, y, p) \in \rho(x, y, z) + \mathcal{V} - \mathcal{V} - \mathcal{V} \]

\[ \subset \rho(x, y, z) + \mathcal{V} - \mathcal{V} - \mathcal{V} \]

\[ = \rho(x, y, z) + \mathcal{V} + \mathcal{V} + \mathcal{V} \]

\[ \subset \mathcal{V}. \]

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Hence \( p \in \bar{\neg} (x, y; V) \), i.e., \( \neg (x, z; V) \subset \neg (x, y; V) \), which completes the proof of the theorem.

The topology generated by the sets \( \neg (x, y; V) \) will be called as the natural topology of the 2-metric space \( X \) over \( E \). Since every metric space is 2-metrizable, this theorem generalizes Antonovskii's result.

2.2. We shall define continuity of the 2-metric \( p \) over \( E \):

**Definition 4.** Let \( U_c \) be any \( c \) neighbourhood of zero in \( E \). For points \( x, y \in X \), we define

\[
U_c (x, y) = \left\{ z \mid p(x, y, z) \in U_c \right\}.
\]

**Definition 5.** Let \( U_c \) be any \( c \)-neighbourhood of zero in \( p \). For arbitrary points \( x, y, z \) in \( (X, p, E) \) and to each point \( x' \) in the neighbourhood of \( U_{c/2} (x, y) \cap U_{c/2} (x, z) \) of \( x \), if

\[
| p(x, y, z) - p(x', y, z) | \ll p(x, y, x') + p(x, z, x') \in U_c,
\]

we say that \( p(x, y, z) \) is **continuous**.

7) Antonovskii, T. Ya.; Boltvanski, V. C. and Sarynaskov, T. A. (1)
**THEOREM 2.** The map $\rho(x, y, z)$ as a function of three variables $x, y$ and $z$ in $(X, \rho, E)$ is continuous if and only if, the following property exists: for arbitrary points $x, y \in X$ and for arbitrary $E$-neighbourhoods of zero $U_E$ in $E$ there exists neighbourhoods $U_x$ of $x$ and $U_y$ of $y$ so that for arbitrary points $x' \in U_x$ and $y' \in U_y$,

$$\rho(x', y') \in U_E.$$

**PROOF:** Let the property hold for a 2-metric space $X$ over $E$.

That is, for arbitrary points $x$ and $y$ in $X$, $\rho(x, x', y')$ as a function of two points $x' = x$ and $y' = y$ is continuous. Therefore it follows that the triple map $\rho(x, y, z)$ over $X$ as a function of three points is continuous.

Conversely, let $\rho(x, y, z)$ be continuous as a function of the three variables $x, y, z$ in $X$.
Let $U^1_x$ and $U^1_z$ be neighbourhoods of $x$ and $z$ respectively such that for each point $x' \in U^1_x$ and $z' \in U^1_z$,

A) \( f(x, y, x') \in U^c_{x'} \), \( f(x, z, x') \in U^c_{z'} \) and

\[
\rho(\, y, \; z, \; z' \,) \in U^c_{z'}. \]

By the property, there exists neighbourhoods $U^2_x$ of $x$ and $U_y$ of $y$ and $U^2_z$ of $z$ such that for $x' \in U^2_x$, $y' \in U_y$ and $z' \in U^2_z$,

B) \( f(y, x', y') \in U^c_{x'} \), \( f(\, y, \; y', \; z' \,) \in U^c_{z'} \) and

\[
\rho(\, z, \; z', \; x' \,) \in U^c_{x'}. \]

Therefore,

\[
\| f(x, \; y, \; z) - f(x', \; y', \; z') \|
\]

\[
\lll \rho(\, x, \; y, \; x' \,) + \rho(\, x, \; z, \; x' \,) + \rho(\, y, \; z, \; z' \,) + \rho(\, y, \; x', \; y' \,) + \rho(\, y, \; y', \; z' \,) + \rho(\, z, \; z', \; x' \,). \]

Hence from A) and B), for arbitrary $x' \in U_x = U^1_x \cap U^2_x$, $y' \in U_y$ and $z' \in U_z = U^1_z \cap U^2_z$. 
we get
\[ |f(x, y, z) - f(x', y', z')| \leq W_c, \]
which proves the theorem.

If we take \( F = \) in the above theorem we get the property \( S \) of 2-metric spaces. 3)

2.3 Let \((X, \rho, \mu)\) be a 2-metric space over the topological semifield \( \mathcal{V} \). Unless mentioned otherwise, we shall denote the 2-metric space \( X \) over \( \mathcal{V} \) by \( X \).

