Chapter 2

Fuzzy Mathematical Models for the Analysis of Fuzzy Systems with Application to Liver Disorders

The main objective of this Chapter is to focus on how to use the model of fuzzy system to solve fuzzy mathematics problems. Some mathematical models based on fuzzy set theory, fuzzy systems and neural network techniques seem very well suited for typical technical problems. This Chapter proposed an extension model of a fuzzy system to $N$-dimension, using Mamdani’s minimum implication, the minimum inference system, and the singleton fuzzifier with the center average defuzzifier. Here construct two different models namely a fuzzy inference system and an adaptive fuzzy system using neural network. Further, extended a theorem for accuracy of the fuzzy system to $N$-dimensions, and provides a medical application of the fuzzy mathematics models. Since, liver is the largest internal member in the human body, so diagnosing liver disorder disease is a high interest to researchers of the fuzzy modeling and the fuzzy system. Therefore, the fuzzy mathematical models are applied on a real data to the Liver Disorder disease. Consequently, a comparison between three models: the FS with Mamdani model, the ST model, and the ANFIS is obtained the best result with the ANFIS. Finally, the programs of these models by using MATLAB created and performed.

This Chapter is based on the papers:


2.1 INTRODUCTION

Fuzzy mathematics provides the starting point and basic language for fuzzy systems and fuzzy modeling (FM), [Ruan and Wang (1997)], while the fuzzy mathematical principles are developed by replacing the sets in classical mathematical theory with fuzzy sets (FSs), [Samandar (2011)]. The concepts and principles in fuzzy mathematics are useful in FSs and adaptive neuro-fuzzy systems (ANFSs), [Singh et al. (2009)], [Shing and Jang (1993)]. Fuzzy variables are processed using a system called a fuzzy inference system (FIS) which involves fuzzification, fuzzy inference, and defuzzification, [Woo et al. and Wang (1998)], [Aik and Jayakumar (2008)]. The FIS collects the rules in the fuzzy rule-base into a mapping from fuzzy set $\tilde{A} \in X$ to fuzzy set $\tilde{B} \in Y$, [Sivanandam and Deepa (2008)].

We must construct interfaces that are the fuzzifier and defuzzifier, between the FIS and the environment because in most applications the input and output of the FS are real valued numbers such our application in this model to Liver Disorders, [Sug (2012) and Gulia et al. (2014)]. The reason to represent a fuzzy system in terms of a neural network is utilize the learning ability of neural networks for improve the performance, such as adaptation of FIS, [kumar and Arumugam (2011)]. If the expert is demonstrating, then measure the inputs and the outputs, that means collects a set of input-output data pairs, [Nayak (2004)], [Hndoosh et al. (2012) and (2013)]. Therefore, the knowledge is transformed into a set of input-output pairs. The task of this Chapter is model a FIS, which describes the input-output behavior that represented by the input-output pairs, and applies the model to Liver Disorders. We will model the FIS by assigning its structure, then tuning its parameters, [Jose et al. (1999)]. To simulate the modeling system, need a mathematical model of the Liver Disorders that is described by linguistic variables and its membership functions (MFs), [Chai et al. (2009)]. Note that the fuzzy modeler can successfully control and handle the real data of
any problem. As well as, the prediction accuracy is improved by defining more FSs for each input variable, [Marza and Seyyedi (2008)]. The advantage of using the FS is the parameters of MFs have clear physical meanings and we have models to choose good initial values for them, [Doğan et al. (2007)]. We can improve the fuzzy if-then rules that model the FS, [Belohlavek and Klir (2011)]. These improved of fuzzy if-then rules may help to demonstrate the model of FS in a user-friendly manner. This Chapter divided into eight Sections. Section 2.1 introduces the fundamental concepts and principles in the general field of fuzzy theory that are particularly useful of the FSs and ANFIS, [Sivanandam and Deepa (2008)]. The second Section introduces basic concepts of fuzzy rule-base with many types of fuzzy rules, [Doğan et al. (2007)], [Jandaghi et al. (2010)]. Section 2.3 explains the general concepts of the fuzzy inference system, that is composition based inference and individual rule-based inference with the computational procedure of them, [Hndoosh et al. (2012) and (2013)], [Shing and Jang (1993)]. Section 2.4 shows the detailed mathematical formulas of the FIS types, [Woo et al. and Wang (1998)], [Ruan and Wang (1997)], [Rameshkumar and Arumugam (2011)], and [Sug (2012)]. Section 2.5 constructs the interfaces between the FIS and the environment using fuzzifier and defuzzifier models, [Chai and Zhang (2009)], [Jandaghi et al. (2010)]. Section 2.6 divided into three parts, the first part proposes and extends the model of the FIS from 2-dimensions to N-dimensions using Mamdani’s minimum implication with the minimum inference system, the singleton fuzzifier, and center average defuzzifier, [Rojas (1996)]. The second part proves the extension theorem for accuracy of the proposed model, [Hndoosh et al. (2012) and (2013)]. This approach requires N-pieces of information for modeling a FIS that satisfying any pre-specified degree of accuracy, [Kamel and Hassan (2009)]. As well as, adapted the structure of the FIS and modeled of an adaptive FIS using a neural network.
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through the third part, [Singh et al. (2009)], [Shing, and Jang (1993)].
Section 2.7 applies all the previous concepts and models with a real
application to Liver Disorders, [Sug (2012) and Gulia et al. (2014)], and we
have structured of the applied model at the first part. Second and third
subsection provide discussion and results of the models of the FIS with
Mamdani and ST models, thus the adaptive FIS using neural network,
respectively. Consequently, obtained good results of the models, and
created programs for the different models using ‘MATLAB’. Concluding
remarks are present through Section 2.8.

2.2 FUZZY RULE-BASE

A fuzzy rule-base consists of a set of fuzzy if-then rules. Specifically, the
fuzzy rule base comprises the following fuzzy if-then rules [4]:

\[ R^{(i)}_x : \text{if } x_1 \text{ is } A^i_1 \text{ and } \ldots \text{ and } x_n \text{ is } A^i_n \text{ then } y \text{ is } B^i \]  

(2.1)

where \( A^i_j \) and \( B^i \) are fuzzy set in \( X_j \subset \mathbb{R} \) and \( Y \subset \mathbb{R} \), respectively, \( x = (x_1, x_2, \ldots, x_n)^T \in X = (X_1 \times X_2 \times \ldots \times X_n) \subset \mathbb{R}^n \) and \( y \in Y \subset \mathbb{R} \) are the
input and output linguistic variables of the fuzzy system, respectively. Let \( I \)
be the number of rules of the fuzzy rule-base; that is, \( i = 1, 2, \ldots, I \) in (2.1).

We call the rules in the form of (2.1) canonical fuzzy if-then rules because
they include many other types of fuzzy rules as, [Jandaghi et al. (2010)],
[Shing and Jang (1993)]:

a) Partial rules:

\[ R^{(i)}_U : \text{if } x_1 \text{ is } A^i_1 \text{ and } \ldots \text{ and } x_m \text{ is } A^i_m \text{ then } y \text{ is } B^i, \text{ where } m < n. \]  

(2.2)

b) Or rules:

\[ R^{(i)}_U : \text{if } x_1 \text{ is } A^i_1 \text{ and } \ldots \text{ and } x_m \text{ is } A^i_m \text{ or } x_{m+1} \text{ is } A^i_{m+1} \text{ and } \ldots \text{ and } x_n \text{ is } A^i_n \text{ then } y \text{ is } B^i, \]  

(2.3)

c) Statement of single fuzzy:

\[ y \text{ is } B^i, \]  

(2.4)

Gradual rules, as:
The colder the $x$, the hotter the $y$, \hspace{2cm} (2.5)

d) Non-fuzzy rules:

If the membership functions of $A^i_j$ and $B^i$ can only take values one or 0, then the rules (2.1) become non-fuzzy rules.

The fuzzy rule-base consists of a set of rules. We can introduce the following concepts:

1. A set of fuzzy if-then rules is complete, if for any $x \in X$, there exists at least one rule in the fuzzy rule base, say rule $R^{(i)}_X$, such that:

$$\mu_{A^i_j}(x_j) \neq 0, \forall j = 1, 2, \ldots, n.$$ \hspace{2cm} (2.6)

That means, at any point in the input space there is at least one rule that "fires"; that is, the membership value of the “if” part of the rule at this point is non-zero.

