Part-III

ALLOCATION OF WEIGHTS IN INDIVIDUAL SCALING
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5.1. Introduction:

Multidimensional scaling (MDS) is a data analysis technique which constructs configuration of \( n \) points in space using some kind of information about distances between the points. Historically MDS technique started with applications in psychology and have recently found relevance in marketing research, advertising research and other social sciences. The fundamental concept of scaling is that, the objects can be represented as points in space in such a way that the similarity/dissimilarity between the objects is directly related to the distances between the objects. A typical problem of MDS was stated by Torgerson (1958) as "Given a set of stimuli, which vary with respect to an unknown number of dimensions, determine (i) the minimum dimensionality of the set and (ii) projections of stimuli (scaled values) on each of the dimensions involved."

It is a common practice in MDS techniques to take the average of observations overall \( N \) individuals and then generalize findings to the average person. This may lead to a straightforward solution but possibly false interpretation when, results for average person do not describe very accurately, the
consistent responses of each individual in the sample. A recent development in MDS is to work with each individual and find out the differences between them for a common configuration. This is known as Individual scaling. In this section, we discuss the problem of individual scaling.

The first attempt to incorporate notions of individual scaling into multidimensional scaling procedure was that of Tucker and Messick (1963). This procedure amounts essentially to a factor analysis of the dissimilarity matrix of N individuals. Configuration and individual differences are obtained as factors of dissimilarity matrix. Carroll and Chang (1970) developed CANDECOMP procedure in which weighted Euclidean distance model was considered. Weights attached to dimensions were interpreted as individual differences. In this procedure weights and configuration are estimated by alternative linear estimation.

Shonemann (1972) presented an algebraic solution for the same problem when exact distances are given. He transformed interpoint distance matrices into pseudo product matrices. From these product matrices he obtained common configuration matrix and matrices of weights for each individual. Using Shonemann's solution as initial solution, Young and Takane (1977) proposed a new procedure ALSCAL - for fitting
the weighted Euclidean distance model. In this procedure weights and configuration matrices are obtained by method of least squares.

The purpose here is to propose a new iterative procedure for estimating configuration and individual differences. This procedure has two phases. In the first phase, weights for each individual are obtained by quadratic programming method. In the second phase, configuration is obtained by solving the system of homogeneous non-linear equations. The problem solved in this thesis is generalization of Shonemann's problem.

5.2. Problem Formulation:

As discussed in Section 5.1, we consider the weighted Euclidean distance model as

\[ d_{jk}^2 = (a_j - a_k)W(a_j - a_k)' \]  \hspace{1cm} (5.2.1)

where \( d_{jk}^2 \) is interpoint squared distance between points \( j \) and \( k \). \( a_j \) and \( a_k \) are points in multidimensional Euclidean space and \( W \) is a matrix of weights attached to dimensions.

Let \( D_i = (d_{ijk}^2) \) be the \( n \times n \) matrix of squared interpoint
distances between \( n \) points for individual \( i \). \( D_i \) for each individual is transformed into pseudopseudo product matrix \( p_i \) by double centering matrix, \((I - \frac{1}{n} J J')\).

\[
p_i = (I - \frac{1}{n} J J')(-\frac{1}{2} D_i)(I - \frac{1}{n} J J')
\]  
\[
(5.2.2)
\]

where

\( I_{n\times n} \) is the Identity matrix and
\( J' = (1, \ldots, 1)_{1 \times n} \).

For details, we refer to Torgerson (1958)

Given \( p_i \) \((i = 1, \ldots, N)\) matrices for \( N \) individuals, the mathematical problem is to find matrix \( X : n \times t \) and diagonal matrices, \( W_i : t \times t \) with

\[
W_i = \text{diag}(W_{i1}, \ldots, W_{it})
\]

such that

\[
\sum_{i=1}^{N} t_{r}(p_i - X W_i X')^2
\]

is minimum subject to the conditions

\[
\frac{1}{N} \sum_{i=1}^{N} W_i = I_{t \times t}
\]

\[
(5.2.4)
\]

and \( W_{ir} \geq 0 \) for \( i = 1, \ldots, N, \)
\( r = 1, \ldots, t. \)

Here \( t_r(A) \) indicates the trace of Matrix \( A \). The objective
Function (5.2.3) is usually called the STRAIN function.

The matrix \( X : n x t \) presents the configuration of \( n \) points in \( t \)-dimensional Euclidean space and \( W_i, \ i = 1, \ldots, N \) are matrices of weights for each individual.

When the optimal value of STRAIN is zero,

\[
p_i = X W_i X', \quad i = 1, \ldots, N \quad (5.2.5)
\]

When this happens, we call this as the exact case and in this case, \( p_i \)'s are exactly decomposed into \( X W_i X' \). While the problem given by (5.2.3) and (5.2.4) refers to the case of fallible data i.e. \( p_i \neq X W_i X' \) for some \( i \). This means that for the given \( p_i, \ i = 1, \ldots, N \), we cannot find \( X \) and \( W_i, \ i = 1, \ldots, N \) such that \( p_i = X W_i X' \) for \( i = 1, \ldots N \).

Here in fallible case, the optimal value of the STRAIN function is strictly positive.

Shonemann (1972) presented an algebraic solution for solving \( N \) equations, given by (5.2.5) in the exact case.

5.3. Shonemann's Algebraic Solution in Exact Case:

For the exact case, Shonemann (1972) presented an algebraic solution for solving \( N \) equations given by (5.2.5), under the condition (5.2.4).
We assume that $p_i's$ $i = 1, \ldots, N$ are gramian matrices. Given $p_i's$, we can find $X^*$ as follows from (5.2.4) and (5.2.5). We have

$$\overline{p} = X \sum_{i=1}^{N} \frac{W_i}{N}$$

$$= X X', \quad \text{by (5.2.4)}$$

$$= X S S' x'.'$$

$$= X*X'^* \quad (5.3.1)$$

where

$X^* = XS$, $SS' = S'S = I$ and $S$ is an arbitrary rotation of $X$.