**DEFINITION 6.** Let \( X \) be a 2-metric space over \( \mathcal{V} \) and \( D \) be a directed set. Every mapping \( x : D \to X \) is a sequence of type \( D \) in \( X \), the image \( x(p) \) of the element \( p \in D \) under the mapping \( x : D \to X \) is \( x_p \).

This definition allows us to write the sequence in the customary form \( \{x_p\} = x, p \in D \).

Let \( D \) and \( E \) be two directed sets. We shall call a mapping \( \mathcal{I} : E \to D \) cofinal if, for an arbitrary element \( p_0 \in D \) there exists an element \( h_0 \in E \) such that \( \mathcal{I}(h) > p_0 \) for \( h > h_0 \). We define a sequence \( x^\mathcal{I} \) of type \( E \) in \( X \) by

3) Gähler, S. (1)
setting, for arbitrary $h \in H$

$$(x^h)_h = x_\varnothing(h)$$

and call the sequence $x^\varnothing$,
a subsequence of the sequence $x$ (corresponding to the cofinal mapping $\varnothing : H \to \mathcal{D}$).

**Definition 7.** A sequence $\{x_p\}$ of type $\mathcal{D}$ in $X$ converges to the point $a \in X$ if, for an arbitrary neighbourhood of zero $U$ in $\mathbb{K}$ being metrized, there exists an element $F_U \in \mathcal{D}$ such that for $p > F_U$ and for arbitrary elements $b$ and $c$ in $X$, with $f(a, b, c) \neq 0$, we have $f(a, b, x_p) \in U$ and $f(a, c, x_p) \in U$.

We shall denote the convergence by the notation

$$\lim_{p \in \mathcal{D}} x_p = a$$

or $x_p \to a$. In other words, if $\lim_{p \in \mathcal{D}} x_p = a$ in $X$ then $\lim_{p \in \mathcal{D}} f(x_p, a, b) = 0$ and $\lim_{p \in \mathcal{D}} f(a, c, x_p) = 0$, in $\mathbb{K}$.

**Definition 8.** A point $a \in X$ is a limit point for the sequence $\{x_p\}$ if, for arbitrary neighbourhood of zero $U$ in the semifield $\mathbb{K}$, and points $b, c$, in $X$ with $f(a, b, c) \neq 0$, and for an index $p \in \mathcal{D}$, there exists an index $p > F$ such that $f(a, b, x_p) \in U$, $f(a, c, x_p) \in U$.

We shall prove:
**Theorem 3.** No sequence can converge simultaneously to two distinct points.

**Proof:** Let us assume, to the contrary, that \( x_p \to a \), \( x_p \to a_0 \), \( a_0 \neq a \).

Hence \( f(a, a_0, b) \notin U \) where \( U \) is an arbitrary neighborhood of zero in the metrizing semifield \( F \). We choose a saturated neighborhood of zero \( W \) in \( E \) such that \( w + w + w \subseteq U \).

By definition 7, there exists an element \( P_a \in D \) such that for arbitrary elements \( b \) and \( c \) in \( X \) with \( f(a, b, c) \neq 0 \),
\[
f(x_p, a, b) \in W, \quad f(x_p, a, c) \in W \quad \text{for} \quad p > P_a.
\]

Similarly, there exists a \( P_b \in D \) such that
\[
\text{for} \quad p > P_b, \quad f(x_p, a_0, b) \in W \quad \text{and} \quad f(x_p, a_0, c) \in W.
\]

Since \( \lim_{p \in D} f(a, a_0, x_p) = 0 \) in \( W \), we have \( f(a, a_0, x_p) \in W \).

Let \( P_0 \in D \) be such that \( P_0 > P_a \), \( P_0 > P_b \) so that for \( p > P_0 \),
\[
f(x_p, a, b) \in W \quad \text{and} \quad f(x_p, a, c) \in W;
\]
\[
f(x_p, a_0, b) \in W \quad \text{and} \quad f(x_p, a_0, c) \in W.
\]
By (2.1.4) we have,

\[ p(a, a_0, b) \leq p(a, a_0, x_p) + p(a, x_p, b) + p(x_p, a_0, b) \]

\[ C + N + \]

\[ C \leq \]

contradicting \( p(a, a_0, b) \neq 0 \), which proves our theorem.