2. A set of fuzzy if-then rules is consistent if there are no rules with the same if parts but different then parts, [Kumar and Arumugam (2011)].

3. A set of fuzzy if-then rules is continuous if there do not exist such neighboring rules whose then part fuzzy sets have empty intersection. That means the input-output behavior of the fuzzy system should be smooth.

2.3 GENERAL CONCEPTS OF FUZZY INFERENCE SYSTEM

In a fuzzy inference system, fuzzy logic principles are used to combine the fuzzy if-then rules in the fuzzy rule-base into a mapping as:

$$FIS: A' \rightarrow B', A' \in X \text{ and } B' \in Y,$$ \hspace{2cm} (2.7)

There are two ways to infer with a set of rules: composition based inference and individual-rule based inference. We now show the details of these two ways; In composition based inference, all rules in the fuzzy rule-base are combined into a single fuzzy relation in $X \times Y$, which is then viewed as a single fuzzy if-then rule. There are two opposite arguments for what a set of rules should mean. The first one views the rules as independent
conditional statements, [Chai and Zhang (2009)], [Doğan et al. (2007)]. If we accept this view, then a reasonable operator for combining the rules is union. The second one views the rules as strongly coupled conditional statements. If we adapt this view, then we should use the operator intersection to combine the rules. We now show the details of these two schemes:

Let \( R_X^{(i)} \) be a fuzzy relation in \( X \times Y \), which represents the fuzzy if-then rule (2.1); that is,

\[
R_X^{(i)} = A_1^i \times \ldots \times A_n^i \rightarrow B^i, 
\]

We know that \( A_1^i \times \ldots \times A_n^i \) is a fuzzy relation in \( X = X_1 \times \ldots \times X_n \) defined by:

\[
\mu_{A_1\times\ldots\times A_n^i}(x_1, \ldots, x_n) = \mu_{A_1^i}(x_1) \ast \ldots \ast \mu_{A_n^i}(x_n), \quad (2.8)
\]

where \( \ast \) represents any \( t \)-norm operator. The implication \( A \rightarrow \text{im} R_X^{(i)} \), is defined according to various implications.

If we accept the first view of a set of rules, then the \( I \) rules in the form of (2.1) are interpreted as a single fuzzy relation \( Q_I \in X \times Y \) defined by:

\[
Q_I = \bigcup_{i=1}^I R_X^{(i)} \quad (2.9)
\]

This combination is called the \textit{Mamdani} combination. If we use the symbol \( \oplus \) to represent the \( s \)-norm (\( s \)-norm is used to represent a union between two fuzzy sets), then (2.9) can be re-written as:

\[
\mu_{Q_I}(x, y) = \mu_{R_X^{(1)}}(x, y) \oplus \ldots \oplus \mu_{R_X^{(i)}}(x, y) \quad (2.10)
\]

For the second view of a set of rules, the \( I \) fuzzy if-then rules of (2.1) are explained as a fuzzy relation \( Q_J \in X \times Y \), which is defined as:

\[
\mu_{Q_J}(x, y) = \mu_{R_X^{(1)}}(x, y) \ast \ldots \ast \mu_{R_X^{(i)}}(x, y) \quad (2.11)
\]

or equivalently,

\[
Q_J = \bigcap_{i=1}^I R_X^{(i)} \quad (2.12)
\]

where \( \ast \) denotes \( t \)-norm (\( t \)-norm is used to represent an intersection of two fuzzy sets). This combination is called the \textit{Godel} combination. Let \( A' \) be an
arbitrary fuzzy set in $X$, and be the input of the fuzzy inference system then by viewing $Q_I$ or $Q_J$ as a single fuzzy if-then rule and using the generalized rule that is defined as:

$$
\mu_B(y) = \frac{\sup_{x \in X} \{\mu_A(x) : \mu_{A \rightarrow B}(x, y)\} \times \mu_{Q_I}(x, y)},
$$

(2.13)

where $A'$ is a fuzzy set, which represents the input of rule $x$ is $A'$, $\times$ can be any operator, $A \rightarrow B (\in X \times Y)$ is a fuzzy relation that represents the antecedent of rule “if $x$ is $A$ then $y$ is $B$”, and $B' \in Y$ is a fuzzy set that represents the consequent of rule $y$ is $B'$, therefore, we obtain the output of the FIS as:

$$
\mu_B(y) = sup_{x \in X} \{\mu_A(x) : \mu_{Q_I}(x, y)\},
$$

(2.14)

when, we use the Mamdani combination, $\times$ $t$-norm operator, or as:

$$
\mu_B(y) = sup_{x \in X} \{\mu_A(x) : \mu_{Q_J}(x, y)\}
$$

(2.15)

when, we use the Godel combination, and $\div$ $s$-norm operator.

In summary, the computational procedure of the composition-based inference is given as follows [Hndoosh et al. (2012) and (2013)], [Woo et al., and Wang (1998)]:

**Step 1:** For the $I$ fuzzy if-then rules in (2.1), determine the MFs as:

$$
\mu_{A_1 \times \ldots \times A_n}(x_1, \ldots, x_n) \text{ for } i = 1, \ldots, I \text{ according to (2.8)}.
$$

**Step 2:** let $FP_1$ and $FP_2$ are fuzzy propositions, $A_1^{i} \times \ldots \times A_n^{i}$ as the $FP_1$ and $B^i$ as the $FP_2$ in the implications, thus determine the MFs as:

$$
\mu_{R_X^i}(x_1, \ldots, x_n, y) = \mu_{A_1^{i} \times \ldots \times A_n^{i} \rightarrow B^i}(x_1, \ldots, x_n, y) \text{ for } i = 1, 2, \ldots, I
$$

according to any one of implications.

**Step 3:** Determine $\mu_{Q_I}(x, y)$ or $\mu_{Q_J}(x, y)$ according to (2.10) or (2.11).

**Step 4:** For given input $A'$, the FIS gives output $B'$ according to (2.14) or (2.15).

In individual-rule based inference, each rule in the fuzzy rule base determines an output fuzzy set and the output of the completely fuzzy
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inference system is the combination of the \( I \) individual fuzzy sets. The combination can be taken either by union or by intersection. The computational procedure of the individual-rule based inference is summarized as follows:

**Steps 1 and 2:** Same as the Steps 1 and 2 for the composition based inference.

**Step 3:** For given input fuzzy set \( A' \in X \), compute the output fuzzy set \( B_i' \in Y \) for each individual \( R_X^{(i)} \) according to the generalized rule (2.13), that is,

\[
\mu_{B_i'}(y) = \sup_{x \in X} \{ \mu_{A_i'}(x) \land \mu_{R_X^{(i)}}(x, y) \}, \text{ for } i = 1, 2, ..., I
\]  

(2.16)

**Step 4:** The output of the FIS is the combination of the \( I \) fuzzy sets \( \{ B_1', ..., B_I' \} \) either by union, that is:

\[
\mu_B(y) = \mu_{B_1'}(y) + ... + \mu_{B_I'}(y)
\]  

(2.17)

or by intersection, that is,

\[
\mu_B(y) = \mu_{B_1'}(y) \ast ... \ast \mu_{B_I'}(y)
\]  

(2.18)

where \( \land \) and \( \ast \) denote \( s\)-norm and \( t\)-norm operators, respectively.

In fuzzy inference, the decision is based on the input information and the knowledge in the form of rules. The inference model is classified based on the form of the consequent. There are two types of fuzzy rules:

1. **Mamdani fuzzy rules:**

   If the rule has fuzzy suggestions that combined by fuzzy connectives in the antecedent and in the consequent, then it is called a Mamdani fuzzy rule that denoted by (2.1). The fuzzy inference of the Mamdani fuzzy rule is done as follows. First, the truth-value of the rule is evaluated depending on the fuzzy connections (and/or) in the if part (antecedent). Since our model has “\( \text{and} \)” only, then the truth-value evaluation is done by [4]:

\[
\vartheta_i = t \left( \mu_{A_1^{(i)}}(x_1), \mu_{A_2^{(i)}}(x_2), ..., \mu_{A_n^{(i)}}(x_n) \right),
\]  

(2.19)
Then the fuzzy set of the output is obtained depending on the truth-value and the fuzzy set of the consequent by the \textit{t-norm} (min). The fuzzy inference produces the fuzzy set of output by:

\[
\mu_{\overline{B}_i}(y) = t(\vartheta_i, \mu_{B_i}(y)), \forall y \in R. \tag{2.20}
\]

The consequents of all the rules are aggregated in the final consequent by the \textit{s-norm} (max) as:

\[
\mu_{\overline{B}}(y) = s(\mu_{B_1}(y), \mu_{B_2}(y), ..., \mu_{B_I}(y)), \tag{2.21}
\]

where \(I\) is a number of the rules, and \(\vartheta_i, \mu_{\overline{B}_i}, \mu_{B_i}\) are functions of the inputs.