From (5.2.5) and (5.3.1), ignoring the condition (5.2.4) temporarily we can find

$$W_i^* = (X^* X^*)^{-1} x'^* p_i X^*(X^* X^*)^{-1} \quad i = 1, \ldots, N$$

which implies that

$$p_i = x^* (x^* x^*)^{-1} x'^* p_i X^*(x^* X^*)^{-1} x'^*.'$$

The latent roots and vectors of one $W_i^*$ are determined up to a joint permutation $Q_i$.

$$W_i^* = (V_i Q_i)(Q_i^t Q_i)(Q_i^t V_i^t)$$

$$= V_i^* R V_i^t \quad i = 1, \ldots, N \quad (5.3.2)$$

where $V_i^* = V_i Q_i$ and $R = Q_i^t W_i Q_i$. 
Since any one of these permutations can be fixed arbitrarily, we set

\[ Q_1 = I \]

Then we have from (5.3.2)

\[ p_1 = x^* w_1^* x^* \]
\[ = x^* v_1 w_1 v_1^* x^* \]
\[ = x w_1 x' \]

where

\[ X = x^* v_1 \quad \text{i.e.} \quad S = v_1^*. \]

Now let \( Q_1 \) be that particular permutation in (5.3.2) which relates the \( i \)th set of vectors \( v_1^* \) to \( v_1 \).

\[ v_1^* = v_1 q_1 \quad \text{or} \quad q_1 = v_1 v_1^* \quad i = 2, \ldots, N. \]

In this case,

\[ p_1 = x^* w_1^* x^* \]
\[ = x v_1^* w_1 v_1 x' \]
\[ = x v_1^* (v_1 q_1) (q_1^* w_1 q_1) (q_1^* v_1^*) v_1 x' \]
\[ = x w_1 x' \quad i = 2, \ldots, N \]

where

\[ v_1^* w_1 v_1^* \] is diagonal matrix for \( i = 2, \ldots, N. \)

This is a stronger condition for obtaining diagonal matrices \( w_1, \quad i = 2, \ldots, N. \)
This means that the set of Eigen vectors of $W_1^*$ for the arbitrarily fixed first subject can be used to eliminate permutations $Q_i$, $i = 2, \ldots, N$ so as to obtain diagonal elements of $W_i$, $i = 2, \ldots, N$ in the correct order relative to that $W_1$. Shonemann (1972) obtained the solution if and only if

1. All matrices $P_i$ ($i = 1, \ldots, N$) are gramian and the same rank.

2. $V_1 W_1^* V_1^T = W_i$, $i = 2, \ldots, N$.

$W_i$'s are diagonal matrices
$V_1$ is the set of Eigen vectors of $W_1^*$.

5.4. **ALSCAL Method:**

A similar problem was solved by Takane, Young, and Jan De Leeu (1977) in the case of fallible data - as mentioned in Section 5.2. They proposed ALSCAL procedure for obtaining configuration and weights for individuals. This procedure contains mainly two phases. In the first phase, optimally scaled distances were obtained from observations by using initial solution given by Shonemann (1972), The criterion function used for optimality was

$$\text{STRESS} = \sum_{i=1}^{N} \sum_{j=1}^{n} \sum_{k=1}^{n} (d_{ijk}^* - d_{ijk}^2)^2$$
where
\[ d_{ijk}^2, \quad i = 1, \ldots, N, \]
\[ j, k = 1, \ldots, n \]
are obtained from initial solution and \( d^*_{ijk} \) are optimally scaled distances.

In the second phase, from optimally scaled distances in Phase 1, configuration \( X \) and \( W_i \) \((i = 1, \ldots, N)\) were obtained by solving the differential equations in alternative least square method.

In ALSCAL procedure, elements of \( W_i, \quad i = 1, \ldots, N \) matrices are obtained by solving partial differential equations. Thus non-negativity constraint on elements of \( W_i \) is likely to be violated in some of the cases. This problem is solved by setting negative weights to zero and these values of \( W_i \)'s are re-estimated under the assumption that positive weights are constants.

5.5. Present Work

The problem considered here is formulated within the STRAIN framework. STRAIN given by \((5.2.3)\) was defined by Carroll and Chang (1970).

We propose an iterative procedure with two phases A and B. In Phase A, \( W_i, \quad i = 1, \ldots, N \) are obtained by solving a quadratic programming problem. Since quadratic programming
problem is solved under the constraint \( W_{ir} \geq 0 \) for 
\( r = 1, \ldots, t, \ i = 1, \ldots, N \), non-negativity constraint 
is never violated. Also matrices, \( W_i \), \( i = 1, \ldots, N \) are 
obtained by obtaining each element of \( W_i \) matrix. Therefore 
non-diagonality of \( W_i \) matrices does not arise as it does in 
Shonemann's solution.

In Phase B, \( X \) is obtained by solving non-linear homogenous 
equations. Different numerical methods are examined for solving these equations. The method is discussed 
particularly for getting one-dimensional and two-dimensional 
configuration with numerical examples and finally the method 
is compared with Shonemann's method.

5.6. Proposed Procedure:

As mentioned before, the two phases of the proposed 
procedure are discussed in detail:

Phase A : To Find Optimal \( W_i \) for Given \( X \):

The Objective function (5.2.3) can be written as

\[
\text{STRAIN} = \sum_{i=1}^{N} t_r(XW_i'X_i) - 2 \sum_{i=1}^{N} t_r(p_iXW_i'X_i) + \sum_{i=1}^{N} t_r(p_i^2)
\]

\[
= \sum_{i=1}^{N} \sum_{j=1}^{n} \sum_{k=1}^{n} (x_jW_i'x_k)^2 - 2 \sum_{i=1}^{N} \sum_{j=1}^{n} \sum_{k=1}^{n} p_{ijk}x_jW_i'x_k
\]

\[+ \sum_{i=1}^{N} t_r(p_i^2) \quad (5.6.1)
\]
where \( x_j \) : \( 1 \times t \) is the \( j \)th row vector of matrix \( X : n \times t \)

\[ j = 1, \ldots, n. \]