Natural topology of the 2-metric space over \( F \) can also be given in terms of the closure axioms. We prove this fact in the following three theorems.

**Theorem 4.** Let \( A \) be an arbitrary set of a 2-metric space \( X \) over the semifield \( F \). Then for any point \( a \in A \) there exists a sequence of type \( \wedge \) (9) which converges to the point \( a \), consisting of points of the set \( A \).

**Proof:** Since \( a \in A \), we have, for any \( \mu \in \wedge \), that the set \( \wedge (a_0, b; \mu) \), which is a neighbourhood of the point \( a \) in the natural topology of the space \( X \), intersects the set \( A \), i.e., there exists a point \( x_\mu \in A \cap \wedge (a_0, b; \mu) \). We thus obtain a sequence \( \{x_\mu\} \) of type \( \wedge \), consisting of points of the set \( A \). This sequence converges to the point \( a \), because;

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9) See introduction page no. 15.
Let \( V \) be an arbitrary neighbourhood of zero in \( E \). Let \( \mu_0 \) be an element of \( \sim \) such that \( \cup \mu_0 \subset V \).

Then, for \( \mu > \mu_0 \), we have:

\[
x_\mu \in \sim \cup (a, b; U_x) \subset \sim \cup (a, b; U_{\mu_0}) \subset \sim (a, b; V),
\]

i.e. for \( b, c \) in \( A \) with \( \rho(a, b, c) \neq 0 \), \( \rho(a, b, x_\mu) \in V \) for \( \mu > \mu_0 \), and, similarly, \( \rho(a, c, x_\mu) \in V \) for \( \mu > \mu_0 \).

Thus, \( x_\mu \to a \).

**THEOREM 5**: Let \( A \) be an arbitrary set of the 2-metric space \( \mathbb{X} \) over the semifield \( E \) and let \( \{ x_\rho \} \) be a sequence of type \( D \), consisting of elements of the set \( A \) and converging to the point \( a \in \mathbb{X} \). Then \( a \in \overline{A} \).

**PROOF**: Let \( V \) be an arbitrary neighbourhood of zero in the semifield \( E \). Let \( b \) and \( c \) be any arbitrary elements in \( A \) with \( \rho(a, b, c) \neq 0 \).

Let \( P_0 \in D \) be such that for \( \rho > P_0 \),

\[
\rho(a, b, x_\rho) \in V \quad \text{and} \quad \rho(a, c, x_\rho) \in V.
\]

Therefore, for an arbitrary element \( x_\rho \), for which \( \rho > P_0 \), is contained in \( A \cap \sim (a, b; V) \) and also in \( A \cap \sim (a, c; V) \), i.e. the sets \( A \cap \sim (a, b; V) \) and \( A \cap \sim (a, c; V) \) are nonempty.
since the neighbourhood $V$ is arbitrary, it follows that $a \in A$.

From theorems 4 and 5, we have.

**Theorem 6.** The point $a$ belongs to the set $A$ if and only if, there exists a sequence of type $\sqsupset$ which converges to the point $a$ and consisting of points of the set $A$.

Theorem 6 completely characterizes the operation of closure in the natural topology of the 2-metric space $X$ over $\mathbb{R}$ and hence it contains a new definition of the natural topology.

**Theorem 7.** Let $X$ and $Y$ be 2-metric spaces over the semifield $\mathbb{R}$ and let $f : X \to Y$ be an arbitrary mapping. The mapping $f$ is continuous (relative to the usual topologies of the spaces $X$ and $Y$) if and only if, for arbitrary sequence $x$ of type $\sqsupset$ which converges in $X$, the sequence $\left\{ f(x_M) \right\}$ converges in $Y$ and $f(\lim_{\sqsupset} x_M) = \lim_{\sqsupset} f(x_M)$.

**Proof:** Let the mapping $f$ be continuous and $\lim_{\sqsupset} x_M = a$. Let $U$ be an arbitrary neighbourhood of zero in $Y$.

Since $f$ is continuous, for the set $\sqsupset (f(a), f(b); U)$, open in $Y$ and, by virtue of the continuity of the mapping $f$, the
set $G = f^{-1}(\sim (f(a), f(b); U))$ is open in $x$. Clearly, $a, b$ are in $G$. Now we choose a neighbourhood of zero $V$ in the semifield $F$ such that $\sim (a, b; V) \subset G$. Then for any element $y \in \sim (a, b; V)$ we have that $f(y) \in \sim (f(a), f(b); U)$.