2. Sugeno fuzzy rules:

When the consequent of the rule is a linear function of the inputs, then it is called a \textit{Sugeno} fuzzy rule. The consequent of (2.1) becomes:

\[
R_X^{(i)}: \text{if } x_1 \text{ is } A_1^i \text{ and ... and } x_n \text{ is } A_n^i \text{ then } y \text{ is } f_i(x_1, x_2, ..., x_n) \tag{2.22}
\]

The output of the \textit{Sugeno} rule is a crisp value of the function. The final output of the fuzzy inference is obtained by a weighted mean as:

\[
y = \frac{\sum_{i=1}^{I} \vartheta_i f_i(x_1, x_2, ..., x_n)}{\sum_{i=1}^{I} \vartheta_i} \tag{2.23}
\]

where \(I\) is a number of the rules and \(\vartheta_i\) is the truth value of the antecedent obtained by (2.19).

2.4 TYPES OF FUZZY INFERENCE SYSTEMS

In this Section, we will show the detailed formulas of the different types of a FIS that are commonly used in fuzzy systems, [Aik and Jayakumar (2008)], [Nayak et al. (2004)], [Rojas (1996)].

1. \textbf{Product Inference System:} In product inference system, we use:

   (i) Individual rule-based inference with union combination (2.17),

   (ii) Mamdani’s product implication,

   \[
   \mu_{QIP}(x, y) = \mu_{A_1}(x) \cdot \mu_{A_2}(y), \; QIP \in X \times Y, \tag{2.24}
   \]

   (iii) Algebraic product for all the \textit{t-norm} operators, and \textit{max} for all the \textit{s-norm} operators.
Specifically, from (2.16), (2.17), (2.24), and (2.8), we may be obtained the product inference system as

\[
\mu_B(y) = \max_{\forall i} \left[ \sup_{x \in X} \mu_A(x) \left( \prod_{j=1}^{n} \mu_{A_j}(x_i) \mu_{B_i}(y) \right) \right]
\]  

(2.25)

That is, given fuzzy set \( A' \in X \), the product inference system gives the fuzzy set \( B' \in Y \).

2. **Minimum Inference System:** In minimum inference system, we use:
   
   (i) Individual rule-based inference with union combination (2.17),
   
   (ii) Mamdani’s minimum implication,
   
   \[
   \mu_{Q_{IM}}(x, y) = \min[\mu_{A_1}(x), \mu_{A_2}(y)], Q_{IM} \in X \times Y
   \]  

(2.26)

(iii) \( \min \) for all the \( t\)-norm operators, and \( \max \) for all the \( s\)-norm operators.

Specifically, from (2.16), (2.17), (2.26), and (2.8) we have

\[
\mu_B(y) = \max_{\forall i} \left[ \sup_{x \in X} \left( \min \left( \mu_A(x), \mu_{A_j}(x_i), \ldots, \mu_{A_n}(x_n), \mu_{B_i}(y) \right) \right) \right]
\]  

(2.27)

That is, given fuzzy set \( A' \in X \), the minimum inference system gives the fuzzy set \( B' \in Y \) according to (2.27).

3. **Lukasiewicz Inference System:** In Lukasiewicz inference system, we use:
   
   (i) Individual rule-based inference with intersection combination (2.18),
   
   (ii) Lukasiewicz implication:
   
   \[
   \mu_{Q_L}(x, y) = \min[1, 1 - \mu_A(x) + \mu_B(y)], Q_L \in X \times Y
   \]  

(2.28)

(iii) \( \min \) for all the \( t\)-norm operators.

Specifically, from (2.18), (2.16), (2.28) and (2.8), we obtain:

\[
\mu_B(y) = \min_{\forall i} \left[ \sup_{x \in X} \min \left( \mu_A(x), \mu_{R_{\chi}^{(0)}}(x, y) \right) \right]
\]

\[
= \min_{\forall i} \left( \sup_{x \in X} \min \left( \mu_A(x), \min \left( 1, 1 - \min_{\forall j} \left( \mu_{A_j}(x_i) \right) + \mu_{B_i}(y) \right) \right) \right)
\]

\[
= \min_{\forall i} \left( \sup_{x \in X} \min \left( \mu_A(x), 1 - \min_{\forall i} \left( \mu_{A_j}(x_i) \right) + \mu_{B_i}(y) \right) \right)
\]  

(2.29)
That is, for given fuzzy set $A' \in X$, the Lukasiewicz inference system gives the fuzzy set $B' \in Y$.

4. **Zadeh Inference System**: In Zadeh inference system, we use:
   (i) Individual rule-based inference with intersection combination (2.18).
   (ii) Zadeh implication:
   $$\mu_{QZ}(x, y) = \max\left[\min(\mu_A(x), \mu_B(y)), 1 - \mu_A(x)\right], QZ \in X \times Y \quad (2.30)$$
   (iii) $\min$ for all the $t$-norm operators.

Specifically, from (2.18), (2.16), (2.30), and (2.8) we obtain:
$$\mu_B(y) = \min_{\forall i} \left\{ \sup_{x \in X} \left( \mu_A(x) \max\left( \min\left( \mu_{A_1}(x_1), \ldots, \mu_{A_i}(x_1), \mu_B(y) \right) \right) \right) \right\} \quad (2.31)$$

5. **Dienes-Rescher Inference System**: In Dienes-Rescher inference system, we use:
   (i) The same operations as in the Zadeh inference system.
   (ii) Dienes-Rescher implication.
   $$\mu_{QD}(x, y) = \max\left[1 - \mu_A(x), \mu_B(y)\right], QD \in X \times Y \quad (2.32)$$
   (iii) $\min$ for all the $t$-norm operators.

Specifically, we obtain from (2.18), (2.16), (2.32), and (2.8) that:
$$\mu_B(y) = \min_{\forall i} \left\{ \sup_{x \in X} \min\left[ \mu_A(x), \max\left(1 - \min_{\forall j} \mu_{A_j}(x_j), \mu_B(y)\right)\right] \right\} \quad (2.33)$$

2.5 **FUZZIFIER AND DEFUZZIFIER MODELS**

In most applications, the input and output of the fuzzy system are real valued numbers. Therefore, we must construct interfaces between the fuzzy inference system and the environment, [Sug (2012)], [Nayak et al. (2004)], [Rojas (1996)], [Sivanandam and Deepa (2008)]. The interfaces are the fuzzifier and defuzzifier as the following:

2.5.1 **The Fuzzifier**

The fuzzifier is defined as:
$$f: x^* \rightarrow A', \text{where } x^*(\text{real} - \text{valued point}) \in X \subset R^n, A' \in X \quad (2.34)$$

There are some criteria of designing the fuzzifier such as the following:
a. The fuzzifier should consider the fact that the input is at the crisp point \( x^* \), that is, the fuzzy set \( A' \) should have large membership value at \( x^* \).

b. If the input to the fuzzy system is corrupted by noise, then it is desirable that the fuzzifier should help to suppress the noise.

c. The fuzzifier should help to simplify the computations involved in the fuzzy inference system.

Now, we can introduce three types of the fuzzifiers:

(i) **Singleton fuzzifier**:

The Singleton fuzzifier mapping is defined as:

\[
S_f : (x^* \in X) \rightarrow (A' \in X),
\]

where \( x^* \) is a real-valued point and \( A' \) is a fuzzy Singleton, is defined as:

\[
\mu_{A}(x) = \begin{cases} 1 & \text{if } x = x^* \\ 0 & \text{otherwise} \end{cases}
\]

(ii) **Gaussian fuzzifier**:

The Gaussian fuzzifier mapping is defined as:

\[
G_f : (x^* \in X) \rightarrow (A' \in X),
\]

where \( A' \) is a fuzzy set, it has the following Gaussian MF:

\[
\mu_{A}(x) = \exp \left( -\left( \frac{x_1-x_1^*}{\alpha_1} \right)^2 \right) \ast \ldots \ast \exp \left( -\left( \frac{x_n-x_n^*}{\alpha_n} \right)^2 \right),
\]

where \( \alpha_j \) are positive parameters and the \( t\)-norm \( \ast \) is usually chosen as an algebraic product or \( \text{min} \).