Now

\[ x_j W_k x_k^t = \sum_{r=1}^{t} x_{jr} x_{kr} W_{ir} \]

\[ (x_j W_k x_k^t)^2 = \sum_{r=1}^{t} x_{jr}^2 x_{kr}^2 W_{ir}^2 + 2 \sum_{r=1}^{t} \sum_{s=1}^{t} x_{jr} x_{kr} x_{js} x_{ks} W_{ir} W_{is}. \]

\[ \sum_{j=1}^{n} \sum_{k=1}^{n} (x_j W_k x_k^t)^2 = \sum_{r=1}^{t} \left[ \sum_{j=1}^{n} \sum_{k=1}^{n} x_{jr}^2 x_{kr}^2 \right] W_{ir}^2 \]

\[ + 2 \sum_{r=1}^{t} \sum_{s=1}^{t} \sum_{j=1}^{n} \sum_{k=1}^{n} x_{jr} x_{kr} x_{js} x_{ks} W_{ir} W_{is}. \]

\[ = \sum_{r=1}^{t} \sum_{j=1}^{n} x_{jr}^2 W_{ir}^2 + 2 \sum_{r=1}^{t} \sum_{s=1}^{t} \sum_{j=1}^{n} x_{jr} x_{js} W_{ir} W_{is}. \]

\[ = (W_{il}, \ldots, W_{it}) \begin{bmatrix} y_1^t y_{1t} & \cdots & y_{1t}^t \\ \vdots & & \vdots \\ y_t^t & \cdots & y_{tt}^t \end{bmatrix} \begin{bmatrix} W_{il} \\ \vdots \\ W_{it} \end{bmatrix} \]
where

\[ y_r = \begin{bmatrix} x_{1r} \\ \vdots \\ x_{nr} \end{bmatrix} \quad r = 1, \ldots, t \text{ is a } r^{th} \text{ column vector of matrix } X = n \times t. \]

Let

\[ u = \begin{bmatrix} y_1y_1 & \cdots & y_1y_t \\ \vdots & \ddots & \vdots \\ y_ty_1 & \cdots & y_ty_t \end{bmatrix}_{t \times t} \]

then we have

\[ \sum_{j=1}^{n} \sum_{k=1}^{n} (x_j w_k x_k')^2 = (w_{11}, \ldots, w_{1t}, \ldots, w_{nt})' u \begin{bmatrix} W_{11} \\ \vdots \\ W_{nt} \end{bmatrix} \quad (5.6.2) \]

\[ \sum_{i=1}^{N} \sum_{j=1}^{n} \sum_{k=1}^{n} (x_j w_k x_k')^2 = (w_{11}, \ldots, w_{nt})' \begin{bmatrix} u' & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & u' \end{bmatrix} \begin{bmatrix} W_{11} \\ \vdots \\ W_{nt} \end{bmatrix} \]

\[ = W H W' \quad (5.6.3) \]

where

\[ W = (w_{11}, \ldots, w_{1t}, \ldots, w_{nt}) \text{ and } H \text{ is partitioned } 1 \times Nt \]

matrix given by
\[
H = \begin{bmatrix}
u & & & 0 \\
& \ddots & & \\
& & \ddots & \\
0 & & & u
\end{bmatrix}
\]

From (5.6.3), we have

\[
\sum_{i=1}^{N} t_r (x_{wi} x')^2 = \sum_{i=1}^{N} \sum_{j=1}^{n} \sum_{k=1}^{n} (x_j w_i x_k')^2 = W^T H W',
\]

(5.6.4)

Consider

\[
t_r p_{ij} x_{wi} x' = \sum_{j=1}^{n} \sum_{k=1}^{n} p_{ijk} x_j w_i x_k' = \sum_{r=1}^{t} \left[ \sum_{j=1}^{n} \sum_{k=1}^{n} p_{ijk} x_j x_k' \right] \mathbf{w}_i^r
\]

\[
= \mathbf{w}_i \mathbf{w}_i^r
\]

\[
= (w_{i1}, \ldots, w_{it}) v_i
\]
\[ V_1 = \begin{bmatrix}
\sum_{j=1}^{n} \sum_{k=1}^{n} p_{ijk} x_{j1} x_{k1} \\
\sum_{j=1}^{n} \sum_{k=1}^{n} p_{ijk} x_{jt} x_{kt} 
\end{bmatrix} \]

\[
\sum_{i=1}^{N} \text{tr} \left( p_i X W_i X^t \right) = (W_{11}, \ldots, W_{Nt}) \]

\[
= W.G. \quad (5.6.5)
\]

where \( G \) is a partitioned matrix given by

\[ G' = \begin{bmatrix} V_1 \mid V_2 \mid \ldots \mid V_N \end{bmatrix} \]

From (5.6.4) and (5.6.5), (5.2.3) can be written as

\[
\text{STRAIN} = WHW' - 2WG + \sum_{i=1}^{N} \text{tr} \left( p_i^2 \right) \quad (5.6.6)
\]

Since \( \sum_{i=1}^{N} \text{tr} \left( p_i^2 \right) \) is known, minimizing (5.6.6), is equivalent to minimizing

\[
WHW' - 2WG \quad (5.6.7)
\]
Further $WHW' \geq 0$, since $\sum_{i=1}^{N} t_{r} (X W_{i} X')^{2} \geq 0$.

Therefore, $H$ is positive semi-definite. Hence (5.6.7) is a quadratic convex function in $W$. Minimizing STRAIN for a given $X$ is the problem of minimizing this convex quadratic function. The complete formulation of the problem is given below:

Minimize

$$WHW' - 2WG$$

subject to

$$\sum_{i=1}^{N} \frac{W_{ir}}{N} = 1, \quad r = 1, \ldots, t$$

$$W_{ir} \geq 0, \quad i = 1, \ldots, N, \quad r = 1, \ldots, t$$

The quadratic programming problem can be solved by available standard methods. Thus, $W_{i}$'s are optimally obtained for given $X$.

Phase B: To Obtain $X$ for Given $W_{i}$:

In this phase, using the value of $W_{i}$'s $i = 1, \ldots, N$ obtained in Phase A, $X$ is estimated so that STRAIN is minimum. A necessary condition for optimum $X$ is obtained in Lemma 5.1.