Now,

$$\lim_{\mu \in \sim} x_\mu = a,$$

therefore there exists elements $b, c$ in $x$ with $f(a, b, c) \neq 0$, and an element $\mu_V \in \sim$ such that

for $\mu > \mu_V$, $f(a, b, x_\mu) \in V$ and $f(a, c, x_\mu) \in V$.

i.e. $x_\mu \in \sim (a, b; V)$ and $x_\mu \in \sim (a, c; V)$ for $\mu > \mu_V$.

Consequently,

for $\mu > \mu_V$

$$f(x_\mu) \in \sim (f(a), f(b); U), f(x_\mu) \in \sim (f(a), f(c); U).$$

therefore the sequence $\{f(x_\mu)\}$ converges to $f(a)$.

Conversely, suppose the mapping $f$ possesses the property that for any sequence $\{x_\mu\}$ of type $\sim$ which converges in $x$, the sequence $\{f(x_\mu)\}$ converges in $V$ to the point $f(\lim_{\mu \in \sim} x_\mu)$. 
f is continuous; for,

let F be an arbitrary closed set of the space Y, and let a be an arbitrary point of the set \( f^{-1}(F) \).

By theorem 4, there exists a convergent sequence \( \{x_\mu\} \) of type \( \bigcap \) consisting of elements of the set \( f^{-1}(\tau) \) such that \( \lim_{\mu \in \bigcap} x_\mu = a \).

Since for any sequence \( \{x_\mu\} \) of type \( \bigcap \) which converges in X, the sequence \( \{f(x_\mu)\} \) converges in Y to the point \( f(\lim_{\mu \in \bigcap} x_\mu) \), we have \( \lim_{\mu \in \bigcap} f(x_\mu) = f(a) \).

But \( f(x_\mu) \in f(f^{-1}(\tau)) \subseteq F \), and hence by theorem 5,

\[ \lim_{\mu \in \bigcap} f(x_\mu) \in \overline{F} = F. \]

Thus, \( f(a) \in F \), or a \( \in f^{-1}(F) \) and hence \( f^{-1}(F) \subseteq f^{-1}(F) \), i.e., the set \( f^{-1}(F) \) is closed.

so, the mapping f is continuous.

which completes the proof of the theorem.

2.4 Cauchy sequences in \( p \)-metric spaces over \( E \).

**Definition 9.** A sequence \( \{x_p\} \) of type D in X is said to be Cauchy if, for arbitrary neighbourhood of zero U in \( E \) and
for arbitrary elements \( b \) and \( c \) in \( X \) with \( f(a, b, c) \neq 0 \),
there exists \( P_U \in D \) such that for \( p, q > P_U \),
\[ f(x_p, x_q, b) \in U, \quad f(x_p, x_q, c) \in U. \]

**Definition 10.** The space \( X \) is complete if an arbitrary Cauchy sequence of type \( \omega \) in it is convergent.

Now we shall prove:

**Theorem 6.** Every convergent sequence is Cauchy.

**Proof:** Let \( \{x_p\} \) be a convergent sequence of type \( D \) in \( X \), converging to the point \( a \).

Let \( U \) be an arbitrary neighbourhood of zero in \( F \) and \( \# \) be a saturated neighbourhood of zero in \( F \) such that \( \& + \& + \& \subseteq U \).

By definition 7, there exists an element \( P_U \in D \) such that for \( p > P_U \) and for arbitrary element \( b, c \) in \( X \) with \( f(a, b, c) \neq 0 \) we have
\[ f(a, b, x_p) \in \&, \quad f(a, c, x_p) \in \&. \]

Also, \( f(x_p, x_q, a) \in \# \) since \( \lim_{p, q \in D} f(x_p, x_q, a) = 0 \) in \( F \).
Hence by (2.1.4),

\[ f(x_p, x_q, b) \leq f(x_p, x_q, a) + f(x_q, a, b) + f(a, x_p, b) \]

\[ c \Delta + a + b \]

\[ C U, \text{ for } p, q > p_u. \]

Similarly, we can show that

\[ f(x_p, x_q, c) \leq U. \]

Hence the theorem.