(iii) **Triangular fuzzifier**:

The triangular fuzzifier mapping is defined as:

\[
T_f : (x^* \in X) \rightarrow (A' \in X),
\]

where \( A' \) is a fuzzy set that has the following triangular MF:

\[
\mu_{A'}(x) = \begin{cases} 
\left(1 - \frac{|x_1-x_1^*|}{\alpha_1}\right) \ast \ldots \ast \left(1 - \frac{|x_n-x_n^*|}{\alpha_n}\right) & \text{if } |x_j - x_j^*| \leq \alpha_j, \\
0 & \text{otherwise}
\end{cases}
\]

where \( \alpha_j \) are positive parameters and the \( t\)-norm \( \ast \) is usually chosen as an algebraic product or \( \text{min} \).
2.5.2 The Defuzzifier

The defuzzifier is defined as:

def: $B' \to y^*$,

where $B' \in Y \subseteq R$ is the output of the fuzzy inference system, $y^* \in Y$ the crisp point. The task of the defuzzifier is specifying a point in $Y$ that is best representing of the fuzzy set $B'$. For all the defuzzifiers, we assume that the fuzzy set $B'$ is obtained from one of the five fuzzy inference systems in the last Section, that is, $B'$ is given by (2.25), (2.27), (2.29), (2.31), or (2.33), [Chai and Zhang (2009)], [Jandaghi et al. (2010)]. From these equations, we note that $B'$ is the union or intersection of $I$ individual fuzzy sets. There are three types of the defuzzifiers as follows:

(i) The Center of Gravity Defuzzifier: The center of gravity defuzzifier specifies the $y^*$ as the center of the area covered by the membership function of $B'$, that is:

$$y^* = \frac{\int_Y y \mu_{B'}(y) \, dy}{\int_Y \mu_{B'}(y) \, dy}$$

(2.38)

where $\int_Y$ is the conventional integral.

In fact, the membership function $\mu_B(y)$ is usually irregular, therefore the integrations in (2.38) are difficult to compute. The next defuzzifier tries to solve this problem with a simpler formula.

(ii) The Center Average Defuzzifier: Because the fuzzy set $B'$ is the intersection or union of $I$ fuzzy sets, a good approximation of (2.38) is the weighted average of the centers of the $I$ fuzzy sets, where the weights equal the heights of the corresponding fuzzy sets. Specifically, let $\bar{y}^i$ be the center of the $i^{th}$ fuzzy set and $w_i$ be its height. Then, the center average defuzzifier determines $y^*$ as:

$$y^* = \frac{\sum_{i=1}^{I} \bar{y}^i w_i}{\sum_{i=1}^{I} w_i}$$

(2.39)
Chapter 2

The center average defuzzifier is the most commonly used defuzzifier in fuzzy systems. It is computationally simple and small changes in $\bar{y}^i$ and $w_i$ result in small changes in $y^\ast$.

(iii) The Maximum Defuzzifier: The maximum defuzzifier chooses the $y^\ast$ as the point in $Y$. We can be defined this type as:

$$hgt(B') = \left\{ y \in Y | \mu_{B'}(y) = \sup_{y \in Y} \mu_{B'}(y) \right\}, \quad (2.40)$$

where $hgt(B')$ is the set of all points in $Y$ at which $\mu_{B'}(y)$ satisfies its maximum value. The maximum defuzzifier, $y^\ast$, defines as an arbitrary element in $hgt(B')$, that is:

$$y^\ast = \text{any point in } hgt(B') \quad (2.41)$$

If $hgt(B')$ contains a single point, then $y^\ast$ is uniquely defined. If $hgt(B')$ contains more than one point, then we may still use (2.40) or use the smallest of maxima, the largest of maxima, or the mean of maxima defuzzifiers. Specifically, the smallest of maxima defuzzifier gives as:

$$y^\ast = \inf \{ y \in hgt(B') \}, \quad (2.42)$$

The largest of maxima defuzzifier gives:

$$y^\ast = \sup \{ y \in hgt(B') \}, \quad (2.43)$$

while, the mean of maxima defuzzifier is defined as:

$$y^\ast = \frac{\int_{hgt(B')} y \, dy}{\int_{hgt(B')} dy} \quad (2.44)$$

where $\int_{hgt(B')}$ is the usual integration for the continuous part of $hgt(B')$ and is the summation for the discrete part of $hgt(B')$. The maximum defuzzifier is computationally simple, but small changes in $B'$ may result in large changes in $y^\ast$. 
2.6 PROPOSAL A GENERAL MODEL OF A FUZZY SYSTEM ON N-DIMENSIONS

This Section proposes a general model of a fuzzy system, that is extension of the work that introduced by Hndoosh et al. (2013) from 2-dimensions to N-dimensions. Consider the general membership function of fuzzy set, $A$, is a continuous function in $R$ given by:

$$
\mu(x; a, b, c, d) = \begin{cases} 
0 & \text{if } x < a \\
 a(x) & \text{if } a \leq x < b \\
 1 & \text{if } b \leq x \leq c \\
 d(x) & \text{if } c < x \leq d \\
 0 & \text{if } d < x 
\end{cases}
$$

(2.45)

where $[a, d] \subset R, a \leq b \leq c \leq d, 0 \leq a(x) \leq 1$ is a non-decreasing function $(a(x) \in [a, b])$, and $0 \leq d(x) \leq 1$ is a non-increasing function $(d(x) \in (c, d])$. If fuzzy sets $A^1, A^2, ..., A^N \in W \subset R$ then, they are called:

1. Complete on $W$, if there exists $A^k$ such that $\mu_{A^k}(x) > 0$, for any $x \in W$.
2. Consistent on $W$, if $\mu_{A^k}(x) = 1$ for some $x \in W$ implies that $\mu_{A^j}(x) = 0$, for all $k \neq j$.
3. Normal, consistent and complete on $W$ with the general MFs, $\mu_{A_j}(x; a_j, b_j, c_j, d_j)$. If $A^1 < A^2 < ... < A^N$, Then $c_j \leq a_{j+1} < d_j \leq b_{j+1}$, for $j = 1, 2, ..., N - 1$.

Next Section proposes a general model depended on N-inputs fuzzy sets.

2.6.1 The Proposed Model

Let $G(x)$ is a mapping defined as the following:

$G(x): X \subset R^n \rightarrow R,$

That is, a function on the compact set $X = [\alpha_1, \beta_1] \times ... \times [\alpha_n, \beta_n]$ and the analytic formula of $G(x)$ be unknown. Suppose that for any $x \in U$, we can obtain $G(x)$. For model a fuzzy system $G(x)$ is a main task and model of a fuzzy system as follows:

**Step 1:**

Suppose the fuzzy sets $A^1_j, A^2_j, ..., A^{N_j}_j \in [\alpha_j, \beta_j], \ \forall N_j \ (j = 1, 2, ..., n)$, which are normal, consistent, and complete with triangular MFs,
\[ \mu_{A_j}(x_j; a_j^1, b_j^1, c_j^1), \ldots, \mu_{N_j}(x_j; a_j^{N_j}, b_j^{N_j}, c_j^{N_j}) \], and \( A_j^1 < A_j^2 < \cdots < A_j^{N_j} \) with \( a_j^1 = b_j^1 \) and \( b_j^{N_j} = c_j^{N_j} = \beta_j \), which:

- \( e_1^1 = \alpha_1, e_1^{N_1} = \beta_1 \), and \( e_j^1 = b_j^1 \) for \( j = 2, 3, \ldots, N_1 - 1 \),
- \( e_2^2 = \alpha_2, e_2^{N_2} = \beta_2 \), and \( e_j^2 = b_j^2 \) for \( j = 2, 3, \ldots, N_2 - 1 \),
- \( \vdots \)
- \( e_n^1 = \alpha_n, e_n^{N_n} = \beta_n \), and \( e_j^1 = b_j^1 \) for \( j = 2, 3, \ldots, N_n - 1 \).