Lemma 5.1: Let $X$ be an optimum solution to (5.2.3) for given $W_{i}$'s satisfying (5.6.9), then a necessary condition satisfied by optimal $X$ is
Proof: The Objective function (5.2.3) can be written as

\[
\text{STRAIN} = \sum_{i=1}^{N} t_{r}(XW_{i}X') - 2 \sum_{i=1}^{N} p_{1}XW_{i}X' + \sum_{i=1}^{N} t_{r}(p_{1})
\]

Let \( x_{j} \) be the \( j \)th row vector of \( X \).

\[
\phi_{jk} = x_{j}W_{i}x'_{k} = \sum_{r=1}^{t} x_{jr} x_{kr} W_{ir}
\]

(5.6.10)

\[
\frac{\partial \phi_{jk}}{\partial x_{j}} = \left( \frac{\partial \phi_{jk}}{\partial x_{j1}}, \ldots, \frac{\partial \phi_{jk}}{\partial x_{jt}} \right)
\]

\[
= (x_{kl} W_{il}, \ldots, x_{kt} W_{it})
\]

\[
= x_{k} W_{i} \quad k = 1, \ldots, n \quad (5.6.11)
\]

Similarly

\[
\frac{\partial \phi_{ii}}{\partial x_{j}} = \frac{\partial x_{j}W_{i}x'_{j}}{\partial x_{j}}
\]

\[
= 2 x_{j} W_{i} \quad j = 1, \ldots, n \quad (5.6.12)
\]

Let \( f_{jk} = (x_{j} W_{i} x'_{k})^{2} \quad j, k = 1, \ldots, n, j \neq k \)

\[
= (\sum_{r=1}^{t} x_{jr} x_{kr} W_{ir})^{2}
\]
\[ \frac{\partial r_{jk}}{\partial x_j} = \left( \frac{\partial r_{jk}}{\partial x_{j1}}, \ldots, \frac{\partial r_{jk}}{\partial x_{jt}} \right) \]

\[ = 2 \left( \sum_{r=1}^{t} x_{jr} x_{kr} W_{ir} \right) \left( x_{kl} W_{il}, \ldots, x_{kt} W_{it} \right) \]

\[ = 2 x_{j} W_{i} x_{k} W_{i} \quad j, k = 1, \ldots, n \quad (5.6.13) \quad j \neq k \]

\[ \frac{\partial r_{jj}}{\partial x_j} = 4 x_{j} W_{i} x_{j} x_{j} W_{i} \quad j = 1, \ldots, n \quad (5.6.14) \]

Now

\[ \frac{\partial}{\partial x_j} \left( x_{W_{i}x'} \right)^2 \]

\[ = \frac{\partial}{\partial x_j} \left( \sum_{j=1}^{n} \sum_{k=1}^{n} \left( x_{j} W_{i} x_{k} \right)^2 \right) \]

\[ = \frac{\partial}{\partial x_j} \left( \sum_{j=1}^{n} (x_{j} W_{i} x_{j})^2 \right) + 2 \sum_{j=1}^{n} \sum_{k=1}^{n} (x_{j} W_{i} x_{k})^2 \]

\[ = \left( x_{W_{i}x'} \right)^2 \quad (5.6.15) \]

From (5.6.13) and (5.6.14), (5.6.15) is written as

\[ \frac{\partial}{\partial x_j} \left( x_{W_{i}x'} \right)^2 \]

\[ = 4 \sum_{k=1}^{n} x_{j} W_{i} x_{j} x_{k} W_{i} \quad (5.6.16) \]

Therefore
\[
\frac{\partial t_r(xw_ix')^2}{\partial x} = \begin{bmatrix}
\frac{\partial t_r(xw_ix')^2}{\partial x_1} \\
\vdots \\
\frac{\partial t_r(xw_ix')^2}{\partial x_n}
\end{bmatrix}
\]

\[
= 4 \begin{bmatrix}
x_1w_1x'_1 & \cdots & x_1w_1x'_n \\
\vdots & \ddots & \vdots \\
x_1w_1x'_n & \cdots & x_nw_1x'_n
\end{bmatrix}
\begin{bmatrix}
x_1w_1 \\
\vdots \\
x_nw_1
\end{bmatrix}
\]

\[
= 4 xx'w_1w_1
\]

(5.6.17)

Now
\[
t_r p_i x w_i x' = \sum_{j=1}^{n} \sum_{k=1}^{n} x_j w_i x'_k p_{ijk}
\]

Using (5.6.11) and (5.6.12) we have

\[
\frac{\partial t_r(p_i x w_i x')}{\partial x} = \begin{bmatrix}
\frac{\partial t_r(p_i x w_i x')}{\partial x_1} \\
\vdots \\
\frac{\partial t_r(p_i x w_i x')}{\partial x_n}
\end{bmatrix}
\]
From (5.6.17) and (5.6.18) we have

\[
\begin{align*}
\frac{\partial (\text{STRAIN})}{\partial X} &= \sum_{i=1}^{N} (4XW_iX'XW_i - 4p_iXW_i) \\
&= 4\sum_{i=1}^{N} (XW_iX'XW_i - p_iXW_i) \\
&= 4\sum_{i=1}^{N} (XW_iX' - p_iXW_i) \tag{5.6.19}
\end{align*}
\]

equating (5.6.19), with zero we have

\[
\sum_{i=1}^{N} (XW_iX' - p_iXW_i) = 0 \tag{5.6.20}
\]

is necessary condition for optimal X to satisfy.

The necessary condition, (5.6.20) can be written in terms of column vectors of X. Let \( X_r : n \times 1 \) be the \( r \)th column vector of X, where
Then

\[ Y_r = \begin{bmatrix} x_{1r} \\ \vdots \\ x_{nr} \end{bmatrix}, \quad r = 1, \ldots, t \]

Using (5.6.21'), (5.6.20), can be written as

\[
X \cdot W_1 = (y_1, \ldots, y_t) \begin{bmatrix} W_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & W_{it} \end{bmatrix}
= (W_{11} y_1, \ldots, W_{it} y_t)
\]  

(5.6.21)

Using (5.6.21), (5.6.20), can be written as

\[
\sum_{i=1}^{N} (XW_1X' - p_1)(W_{11} y_1, \ldots, W_{it} y_t) = 0
\]

i.e.

\[
\sum_{i=1}^{N} XW_1X'W_{11} y_1 = \sum_{i=1}^{N} p_1W_{11} y_1
\]

\[
\sum_{i=1}^{N} XW_1X'W_{it} y_t = \sum_{i=1}^{N} p_1W_{it} y_t
\]

i.e.

\[
X \sum_{i=1}^{N} W_{11} W_{11} y_1 = \sum_{i=1}^{N} p_1W_{11} y_1
\]

\[
\sum_{i=1}^{N} w_{11} W_{11} y_1 = \sum_{i=1}^{N} p_1W_{11} y_1
\]