2.5 We will define equivalent Cauchy sequences and prove that the relation so defined is an equivalence relation.

**Definition 11.** Let \( x = \{x_p\} \) and \( y = \{y_q\} \) be two Cauchy sequences of type \( D \) and \( E \) in \( \mathcal{R} \). The sequences \( x \) and \( y \) are equivalent if, for arbitrary neighbourhood of zero \( U \) in \( F \), there exists elements \( p_U \in D \) and \( q_U \in H \) such that for \( p > p_U \), \( q > q_U \) and for all \( a \in X \) we have

\[ f(x_p, y_q, a) \leq U. \]

**Theorem 9.** The relation so defined is an equivalence relation.
PROOF: It suffices to show the transitivity.

Let \( \{x_p\}, \{y_q\}, \{z_r\} \) be Cauchy sequences of the type D, H and S respectively.

Let \( \{y_q\} \) be equivalent to \( \{x_p\} \) and \( \{y_q\} \) be equivalent to \( \{z_r\} \). We will show that \( \{x_p\} \) and \( \{z_r\} \) are mutually equivalent:

Let \( W \) be a saturated neighbourhood of zero in \( E \), such that \( W + W + W \subseteq U \), for an arbitrary neighbourhood of zero \( U \) in \( E \).

Let \( P_w \in D \), \( Q_w^{(1)} \in H \) such that

for \( p > P_w \), \( q > Q_w^{(1)} \) and for all \( a \in X \), \( \rho(x_p, y_q, a) \subseteq W \).

Furthermore, let \( Q_w^{(2)} \in H \) and \( r_w \in S \) be such that

for \( q > Q_w^{(2)} \) and \( r > r_w \), \( \rho(y_q, z_r, a) \subseteq W \).

For arbitrary \( p \in D \) and \( r \in S \) such that \( p > P_w \), \( r > r_w \), we choose an arbitrary \( q \in H \) so that \( q > Q_w^{(1)} \) and \( q > Q_w^{(2)} \).
Since \( \lim_{p \in D} f(x_p, y_q, z_r) = 0 \) in \( F \), we have \( f(x_p, y_q, z_r) \in \mathbb{N} \).

Hence by (2.1.4),

\[
\hat{f}(x_p, z_r, a) = f(x_p, z_r, y_q) + f(x_p, y_q, a) + f(y_q, z_r, a)
\]

\( \mathbb{N} + \mathbb{N} + \mathbb{N} \)

\( \subset U \).

Since \( a \) is arbitrary, we have, by definition 11, \( \{x_p\} \) is equivalent to \( \{z_r\} \), thereby proving the theorem.

**THEOREM 10.** If \( \{x_p\} \) converges, then any sequence equivalent to it converges to the same point.

**PROOF:** Let \( \{x_p\} \) and \( \{y_q\} \) be sequences of type D and U in \( X \) respectively.

Let \( \lim_{p \in D} x_p = a \) and \( \{y_q\} \) be equivalent to \( \{x_p\} \).

For an arbitrary neighbourhood of zero, \( U \) in \( F \), let \( \# \) be a saturated neighbourhood of zero in \( F \) such that \( \# + \# + \# \subset U \).
There exists \( p_0 \in D \) and \( q_0 \in \mathbb{N} \) such that for 
\( p > p_0 \), \( q > q_0 \) and for all \( x \in X \) we have \( f(x_p, y_q, x) \in \# \),
in particular for \( b \in X \), \( f(x_p, y_q, b) \in \# \).

Further, we choose an element \( p_\# \in D \) such that 
for elements \( b \) and \( c \in X \) with \( f(a, b, c) \neq 0 \), we have
for \( p > p_\# \),

\[ f(x_p, b, a) \in \# \quad \text{and} \quad f(x_p, c, a) \in \# . \]

For \( q \in \mathbb{N} \), \( q > q_0 \), \( p \in D \) be such that \( p \) is
arbitrary, \( p > p_0 \), \( p > p_\# \).

Since \( \lim_{q \in \mathbb{N}} f(y_q, b, x_p) = 0 \) in \( F \), we have
\( p \in D \)

\[ f(y_q, b, x_p) \in \#. \]

Hence, by (2.1.4),

\[ f(y_q, b, a) \ll f(y_q, b, x_p) + f(y_q, x_p, a) + f(x_p, b, a) \]

\[ \in \# + s + s \]

\[ \subset \mathbb{U} . \]
Similarly, we can show that \( f(y_q, c, a) \in U \).

Therefore, \( \lim_{q \to d} y_q = a \) which proves the theorem.

**Theorem 11.** For every Cauchy sequence \( \{x_p\} \) of type \( \mathcal{B} \) in \( X \) there exists a Cauchy sequence of type \( \mathcal{C} \) which is equivalent to it.