**Step 2:**

Construct \( I = N_1 \times N_2 \times \ldots \times N_n \) fuzzy if-then rules in the following form:

\[ R_{X}^{j_1\ldots j_n} : \text{if } x_1 \text{ is } A_1^{j_1} \text{ and } x_2 \text{ is } A_2^{j_2} \text{ and } \ldots \text{ and } x_n \text{ is } A_n^{j_n} \text{ then } y \text{ is } B^{j_1\ldots j_n}, \]  

where \( j_1 = 1, 2, \ldots, N_1, \ j_2 = 1, 2, \ldots, N_2, \ldots, j_n = 1, 2, \ldots, N_n \), and the center of the output fuzzy set \( B^{j_1\ldots j_n} \), denoted by \( \tilde{y}^{j_1\ldots j_n} \), is chosen as:

\[ \tilde{y}^{j_1\ldots j_n} = G(e_1^{j_1}, \ldots, e_n^{j_n}) \]  

This is the case when (2.47) depends on the Mamdani fuzzy rule, [Chai et al. (2009)], and the antecedent of our model is connected by “and”, [Sivanandam and Deepa (2008)], [Marza and Seyyedi (2008)], then the evaluation of truth-value is given by:

\[ \vartheta_i = \tau \left( \mu_{A_1^{j_1\ldots j_n}}(x_1), \mu_{A_2^{j_1\ldots j_n}}(x_2), \ldots, \mu_{A_n^{j_1\ldots j_n}}(x_n) \right) \]  

Therefore, from \( \mu_{B^{j}}(y) = \tau \left( \vartheta_i, \mu_{B^{j}}(y) \right), \ \forall \ y \in R \), the fuzzy inference produces the output fuzzy set by:

\[ \mu_{B^{j_1\ldots j_n}}(y) = \tau \left( \vartheta_i, \mu_{B^{j_1\ldots j_n}}(y) \right), \ \forall \ y \in R \]  

The consequents of all the rules are aggregated by the ‘\( max \)’ function as:

\[ \mu_{B^{j_1\ldots j_n}}(y) = s(\mu_{B^{j_1\ldots j_n}}(y), \mu_{B^{j_1\ldots j_n}}(y), \ldots, \mu_{B^{j_1\ldots j_n}}(y)) \]  

However, when the consequent of rule is a linear function, then the output model is called the Sugeno-rule output that given as the following:

\[ \tilde{y}^{j_1\ldots j_n} = f_i(x_1, x_2, \ldots, x_n), \]  

where \( f_i \) is linear function base on \( x_j \), that is defined as:

\[ f_i(x_1, x_2, \ldots, x_n) = a_1^i x_1 + a_2^i x_2 + \cdots + a_n^i x_n + a_{n+1}^i, \]
where $a^j_i$ are the parameters, and it can be computed by the least square model.

**Step 3:**

Constructing the fuzzy system $f(x)$ from the $N_1 \times N_2 \times \ldots \times N_n$ rules of (2.48) using Mamdani’s minimum implication (MMI) (2.54) with the minimum inference system (MIS) (2.55), the singleton fuzzifier (SF) (2.56), and the center average defuzzifier (CAD) (2.57), as follows:

\[
\mu_{QIM}(x, y) = \min \{\mu_{A_1}(x), \mu_{A_2}(y)\}, \quad Q_{IM} \in X \times Y, \quad (2.54)
\]

\[
\mu_{B'}(y) = \max_{\forall i} \left[ \sup_{x \in X} \min \left( \mu_{A'_i}(x), \mu_{A_i}(x), \ldots, \mu_{A_n}(x), \mu_{B_i}(y) \right) \right], \quad (2.55)
\]

\[
\mu_{A'_i}(x) = \begin{cases} 1 & \text{if } x = x^* \\ 0 & \text{otherwise} \end{cases}, \quad (2.56)
\]

\[
y^* = \frac{\sum_{i=1}^{N_1} \sum_{j=1}^{N_j} w_i}{\sum_{i=1}^{N_1} w_i} \sum_{j=1}^{N_j} \left( \min \left( \mu_{A_1}(x_1), \ldots, \mu_{A_n}(x_n) \right) \right), \quad (2.57)
\]

Therefore, we obtain:

\[
f(x) = \frac{\sum_{j_1=1}^{N_1} \ldots \sum_{j_n=1}^{N_n} \sum_{j_1=1}^{N_1} \ldots \sum_{j_n=1}^{N_n} \left( \min \left( \mu_{A_1}(x_1), \ldots, \mu_{A_n}(x_n) \right) \right)}{\sum_{j_1=1}^{N_1} \ldots \sum_{j_n=1}^{N_n} \left( \min \left( \mu_{A_1}(x_1), \ldots, \mu_{A_n}(x_n) \right) \right)}, \quad (2.58)
\]

Since the fuzzy sets $A_1^1, \ldots, A_j^N_j$ are complete at every $x \in X$, then there exist $j_1, j_2, \ldots, j_n$ such that:

\[
\min \left( \mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots, \mu_{A_n}(x_n) \right) \neq 0. \quad (2.59)
\]

Consequently, the fuzzy system (2.58) is well defined. From step 2, we note that the antecedent of the rules (2.48) constitute all the possible sets of the fuzzy sets defined for each input variable, [Hndoosh (2013)]. The total number of rules is $N_n$, that increases exponentially with the dimension of the input space, [Jandaghi (2010)].

In the second part, the accuracy of the proposed model $f(x)$ on the unknown function $G(x)$ is proved, [Nayak (2004)], [Kamel and Hassan (2009)]. Here extended the accuracy of the fuzzy system from 2-dimensions to N-dimensions.
2.6.2 Theorem 2.1 (An Accuracy of the Proposed Model)

Let $f(x)$ be the fuzzy system in (2.58) and $G(x)$ be the unknown function in (2.48). If $G(x)$ is continuously differentiable on $X = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \times \ldots \times [\alpha_n, \beta_n]$, then:

$$
\|G - f\|_\infty \leq \left\| \frac{\partial G}{\partial x_1} \right\|_\infty h_1 + \left\| \frac{\partial G}{\partial x_2} \right\|_\infty h_2 + \ldots + \left\| \frac{\partial G}{\partial x_n} \right\|_\infty h_n,
$$

(2.60)

where the infinite norm $\| . \|_\infty$ is defined as:

$$
\|d(x)\|_\infty = \sup_{x \in X} |d(x)| \quad \text{and} \quad h_j = \max_{1 \leq k \leq N_j} |e_j^{k+1} - e_j^k|, (j = 1, 2, \ldots, n).
$$

Proof:

Let $X^{j_1 \ldots j_n} = [e_1^{j_1}, e_1^{j_1+1}] \times [e_2^{j_2}, e_2^{j_2+1}] \times \ldots \times [e_n^{j_n}, e_n^{j_n+1}]$, where $j_1 = 1, 2, \ldots, N_1 - 1$, $j_2 = 1, 2, \ldots, N_2 - 1$, $\ldots$, $j_n = 1, 2, \ldots, N_n - 1$. Since $[\alpha_j, \beta_j] = [e_1^j, e_1^2] \cup [e_2^j, e_2^3] \cup \ldots \cup [e_N^{N_j-1}, e_N^{N_j}], j = 1, 2, \ldots, n$, then:

$$
X = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \times \ldots \times [\alpha_n, \beta_n] = \bigcup_{j_1=1}^{N_1-1} \bigcup_{j_2=1}^{N_2-1} \ldots \bigcup_{j_n=1}^{N_n-1} X^{j_1 \ldots j_n},
$$

(2.61)

which implies that for any $x \in X$, there exists $X^{j_1 \ldots j_n}$ such that $x \in X^{j_1 \ldots j_n}$.