(5.6.22)

\[
X \sum_{i=1}^{N} W_{it} W_{it} y_t = \sum_{i=1}^{N} p_1W_{it} y_t
\]

\[
\sum_{i=1}^{N} w_{it} W_{it} y_t = \sum_{i=1}^{N} p_1W_{it} y_t
\]

(5.6.23)

Let \( \Delta_r = \sum_{i=1}^{N} W_{ir} \) for \( r = 1, \ldots, t \)

\[
= (z_{rs})_{t \times t}
\]
and $Q_r = \sum_{i=1}^{N} p_i W_{ir}$

$$= (q_{jk})_{mn}$$ for $r = 1, \ldots, t$

Then (5.6.23) becomes

$$X \Delta_1 X' y_1 = Q_1 y_1$$

$$\vdots$$

$$X \Delta_t X' y_t = Q_t y_t$$

Again from (5.6.21), (5.6.24) can be written as

$$z_{11} y_1 y_1' y_1 + \cdots + z_{1t} y_t y_t' y_1 = Q_1 y_1$$

$$\vdots$$

$$z_{tt} y_t y_t' y_t = Q_t y_t$$

Solving the system of equations (5.6.25), for given $W_i$, $i = 1, \ldots, N$. i.e. $\Delta_r$, $r = 1, \ldots, t$ we obtain $X$ which is locally optimum solution to (5.2.3).

Remarks:

1. Trivial solution of the equations in (5.6.25) is $X = 0$. However, we are looking for non-zero solution.

2. If $y_r' y_r = 1$, $r = 1, \ldots, t$, then (5.6.25) is the system of linear equations of $y_r'$, $r = 1, \ldots, t$. 
But, we do not take \( y^*_r \), \( y_r = 1, r = 1, \ldots, t \), since this is not the constraint imposed on \( X \) in the problem.

3. \((5.6.25)\) is the system of non-linear homogenous equations of third degree.

The equations in \((5.6.25)\) can be solved for example, by Newton-Raphson method or some other numerical analysis method. Particular cases for \( t = 1 \), and \( t = 2 \) are discussed with numerical examples. For \( t = 2 \), we have examined four different approaches.

For the Case \( t = 1 \):

In this case, \( W_i = W_{i1} \) for \( i = 1, \ldots, N \) and

\[
X = \begin{bmatrix} x_{11} \\ \vdots \\ x_{n1} \end{bmatrix} \quad \text{and} \quad X'X = \sum_{j=1}^{n} x^2_j \]

\[
\Delta_1 = \sum_{i=1}^{N} W^2_{i1} = z_{11} \quad \text{and} \quad Q_1 = \sum_{i=1}^{N} p_i W_{i1}.
\]

From \((5.6.25)\), we have

\[
(\sum_{i=1}^{N} W^2_{i1}) \begin{bmatrix} x_{11} \\ \vdots \\ x_{n1} \end{bmatrix} = Q_1 \begin{bmatrix} x_{11} \\ \vdots \\ x_{n1} \end{bmatrix}
\]
i.e., \[ Q_1 - \left( \sum_{i=1}^{N} W_{11}^2 \right) \left( \sum_{j=1}^{n} x_{j1}^2 \right) I \] \( X = 0 \) \ (5.6.26)

i.e. \[ Q_1 - z_{11} \left( \sum_{j=1}^{n} x_{j1}^2 \right) I \] \( X = 0. \)

This implies that \( X \) is Eigen vector of \( Q_1 \) corresponding to the Eigen value, \( \left( \sum_{i=1}^{N} W_{11}^2 \right) \left( \sum_{j=1}^{n} x_{j1}^2 \right) \). We find all the possible Eigen values of \( Q_1 \) and corresponding Eigen vectors. Then we find the ratio,

\[
\frac{\text{Eigen Value}}{z_{11}} = R.
\]

We select the Eigen vector corresponding the Eigen value for which \( \sum_{j=1}^{n} x_{j1}^2 = R \). The following example 5.1 indicates the determination of optimal \( X \) for given \( W_1 \) in one dimensional case.

**Example 5.1:** Let \( N = 3, \ n = 4 \). We consider the following \( P_1, P_2 \) and \( P_3 \) for three individuals for given \( W_1, W_2 \) and \( W_3 \).

\[
P_1 = \begin{bmatrix}
6.86 & 10.69 & 3.05 & 2.20 \\
10.69 & 16.76 & 4.6 & 4.07 \\
3.05 & 4.6 & 1.57 & 0.13 \\
2.20 & 4.07 & 0.13 & 4.2
\end{bmatrix}
\]
Eigen values and corresponding Eigen vectors of $Q_1$ are found. We take eigen vector for which

$$
\text{Eigen value } z_{11} = \sum_{j=1}^{n} x_j^2
$$

For this example we take the eigen vector corresponding eigen value $= 64.68$ for which the above condition is satisfied. Here

$$
\frac{64.68}{z_{11}} = \frac{64.68}{3.4141} = 18.94.
$$
For the eigen vector \( X = (2.18, 3.2, 56, 0.91, 1.19) \)

we have \( \sum_{j=1}^{n} x_{j}^2 = 18.94 \).

Thus, \( X \) is solution to (5.6.26) for the case \( t = 1 \).

For the Case \( t = 2 \):

In this case we have \( X_{n \times 2} = (X_1, X_2) \)

\[
W_i = \begin{bmatrix} W_{i1} & 0 \\ 0 & W_{i2} \end{bmatrix} \quad i = 1, \ldots, N
\]

\[
Q_1 = \sum_{i=1}^{N} P_i W_{i1} \quad Q_2 = \sum_{i=1}^{N} P_i W_{i2}
\]

\[
\Delta_1 = \begin{bmatrix} z_{11} & 0 \\ 0 & z_{12} \end{bmatrix} \quad \Delta_2 = \begin{bmatrix} z_{12} & 0 \\ 0 & z_{12} \end{bmatrix}
\]

From (5.6.25) we have

\[
\begin{align*}
z_{11} y_1 y_1' + z_{12} y_2 y_2' y_1 &= Q_1 y_1 \\
z_{12} y_1 y_2' + z_{22} y_2 y_2' y_2 &= Q_2 y_2
\end{align*}
\]

(5.6.27)

For solving the equations (5.6.27) we have examined methods of successive approximation and Newton-Raphson method.
Methods of Successive Approximation:

Several iterative methods were tried to solve (5.6.27). We report a few of them.