**Proof:** Let \( \mu \) be an arbitrary element of the set \( \mathcal{C} \). We choose an element \( P \mu \in D \) such that

\[
\{x_p, x_p, \ldots, a\} \in U \mu \quad \text{and} \quad \{x_p, x_p, b\} \in U \mu
\]

for arbitrary \( a, b \in X \), and for \( p', p'' > P \mu \).

We shall define by induction on \( |\mu| \) certain elements of the set \( D \):

For \( |\mu| = 1 \) we set \( P^{\prime}_\mu = P \mu \).

Suppose that the element \( P^{\prime}_\mu \) have already been defined for all \( \mu \) satisfying the condition \( |\mu| < m \) and let \( \nu \in \mathcal{C} \) be an element such that \( |\nu| = m \). There exists only finite number of elements \( \mu \in \mathcal{C} \) such that \( \mu < \nu \).

Suppose these are the elements \( \mu_1, \mu_2, \ldots, \mu_k \).
It is obvious that $|y_i| < |y^i|$, $i = 1, \ldots, k$, and hence the elements $P_{\mu_i}$, $P_{\mu_{i+1}}$, \ldots, $P_{\mu_k}$ are already defined. As for $P_{\mu}$, we choose an arbitrary element of the set $E$ which satisfies the conditions:

$$P_{\mu} \geq P_{\mu'} ; P_{\mu} \geq P_{\mu_i}, i = 1, \ldots, k.$$ 

The induction just carried out allows us to define the elements $P_{\mu^i}$ for all $\mu \in \mathcal{M}$; moreover, the following relations are satisfied:

$$P_{\mu^i} \geq P_{\mu}, P_{\mu^i} \geq P_{\mu^i}, \text{for } \mu^i > \mu, (\mu, \mu^i \in \mathcal{M}).$$

We define

$$y_{\mu} = x_{P_{\mu}}, \mu \in \mathcal{M}.$$ 

We will show that the sequence $\{y_{\mu}\}$ of type $\mathcal{M}$ is equivalent to the sequence $\{x_{\mu}\}$:

For an arbitrary neighbourhood of zero $U$ in $E$ and $W$ a saturated neighbourhood of zero in $F$, such that $U + W + W \subset W$. There exists an element $P_0 \in U$ such that for arbitrary $a, b \in X$ and for $p^i, p^i' > P_0$,

$$p(x_p, \ldots, x_p, a) \in \mathcal{M} \text{ and } p(x_p, \ldots, x_p, b) \in \mathcal{M}. $$
Moreover, we choose an element \( \mu' \in \mathcal{U} \) such that \( U_{\mu'} \subset \mathcal{U} \).

Finally, we choose an arbitrary element \( p' \in \mathcal{P} \) satisfying the inequalities \( p' > p_0', p' > P_{\mu'} \). Then, for \( p > p_0', \mu > \mu' \), we have,

\[
\mu \cdot \langle x_{p'}, y_{\mu'}, a \rangle < \mu \cdot \langle x_{p'}, y_{\mu'}, a \rangle + \mu \cdot \langle x_{p'}, x_{p'}, a \rangle + \mu \cdot \langle y_{\mu'}, x_{p'}, a \rangle
\]

\( \in \mathcal{U} \) since \( F_{\mu'} \geq F_{p'} > P_{\mu'} \) and \( U_{\mu'} \subset \mathcal{U} \).

Therefore \( \mu \cdot \langle x_{p'}, y_{\mu'}, a \rangle \in \mathcal{U} \) for \( p > p_0', \mu > \mu' \).

Similarly, we can show that

\[
\mu \cdot \langle x_{p'}, y_{\mu'}, b \rangle \in \mathcal{U}
\]

for \( p > p_0', \mu > \mu' \).

Hence the sequences \( \{x_p\} \) and \( \{y_\mu\} \) are equivalent.

This completes the proof of the theorem.

**Theorem 12.** Let \( x \) be a complete 2-metric space over \( F \). Then every Cauchy sequence \( \{x_p\} \) of arbitrary type \( \mathcal{P} \) is convergent in \( x \).
**Proof**: Let \( \{y_n\} \) be a Cauchy sequence of type -- in \( X \), equivalent to the sequence \( \{x_p\} \) (Theorem 11). Since the space \( X \) is complete, the sequence \( \{y_n\} \) converges to some point \( a \). Then from Theorem 10, we see that the sequence \( \{x_p\} \) also converges to the point \( a \).

This completes the study of Cauchy sequences in 2-metric spaces over topological semifields.