Now suppose $x \in X^{j_1 \ldots j_n}$, that is $x_1 \in [e_1^{j_1}, e_1^{j_1+1}]$, $x_2 \in [e_2^{j_2}, e_2^{j_2+1}]$, $\ldots$, $x_n \in [e_n^{j_n}, e_n^{j_n+1}]$. (since $x$ is fixed, then $j_n$ are also fixed in the sequel). Since the fuzzy sets $A_1^1, A_1^2, \ldots, A_1^{N_1}$ are normal, consistent, and complete, at least one and at most two $\mu_{A_1^{k_1}}(x_1)$ are non-zero for $k_1 = 1, \ldots, N_1$. From the definition of $e_1^{k_1}(k_1 = 1, 2, \ldots, N_1 - 1)$, these two possible non-zero $\mu_{A_1^{k_1}}(x_1)$'s are $\mu_{A_1^{k_1}}(x_1)$ and $\mu_{A_1^{k_1+1}}(x_1)$. Similarly, the two possible non-zero $\mu_{A_n^{k_n}}(x_n)$'s, $(k_n = 1, \ldots, N_n)$ are $\mu_{A_n^{j_n}}(x_n)$ and $\mu_{A_n^{j_n+1}}(x_n)$. Hence, the fuzzy system $f(x)$ of (2.61) is simplified to:

$$
f(x) = \frac{\sum_{k_1=j_1}^{j_1+1} \sum_{k_n=j_n}^{j_n+1} \prod_{k=1}^{k_n} \min \left( \mu_{A_1^{k_1}}(x_1), \mu_{A_2^{k_2}}(x_2), \ldots, \mu_{A_n^{k_n}}(x_n) \right)}{\sum_{k_1=j_1}^{j_1+1} \sum_{k_n=j_n}^{j_n+1} \min \left( \mu_{A_1^{k_1}}(x_1), \mu_{A_2^{k_2}}(x_2), \ldots, \mu_{A_n^{k_n}}(x_n) \right)}
$$

(2.62)

(93)
From (2.48), we obtain:

\[
f(x) = \sum_{k_1=j_1}^{j_1+1} \ldots \sum_{k_n=j_n}^{j_n+1} \left[ \min \left( \mu_{A_1}(x_1), \ldots, \mu_{A_n}(x_n) \right) \right] * G(e_1^{k_1}, \ldots, e_n^{k_n})
\]  

(2.63)

\[
\sum_{k_1=j_1}^{j_1+1} \ldots \sum_{k_n=j_n}^{j_n+1} \min \left( \mu_{A_1}(x_1), \ldots, \mu_{A_n}(x_n) \right)
\]

(2.64)

From (2.68), we can conclude that fuzzy systems in the form of (2.62).
Specifically, since $\|\frac{\partial G}{\partial x_1}\|, \|\frac{\partial G}{\partial x_2}\|, ..., \|\frac{\partial G}{\partial x_n}\|$ are finite numbers for any given $\varepsilon > 0$, we can choose $h_1, h_2$ and $h_n$ small enough such that $\|\frac{\partial G}{\partial x_1}\| \cdot h_1 + \|\frac{\partial G}{\partial x_2}\| \cdot h_2 + \ldots + \|\frac{\partial G}{\partial x_n}\| \cdot h_n < \varepsilon$. Thus, from (2.60) we obtain:

$$\sup_{x \in X} |G - f| = \|G - f\|_\infty < \varepsilon$$

(2.69)

From (2.68), we can show that, in order to model a fuzzy system with a pre-specified accuracy, we must know the bounds of the derivatives of $G(x)$ with respect to $x_1, x_2, ..., x_n$, that is, $\|\frac{\partial G}{\partial x_1}\|, \|\frac{\partial G}{\partial x_2}\|, ..., \|\frac{\partial G}{\partial x_n}\|$. In the model process, we need to know the value of, $G(x)$ at $x = (e_1^1, e_2^j, ..., e_n^j)$ for $j_1 = 1, ..., N_1, j_2 = 1, ..., N_2, ..., j_n = 1, ..., N_n$ (2.70)

Therefore, this approach requires these $N$ pieces of information in order for the model fuzzy system, to satisfy any pre-specified degree of accuracy.

### 2.6.3 Model of an Adaptive Fuzzy System Using a Neural Network

In this Section, we adapt the structure of the fuzzy system that is specified with the structure of some parameters [Singh et al. (2009)], [Shing and Jang (1993)]. We specify the structure of the fuzzy system to be modeled. Here, we choose the fuzzy system with a MIS, a SF, a CAD, and a Triangular MF [Samandar (2011)], then, we obtain:

$$f(x) = \sum_{i=1}^{I} \bar{y}^i \left( \min_{v_j} \mu_{A_i}(x_j) \right) = \frac{\sum_{i=1}^{I} \bar{y}^i \left[ \min_{v_j} \left( \max_{v_j} \left( \min_{v_j} \left( \frac{x_j - a_j^i}{b_j^i - a_j^i}, \frac{c_j^i - x_j}{c_j^i - b_j^i} \right), 0 \right) \right) \right]}{\sum_{i=1}^{I} \left[ \min_{v_j} \left( \max_{v_j} \left( \min_{v_j} \left( \frac{x_j - a_j^i}{b_j^i - a_j^i}, \frac{c_j^i - x_j}{c_j^i - b_j^i} \right), 0 \right) \right) \right]},$$

(2.71)

where $I$ is fixed, and $\bar{y}^i, a_j^i, b_j^i, c_j^i$ are free parameters. The fuzzy system (2.71) has not been modeled because the parameters $\bar{y}^i, a_j^i, b_j^i, c_j^i$ are not specified [Hndoosh (2013), Woo et al. and Wang (1998)]. In order to determine these parameters in some optimal manner, it is helpful to represent the fuzzy system $f(x)$ of (2.71) as a feed-forward network [Jose et al. (1999)]. Specifically, the mapping from the input $x \in U \subset \mathbb{R}^n$ to the
output \( f(x) \in V \subset R \) can be performed according to operations, [Doğan et al. (2007)]. Note that, the input \( x \) is passed through a minimum triangular operator to become:

\[
z^i = \max \left( \min_{vj} \left( \frac{x_j - a^i_j}{b^i_j - a^i_j}, \frac{c^i_j - x_j}{c^i_j - b^i_j} \right), 0 \right);
\]

where \( z^i \) are passed through a summation operator.

Let \( K = \sum_{i=1}^l z^i \) and \( L = \sum_{i=1}^l \bar{y}^i z^i \); therefore, the output of the fuzzy system is computed as \( f(x) = L/K \). Consequently, we summarize the procedures to model a fuzzy system that depends on layers of network as the following:

**Step 1: Structure specification and initial parameters**

Select the fuzzy system (2.71) and determine the number of rule [Rameshkumar and Arumugam (2011)]. The larger number of rule, results more parameters and more computation, but gives better accuracy. Specify the initial parameters \( \bar{y}^i(0), a^i_j(0), b^i_j(0), c^i_j(0) \), then the initial fuzzy system becomes as in (2.72). These initial para-meters may be determined according to the linguistic rules from human experts as in our application.

\[
f(x) = \frac{\sum_{j_1=1}^{N_1} \cdots \sum_{j_n=1}^{N_n} \bar{y}^{j_1,j_n}(0) \left[ \min \left( \max \left( \min_{v_k} \left( \frac{x^p_{k0} - a^{j_1,j_n}_k(0)}{b^{j_1,j_n}_k(0) - a^{j_1,j_n}_k(0)}, \frac{c^{j_1,j_n}_k(0) - x^p_{k0}}{c^{j_1,j_n}_k(0) - b^{j_1,j_n}_k(0)} \right), 0 \right) \right] \right]}{\sum_{j_1=1}^{N_1} \cdots \sum_{j_n=1}^{N_n} \left[ \min \left( \max \left( \min_{v_k} \left( \frac{x^p_{k0} - a^{j_1,j_n}_k(0)}{b^{j_1,j_n}_k(0) - a^{j_1,j_n}_k(0)}, \frac{c^{j_1,j_n}_k(0) - x^p_{k0}}{c^{j_1,j_n}_k(0) - b^{j_1,j_n}_k(0)} \right), 0 \right) \right] \right)}
\]

(2.72)

**Step 2: Calculating the outputs of the fuzzy system**

For a given inputs-output pair \( (x^p_{k0} ; y^p_0) \), \( p = 1, 2, \ldots, k = 1, 2, \ldots, n \) and at the \( q^{th} \) training stage, \( q = 0, 1, \ldots \), present \( x^p_{k0} \) to the input layer of the fuzzy system in Figure 2.1 and compute the outputs of layers, and therefore, we compute:
First output: Every node produces MF of an input parameter. The node output $o_1^{j_i}$ is explained by:

$$o_1^{j_1} = \mu_{A_1^{j_1}}(x_1); \ o_1^{j_2} = \mu_{A_2^{j_2}}(x_2), \ldots, \text{and} \ o_1^{j_n} = \mu_{A_n^{j_n}}(x_n),$$ \hspace{1cm} (2.73)

where $x_1, x_2, \ldots, x_n$ are the inputs; $\mu_{A_1^{j_1}}, \mu_{A_2^{j_2}}, \ldots, \mu_{A_n^{j_n}}$ are linguistic fuzzy sets related with nodes, and $o_1^{j_i}$ is the degree of MFs of a fuzzy set.