Approach 1:

Let \( \beta_1 = y_1', y_1, \quad \beta_2 = y_2', y_2 \) and

\[ \hat{\mu} = y_1', y_2 = y_2', y_1. \]

From (5.6.27) we have

\[
\begin{align*}
    z_{11} \beta_1 y_1 + z_{12} \mu y_2 &= Q_1 y_1 \\
    z_{12} \mu y_1 + z_{22} \beta_2 y_2 &= Q_2 y_2
\end{align*}
\]

(5.6.28) implies that

\[
\begin{align*}
    y_2 &= \frac{Q_1 - z_{11} \beta_1 I}{z_{12}} \cdot y_1 \\
    y_1 &= \frac{Q_2 - z_{22} \beta_2 I}{z_{12}} \cdot y_2
\end{align*}
\]

(5.6.29)

For some given values of \( y_1 = y_1^0, \ y_2 = y_2^0 \), find new \( y_1 \) and \( y_2 \) say \( y_1, \ y_2 \). Repeat the procedure by using \( y_1 \) and \( y_2 \).

Continue the procedure till it converges. For the following numerical example this method was tried but it did not converge.
Example 5.2: The numerical example of the exact case given by Shonemann (1972), was taken for testing the method. Here $N = 3$, $n = 4$. Actual values of $X$ and $W_1$, $W_2$ and $W_3$ are given as

$$X = \begin{bmatrix} 1.0 & 2.0 \\ 2.0 & 3.0 \\ 0.0 & 1.0 \\ 2.0 & 0.0 \end{bmatrix}$$

$W_1 = \text{diag}(1.0, 1.5)$, $W_2 = \text{diag}(0.7, 0.5)$, $W_3 = \text{diag}(1.3, 1.0)$.

$$P_1 = \begin{bmatrix} 7.0 & 11.0 & 3.0 & 2.0 \\ 17.0 & 4.5 & 4.5 \\ 1.5 & 0.0 \\ 4.0 & & & \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 2.7 & 4.4 & 1.0 & 1.4 \\ 7.3 & 1.5 & 2.8 \\ 0.5 & 0.0 \\ & & & 2.8 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 5.3 & 8.6 & 2.0 & 2.6 \\ 14.2 & 3.0 & 5.2 \\ 1.0 & 0.0 \\ & & & 5.2 \end{bmatrix}$$
The initial values of $y_1$ and $y_2$ were taken very close to the actual values:

$$y_1 = \begin{bmatrix} 0.99 \\ 2.00 \\ 0.00 \\ 2.00 \end{bmatrix}, \quad y_2 = \begin{bmatrix} 2 \\ 3 \\ 0.99 \\ 0 \end{bmatrix}$$

The new values of $y_1$ and $y_2$ are calculated by the method. Repeating the procedure, after ten iterations we got the values of $y_1$ and $y_2$ as

$$y_1 = \begin{bmatrix} -1.87 \times 10^{20} \\ -2.85 \times 10^{19} \\ -6.96 \times 10^{19} \\ -9.58 \times 10^{19} \end{bmatrix}, \quad y_2 = \begin{bmatrix} 8.7 \times 10^{-9} \\ 1.44 \times 10^{-8} \\ 3.045 \times 10^{-9} \\ 5.27 \times 10^{-9} \end{bmatrix}$$

Thus, the values of $y_1$ and $y_2$ are far from the actual values. Even in two successive iterations the values are not identical.

**Approach 2:**

We examine second method of successive approximations in which we take initial value of $y_1$ and find $y_2$. Then for given $y_2$, we find $y_1$ and so on. From (5.6.29), we have
(a) \[ \mu y_2 = \frac{Q_1 - z_{11} \beta_1 I}{z_{12}} y_1 \]

\[ \mu y_1^2 = y_1 \frac{(Q_1 - z_{11} \beta_1 I)}{z_{12}} y_1 \]

\[ \mu^2 = y_1 \frac{(Q_1 - z_{11} \beta_1 I)}{z_{12}} y_1 \]

For some given value of \( y_1 \), find \( \mu^2 \) and \( \mu = \pm \sqrt{\mu^2} \). Using this value of \( \mu \), we find \( y_2 \) from (5.6.30).

(b) From (5.6.29) we also have

\[ \mu y_1 = \frac{Q_2 - z_{22} \beta_2 I}{z_{12}} \cdot y_2 \] \hspace{1cm} (5.6.31)

i.e. \( \mu^2 = y_2 \frac{Q_2 - z_{22} \beta_2 I}{z_{12}} y_2 \)

Substitute the value of \( y_2 \) obtained in (a) and find new

\[ \mu = \pm \sqrt{\mu^2} \]

Hence \( y_1 \) is obtained from (5.6.31).
Repeating (a) and (b) till the procedure converges to a desired results we get \( y_1 \) and \( y_2 \). Here again, it is found that for the following numerical example the process did not converge. We consider the same numerical example 5.2. The initial value of \( y_1 \) is taken as

\[
y_1' = (1, 1.9, 0, 1.8).
\]

For this value of \( y_1' \), \( y_2 \) is obtained as

\[
y_2 = (1.93, 3.025, 0.93, 0.33).
\]

Repeating (a) and (b) as discussed above, after 30 iterations, we get

\[
y_1' = (2.246, 1.8, -0.43, -1.83)
\]

\[
y_2' = (1.33, 3.10, 1.135, 1.279).
\]

This is the example in exact case and we start with \( y_1 \) which is very close to the true value of \( y_1 \). Even then after 30 iterations resulting \( y_1 \) and \( y_2 \) are far from actual \( y_1 \) and \( y_2 \). No two iterations showed identical values. So we conclude that process does not converge to a specific point.

**Approach 3:**

In this method, we solve the equations by finding eigen vectors of the matrix which is function of \( Q_1 \), \( Q_2 \) and also of
\[ z_{11}, z_{12} \text{ and } z_{22}. \text{ From (5.6.29) we have} \]
\[
[(Q_2 - z_{22} \beta_2 I)(Q_1 - z_{11} \beta_1 I) - z_{12} \beta_2 I] y_1 = 0 \quad (5.6.32)
\]
and
\[
[(Q_1 - z_{11} \beta_1 I)(Q_2 - z_{22} \beta_2 I) - z_{12} \beta_2 I] y_2 = 0 \quad (5.6.33)
\]
(5.6.32) implies that \( y_1 \) is eigen vector of the matrix.

\[ R = Q_2 Q_1 - z_{22} \beta_2 Q_1 - z_{11} \beta_1 Q_2 \quad (5.6.34) \]

corresponding the eigen value \( \frac{z_{12} \beta_2}{z_{11} \beta_1} - z_{11} \beta_1 z_{22} \beta_2 \).

(5.6.33) implies that \( y_2 \) is eigen vector of the matrix.

\[ R = Q_1 Q_2 - z_{22} \beta_2 Q_1 - z_{11} \beta_1 Q_2 \text{ corresponding the same eigen value.} \]

If \( \beta_1 \) and \( \beta_2 \) are known, we can find \( y_1 \) and \( y_2 \) satis­

fying (5.6.32) and (5.6.33).

Initial values of \( \beta_1 \) and \( \beta_2 \) are obtained from \( \bar{P} \). We know that in exact case,

\[
tr. \bar{P} = tr. x' x' = tr. x' x
\]
\[
= \sum_{j=1}^{n} x^2_j 1 + \sum_{j=2}^{n} x^2_j 2
\]
\[
= y_1' y_1 + y_2' y_2
\]
\[
= \beta_1 + \beta_2. \quad (5.6.35)
\]
Substituting these values of \( \beta_1 \) and \( \beta_2 \), new \( y_1 \) and \( y_2 \) are found as eigen vectors of \( R \) and \( R \) such that \( y_1' y_1 = \beta_1 \) and \( y_2' y_2 = \beta_2 \) corresponding the eigen value,

\[
z_{12}^2 \mu^2 = z_{11} \beta_1 z_{22} \beta_2 = E
\]

i.e. \( \mu^2 = \frac{E + z_{11} \beta_1 z_{22} \beta_2}{z_{12}^2} \)

From the set of eigen vectors with \( y_1' y_1 = \beta_1 \) and \( y_2' y_2 = \beta_2 \) we select those for which \( (y_1' y_2)^2 = \mu^2 \). Thus solution to \((5.6.27)\) is obtained.

**Example 5.31** We take the numerical example from Shonemann's (1972) paper for fallible case. Here \( N = 3, n = 4, t = 2 \). \( \beta_1, \beta_2 \) and \( \beta_3 \) are as given as example 5.1.

We take \( W_1 = \text{diag.} (0.97, 1.5), W_2 = \text{diag} (0.73, 0.49) \), \( W_3 = \text{diag.} (1.3, 1.0) \).

From \((5.6.35)\), we have \( \beta_1 = 8.4141, \beta_2 = 14.07 \). From \((5.6.34)\) we obtain

\[
R = \begin{bmatrix}
-505.24 & -746.35 & -243.58 & -89.37 \\
-740.87 & -1163.27 & -305.70 & -362.82 \\
-249.84 & -317.81 & -162.20 & +138.01 \\
-77.52 & -348.01 & +146.94 & -856.75
\end{bmatrix}
\]
Eigen value + $z_{11} z_{22} \beta_1 \beta_2$

$\mu^2 = \frac{z_{12}^2}{z_{12}^2}$

Since $\mu$ is not a complex number, we take only those eigen values for which $\mu^2$ is positive. In this example, we have $\mu^2$ positive in the case of three eigen values, they are

<table>
<thead>
<tr>
<th>Eigen Value</th>
<th>$\mu^2$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-343.84</td>
<td>47.85</td>
<td>$\pm 6.92$</td>
</tr>
<tr>
<td>- 0.80</td>
<td>134.94</td>
<td>$\pm 11.61$</td>
</tr>
<tr>
<td>- 2.82</td>
<td>134.65</td>
<td>$\pm 11.60$</td>
</tr>
</tbody>
</table>

Now we take those eigen vectors for which $y_1'y_1 = \beta_1 = 8.41$. Then we find eigen vectors of $R$ and take those eigen vectors for which $y_2'y_2 = \beta_2$. (Eigen values of $R$ and $R'$ are the same).

Following are the eigen vectors for which $y_1'y_1 = 8.41$, $y_2'y_2 = 14.07$. 
\[ y_1 \quad y_2 \]
\begin{align*}
(1) & \quad -0.7401 & -0.9131 \\
& \quad -0.3363 & -0.3617 \\
& \quad -0.9575 & -1.2231 \\
& \quad 2.6143 & +3.4075 \\
(2) & \quad -2.2308 & -2.8883 \\
& \quad 0.9879 & -1.2746 \\
& \quad 1.5676 & 2.0242 \\
& \quad 0.0694 & 0.0876 \\
(3) & \quad 0.924 & 1.1884 \\
& \quad -1.4375 & -1.8713 \\
& \quad 2.1741 & 2.81 \\
& \quad 0.8761 & 1.125 \\
\end{align*}

From these set of \( y_1 \) and \( y_2 \) vectors we select those ones for which \( y_1 y_2 = \mu \). We find that for the third vector \( y_1 \) and \( y_2 \). We have

\[ y_1' y_2 = 11.52 \]

which is approximately equal 11.60. Therefore the solution is

\[ X = \begin{bmatrix} 0.924 & 1.188 \\ -1.437 & -1.871 \\ 2.174 & 2.810 \\ 0.876 & 1.125 \end{bmatrix} \]

for given \( W_1 \) as before.
5.7. Algorithm to Find X and \( W_i \) and Examples*

The complete Algorithm for solving the problem given by (5.2.3) and (5.2.4) can be given as follows.

1. Find \( \overline{P} = \left( \sum_{i=1}^{N} P_i \right) / N \) and then gram factors of \( \overline{P} \) as
   \[ \overline{P} = X_0 X_0' \],
   take gram factor \( X_0 \) of \( \overline{P} \) as initial value of \( X \).
   i.e. \( X = X_0 \).

2. For the given \( X_0 \) find
   \[ W = \begin{bmatrix} W_{11} & \cdots & W_{1t} & W_{21} & \cdots & W_{2t} & \cdots & W_{nt} \end{bmatrix} \]
   by quadratic programming method available as given in Phase A. [Here first \( t \)-elements are for first individual indicating elements of \( W_{ij} = \text{diagonal}(W_{ij}, \ldots, W_{ij}) \).
   Then for second individual and so on]. Suppose \( W = W_0 \) is the solution.