Second output: Every node is a fixed node, whose output is the minimum of all MFs:

$$o_2^{j_1 \cdots j_n} = z_1^{j_1 \cdots j_n}, \ z_1^{j_1 \cdots j_n} = \min_{v_j} \left( \mu_{A_{j_1 \cdots j_n}}(x_j) \right),$$ \hspace{1cm} (2.74)

where $\mu_{A_{j_1 \cdots j_n}}$ is declared by triangular MF, and then we obtain:

$$z_1^{j_1 \cdots j_n} = \min_{v_j} \left( \max \left( \min_{v_k} \left( \frac{x_{k0} - a_{k0}^{j_1 \cdots j_n}(q)}{b_{k0}^{j_1 \cdots j_n}(q) - a_{k0}^{j_1 \cdots j_n}(q)}, \frac{c_{k0}^{j_1 \cdots j_n}(q) - x_{k0}^{p}}{c_{k0}^{j_1 \cdots j_n}(q) - b_{k0}^{j_1 \cdots j_n}(q)} \right), 0 \right).$$ \hspace{1cm} (2.75)

Third output: Depending on (2.75), $j_1 \ldots j_n^{th}$ node calculates all rules as:

$$o_3^{j_1 \cdots j_n} = z_1^{j_1 \cdots j_n}(x) = \frac{z_1^{j_1 \cdots j_n}}{\sum_{j_1=1}^{N_1} \cdots \sum_{j_n=1}^{N_n} z_1^{j_1 \cdots j_n}}$$ \hspace{1cm} (2.76)

Fourth output: Every node $j_1 \ldots j_n$ is an adaptive node with a node MF of output.

$$o_4^{j_1 \cdots j_n} = z_1^{j_1 \cdots j_n} \bar{y}^{j_1 \cdots j_n},$$ \hspace{1cm} (2.77)

where $\bar{y}^{j_1 \cdots j_n} = f_{j_1 \cdots j_n}(x_1, x_2, \ldots, x_n)$, and from (2.53), we get:

Figure 2.1: Network representation of the fuzzy system.
\[
\tilde{y}_{j_1\ldots j_n} = \alpha_{1j_1\ldots j_n} x_1 + \alpha_{2j_1\ldots j_n} x_2 + \cdots + \alpha_{nj_1\ldots j_n} x_n + \alpha_{n+1j_1\ldots j_n},
\]

(2.78)

where \( \alpha_{j_1\ldots j_n} (j = 1, \ldots, n + 1.) \), is the parameter set of the node.

**Fifth output:** The single node is a fixed node labeled \( \sum \), which computes the final output as the summation of all result \( \alpha_{4j_1\ldots j_n} \)

\[
\alpha_{5j_1\ldots j_n} = \sum_{j_1=1}^{N_1} \ldots \sum_{j_n=1}^{N_n} z_{j_1\ldots j_n} (\alpha_{1j_1\ldots j_n} x_1 + \cdots + \alpha_{nj_1\ldots j_n} x_n + \alpha_{n+1j_1\ldots j_n})
\]

(2.79)

Suppose,

\[
K = \sum_{j_1=1}^{N_1} \ldots \sum_{j_n=1}^{N_n} z_{j_1\ldots j_n}
\]

\[
L = \sum_{j_1=1}^{N_1} \ldots \sum_{j_n=1}^{N_n} z_{j_1\ldots j_n} (\alpha_{1j_1\ldots j_n} x_1 + \cdots + \alpha_{nj_1\ldots j_n} x_n + \alpha_{n+1j_1\ldots j_n})
\]

(2.80)

Consequently, the final output is obtained as:

\[
f(x) = \frac{L}{K}.
\]

(2.81)

Here noted that \( \tilde{y}_{j_1\ldots j_n} \) are free parameters to be modeled. When select the initial parameters \( \theta(1) \), and there are linguistic rules from experts, then choose \( \tilde{y}_{j_1\ldots j_n}(1) \) to be the centers of the *then* part fuzzy sets in these linguistic rules; otherwise, choose \( \theta(1) \) arbitrarily in the output space \( Y \subset R \). In this way, we can say that the initial fuzzy system is constructed from experts.

**Step 3: Update the parameters**

Use the training algorithm to compute the updated parameters \( \tilde{y}_{j_1\ldots j_n}(q + 1), a_{kj_1\ldots j_n}(q + 1), b_{kj_1\ldots j_n}(q + 1), c_{kj_1\ldots j_n}(q + 1) \), where \( y = y^p_0 \), and \( z_{j_1\ldots j_n}, K, L \) and \( f \) equal to those that computed in step 2, i.e., compute the new parameters \( \theta \) using the least squares model as:

\[
\theta(p + 1) = \theta(p) + t(p + 1) \left[ y^p_0 - \left( z(x^p_{k0}) \right)^T \theta(p) \right]
\]

(2.82)

in which,
\[
t(p + 1) = \frac{P(p + 1) z(x_k^p)}{\left[ P(p + 1) z(x_k^p) (z(x_k^p))^T + 1 \right]^T},
\]

\[
P(p + 1) = P(p) - \frac{P(p) z(x_k^p)}{\left[ P(p) z(x_k^p) (z(x_k^p))^T + 1 \right]^T} P(p) \left[ z(x_k^p) \right]^T
\]

When \( p = 1 \), note that \( \theta(1) \) is chosen using step 2, and \( P(1) \) is a large constant. The modeled fuzzy system in (2.71) with the parameters \( \tilde{y}^{j_1...j_n} \) is equal to the corresponding elements in \( \theta(p) \).

**Step 4:** Repeat by going to step 2 with \( q = q + 1 \), until the error \( |f - y_0^p| < \varepsilon \), or until the \( q \) equals a pre-specified number.

**Step 5:** Repeat by going to step 2 with \( p = p + 1, p = 1, 2, \ldots \); that is, update the parameters using the next input-output pair \( (x_{k0}^{p+1}; y_0^{p+1}) \).

Keep in mind that the parameters \( \tilde{y}^{j_1...j_n} \) are the centers of the fuzzy sets in the consequent parts of the rules, and the parameters \( a_j^{j_1...j_n} \) and \( c_j^{j_1...j_n} \) are the left and right base points, \( b_j^{j_1...j_n} \), the centers of the triangular fuzzy sets in the antecedent parts of the rules [Rojas (1996)]. We can improve the fuzzy if-then rules that modeled the fuzzy system, and improved fuzzy if-then rules may help to explain the model fuzzy system in a user-friendly manner [Aik and Jayakumar (2008)].

### 2.7 APPLICATION

Liver is the largest internal member in the human body, and it is known that the member is responsible for more than one hundred functions of human body. The complexity of this member makes it easily affected by disease of disorder. Therefore, diagnosing liver disorder disease (LDD) is a high interest to researchers and doctors, and fuzzy system has been a good intelligent model to diagnose such disease [Sug (2012) and Gulia et al. (2014)]. The fuzzy system has very good property that the model is easy to understand. This property of fuzzy system is important in case that human
should understand the knowledge structures fully. This is one of the main reasons why fuzzy system is accepted in medical domain. There are six continuous attributes as dependent attributes, (Table 1 for detail of the attributes). The first five variables are all blood tests that are thought to be sensitive to liver disorders that might result from excessive alcohol consumption. Each line in the LDD_data constitutes the record of a single male individual.

2.7.1 Structure of the Model
Consider the multi-inputs $x_i$, ($i = 1, 2, \ldots, 6$), with output $y$ (disorder types of liver that contains simple liver disorder or acute liver disorder). A fuzzy inference system (FIS) can be defined as:

$$FIS : X \rightarrow Y,$$

where $X \subset R^n$ and $Y \subset R$. (2.85)

The fuzzy system is composed of a fuzzifier, fuzzy rule-base, fuzzy inference, and defuzzifier. In order to apply a steps of FIS systematically, the inputs $x_1, x_2, \ldots, x_6$ with output $y$ must be described as the following:

1. **Inputs**
Inputs-data have treated and measured, so that, it becomes restricted between 0 and 1. The first five variables have five different degrees of linguistic variables: Low (L), Low Medium (LM), Medium (M), High Medium (HM) and High (H), while sixth input is represented by five linguistic variables, Less (Le), Less Average (LA), Average (A), More Average (MA) and More (Mo).