3. Using the solution in (2) i.e. \( W_0 \) find \( X \) by solving the equations in (5.6.27) by Newton-Raphson method.

4. Repeat the Steps (2) and (3) till process converges to a desired accuracy.

In the previous section, we have given a few approaches to find \( X \) for given \( W_1 \) for \( t = 1 \) and \( t = 2 \) cases.
In this section we give an example of finding $X$ and $W_i$ from $P_i$. That is, we give examples for $t = 1$ and $t = 2$ which includes iterations of Phase A and Phase B. Actually we take example from Shonemann's paper for fallible data in the case of $t = 2$.

He has given the example with $N = 3$, $n = 3$. For the same $P_1$, $P_2$, and $P_3$ we find one-dimensional and two dimensional configurations.

To Obtain One Dimensional Configuration:

$N = 3$, $n = 4$. $P_1$, $P_2$ and $P_3$ are given in Example 5.1 in Section 5.6.

**Phase A:**

Initial value of $X$ is taken as

$$X = \begin{bmatrix} 2.1 \\ 3.2 \\ 1.0 \\ 1.07 \end{bmatrix}$$

and $W_{11}$, $W_{21}$ and $W_{31}$ are found by using method to quadratic programming problem as

$$W_{11} = 1.3891, \quad W_{21} = 0.4994, \quad W_{31} = 1.1115$$

$$W_{11} + W_{21} + W_{31} = 3.0000.$$
Phase B:
Using the above values of $W_{11}$, $W_{21}$ and $W_{31}$, $Q_1 = \frac{3}{\Sigma P_i W_{i1}}$ is calculated.

The eigen values and corresponding eigen vectors of $Q_1$ are found. We select the eigen vector for which

$$\frac{\text{Eigen Value}}{3 \Sigma W_{i1}^2} = \frac{1}{\Sigma x_j^2}$$

In this example we have

$$\frac{64.68}{3.4144} = 18.94$$

Corresponding this Eigen value, the eigen vector is

$$X = \begin{bmatrix} 2.18 \\ 3.456 \\ 0.91 \\ 1.19 \end{bmatrix}$$

$$\Sigma x_j^2 = 18.94.$$ 

Repeating phase A and phase B, after some iterations, the values of $W_{i1}$ and $X$ remained the same. Thus the solution is

$$X = \begin{bmatrix} 2.197 \\ 3.482 \\ 0.911 \\ 1.202 \end{bmatrix}$$
To Obtain Two Dimensional Configuration:
For the same example, we now find the configuration on two dimensions and compare the solution with the solution obtained by Shonemann (1972).

Phase A:
We find $P$ and its gram factors to take as initial value of $X$. Here

$$X = \begin{bmatrix}
-0.43 & 2.18 \\
-0.18 & 3.48 \\
-0.58 & 0.87 \\
1.03 & 1.27
\end{bmatrix}$$

With this initial value we find $W_1 : 2 \times 2$, $W_2 : 2 \times 2$, $W_3 : 2 \times 2$ matrices of weights attached to dimensions for three individuals

$$W_1 = \text{diag}(1.13, 1.33), \quad W_2 = \text{diag}(0.63, 0.57), \quad W_3 = \text{diag}(1.24, 1.10).$$

Phase B:
Using the above values of $W_i$ ($i = 1, 2, 3$) we find configuration $X$ by solving the homogeneous equations given in (5.6.27) by Newton-Raphson method as

$$W_{11} = 1.34, \quad W_{21} = 0.56, \quad W_{31} = 1.1$$

$\text{STRAIN} = 45.65.$
Solution obtained by Shonemann is given by

\[
X = \begin{bmatrix}
.596 & 2.158 \\
1.374 & 3.217 \\
-0.115 & 1.0485 \\
2.049 & 0.366 \\
\end{bmatrix}
\]

Repeating Phase A and Phase B we notice that after thirty iterations, the values of \( w_1 \) and \( X \) remain the same up to two decimal points. The final solution for \( X \) and \( w_1 \) \((i = 1, 2, 3)\) is

\[
X = \begin{bmatrix}
0.90 & 2.03 \\
1.81 & 2.98 \\
0.64 & 1.03 \\
2.03 & 0.10 \\
\end{bmatrix}
\]

\[
w_1 = \begin{bmatrix}
0.97 & 0 \\
0 & 1.50 \\
\end{bmatrix}
\]

\[
w_2 = \begin{bmatrix}
0.73 & 0 \\
0 & 0.49 \\
\end{bmatrix}
\]

\[
w_3 = \begin{bmatrix}
1.3 & 0 \\
0 & 1.0 \\
\end{bmatrix}
\]

\[
\text{STRAIN} = \text{tr} \left( \frac{1}{3} \sum_{i=1}^{3} (P_i - X w_i X')^2 \right)
\]

\[
= 0.9442
\]

Solution obtained by Shonemann is given by

\[
X = \begin{bmatrix}
0.91 & 2.03 \\
1.86 & 2.95 \\
0.02 & 1.04 \\
2.07 & 0.1 \\
\end{bmatrix}
\]
Here in the case of $W_2$ and $W_3$, the non-diagonal elements are small and non-zero instead of exactly zero. If we take them as zero then the STRAIN value corresponding to Shonemann's solution is

$$\text{STRAIN} = 1.2194.$$ 

Thus STRAIN obtained by the new proposed method is considerably smaller than the one given by Shonemann (1972). It is to be recalled that Shonemann proposed the solution for the exact case as an approximation to the fallible case. The approach suggested in our work is applicable to both exact and fallible cases. In fact, one does not know to start with whether one has the exact case or the fallible case.

The thesis contains the listing of two programmes:

(i) DISTNC, for obtaining $P_i$, $i = 1, \ldots, N$ matrices from observations.

(ii) NEWTON, Newton-Raphson method for solving equations given by (5.6.27) for $t = 2$. 

\[ W_1 = \begin{bmatrix} 0.98 & 0 \\ 0 & 1.51 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0.7 & 0.02 \\ 0.02 & 0.47 \end{bmatrix}, \quad W_3 = \begin{bmatrix} 1.32 & -0.02 \\ -0.02 & 1.02 \end{bmatrix} \]