2. **Output**
Output-data are represented the disorder types of liver that contains two types: simple liver disorder (SLD) or acute liver disorder (ALD), see Figure 2.2.
2.7.2 Discussion and Results for the Model of the Fuzzy System with Mamdani and ST Models

In this part, we build the proposed fuzzy system systematically on our application:

Table 2.1: The Meaning of Variables

<table>
<thead>
<tr>
<th>Variable</th>
<th>Variable name</th>
<th>Meaning</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>mcv</td>
<td>mean corpuscular volume</td>
<td>[79,103]</td>
</tr>
<tr>
<td>$x_2$</td>
<td>alkphos</td>
<td>alkaline phosphotase</td>
<td>[35,109]</td>
</tr>
<tr>
<td>$x_3$</td>
<td>sgpt</td>
<td>alanine aminotransferase</td>
<td>[5,155]</td>
</tr>
<tr>
<td>$x_4$</td>
<td>sgot</td>
<td>aspartate aminotransferase</td>
<td>[11,68]</td>
</tr>
<tr>
<td>$x_5$</td>
<td>gammagt</td>
<td>gamma-glutamyl transpeptidase</td>
<td>[5,297]</td>
</tr>
<tr>
<td>$x_6$</td>
<td>drinks</td>
<td>number of half-pint equivalents of alcoholic beverages drunk per day</td>
<td>[0,20]</td>
</tr>
</tbody>
</table>

Figure 2.2 a: The FIS of LD using Mamdani model

Figure 2.2 b: The FIS of LD using ST model

Step 1:

Define $N_j, (j = 1,2, ..., 6)$, where $N_j = 1,2,3$. Let $A^1_j, ..., A^6_j$ is a fuzzy sets for $x_i$, with triangular MFs, $\mu_{A^1_j}(x_1; a^1_1, b^1_1, c^1_1), ..., \mu_{A^6_j}(x_6; a^5_6, b^5_6, c^5_6)$, and $A^1_j < A^2_j < \cdots < A^{N_j}_j$ wit $a^j_1 = b^j_1 = 0$ and $b^{N_j}_j = c^{N_j}_j = 1$.

Therefore, we can define:

- $e^1 = 0, e^5 = 1$, and $e^2 = b^2_1, e^3 = b^3_1, e^4 = b^4_1$
- $e^2 = 0, e^3 = b^3_1, e^4 = b^4_2, e^5 = b^5_1$
- $e^3 = 0, e^4 = b^4_1, e^5 = b^5_2, e^6 = b^6_1$
- $e^4 = 0, e^5 = b^5_1, e^6 = b^6_2, e^7 = b^7_1$
- $e^5 = 0, e^6 = b^6_1, e^7 = b^7_2, e^8 = b^8_1$
- $e^6 = 0, e^7 = b^7_1, e^8 = b^8_2$

Here the Singleton fuzzifier is defined as: SF: $x^* \rightarrow A'$.
where \( x^* = \{x_1, \ldots, x_6\} \), and the software ‘MATLAB’ is used to create the programs of our application. Six vectors of inputs with one vector of output are loaded for 100 observations, see Figure 2.3. For example, the first observation is represented as \([mcv=0.8544, \text{alkphos}= 0.5088, \text{sgpt}=0.12903, \text{sgot}= 0.25, \text{gammagt}=0.0303, \text{and drinks }=0.025\]).

Consequently, we have built the first rule as: \([\text{If (mcv is M) and (alkphos is M) and (sgpt is L) and (sgot is LM) and (gammagt is L) and (drinks is L) Then y is SLD}]\].

**Step 2:**

Note that, the fuzzy rule-base consists of \((I = 5^6)\) fuzzy if-then rules and the centers of \((f_{j_1\cdots j_6,l}(x_1, \ldots, x_6))\) are evaluated at the 729 points, and therefore, we obtain:

\[
R^i,: \begin{cases} 
\text{if } x_1 \text{ is } A_{j_1,1}^i \text{ and } x_2 \text{ is } A_{j_2,1}^i \text{ and } \ldots \text{ and } x_6 \text{ is } A_{j_6,1}^i \text{ then } y \text{ is } B_{j_1\cdots j_6,1}^i \\
\text{if } x_1 \text{ is } A_{j_1,2}^i \text{ and } x_2 \text{ is } A_{j_2,2}^i \text{ and } \ldots \text{ and } x_6 \text{ is } A_{j_6,2}^i \text{ then } y \text{ is } B_{j_1\cdots j_6,2}^i \\
\quad \quad \vdots \\
\text{if } x_1 \text{ is } A_{j_1,l}^i \text{ and } x_2 \text{ is } A_{j_2,l}^i \text{ and } \ldots \text{ and } x_6 \text{ is } A_{j_6,l}^i \text{ then } y \text{ is } B_{j_1\cdots j_6,l}^i
\end{cases}
\] (2.87a)

\[
R^i,: \begin{cases} 
\text{if } x_1 \text{ is } A_{j_1,1}^i \text{ and } \ldots \text{ and } x_6 \text{ is } A_{j_6,1}^i \text{ then } y \text{ is } f_{j_1\cdots j_6,1}(x_1, \ldots, x_6) \\
\text{if } x_1 \text{ is } A_{j_1,2}^i \text{ and } \ldots \text{ and } x_6 \text{ is } A_{j_6,2}^i \text{ then } y \text{ is } f_{j_1\cdots j_6,2}(x_1, \ldots, x_6) \\
\quad \quad \vdots \\
\text{if } x_1 \text{ is } A_{j_1,l}^i \text{ and } \ldots \text{ and } x_6 \text{ is } A_{j_6,l}^i \text{ then } y \text{ is } f_{j_1\cdots j_6,l}(x_1, \ldots, x_6)
\end{cases}
\] (2.87b)
The model (2.87a) is a system of Mamdani fuzzy rules, while the model (2.87b) is a system of Sugeno fuzzy rules. The sets $A$ and $B$ are fuzzy sets, $x_h$ ($h = 1, ..., 6$) are input variables, $y$ is the output variable, and $i$ ($i = 1, ..., I$) is the number of rules. The fuzzy set $A$ consists of $(A_1^i, ..., A_6^i)$, fuzzy subsets. It is called linguistic terms that represented by triangular MFs as (2.1) when $b = c$.

The fuzzy operator “and” ($t$-norm) is used to connecting between linguistic terms in each rule of the model. The function $f^{j_1...j_6,l}(x_1, ..., x_6)$ is a linear function depends on inputs $x_k$ that defined using (2.53). The first five linguistic terms are represented by $A_h^{j_i} (j_i = 1, 2, ..., 5)$ that depends on linguistic variable $x_h$ ($h = 1, ..., 5$), for e.g. the linguistic term for first input variable defined as the following:

$A_1^1$ ≡ “Low”, $(\mu_{A_1^1}(x_1; 0,0,0.18))$,  
$A_1^2$ ≡ “LowMedium”≡ $(\mu_{A_1^2}(x_1; 0.15,0.25,0.4))$,  
$A_1^3$ ≡ “Mediam”≡ $(\mu_{A_1^3}(x_1; 0.35,0.25,0.65))$,  
$A_1^4$ ≡ “HighMedium”≡ $(\mu_{A_1^4}(x_1; 0.6,0.73,0.85))$ and  
$A_1^5$ ≡ “High”≡ $(\mu_{A_1^5}(x_1; 0.82,1,1))$.

While, we have represented the sixth linguistic term $A_6^{j_i} (j_i = 1, ..., 5)$ that depends on linguistic variable $x_6$ as the following:

$A_6^1$ ≡ “Less”≡ $(\mu_{A_6^1}(x_6; 0,0,0.1))$,  
$A_6^2$ ≡ “Less Average”≡ $(\mu_{A_6^2}(x_6; 0.08,0,2,0.3))$,  
$A_6^3$ ≡ “Average”≡ $(\mu_{A_6^3}(x_6; 0.25,0.5,0.7))$,  
$A_6^4$ ≡ “More Average”≡ $(\mu_{A_6^4}(x_6; 0.6,0.7,0.83))$, and  
$A_6^5$ ≡ “More”≡ $(\mu_{A_6^5}(x_6; 0.8,1,1))$.

Moreover, the output is described by triangular MFs, $(\mu_{B^{j_1...j_6}}(y))$, and its linguistic term that represented as the following:

$B_1$ ≡ “SLD”, $(\mu_{B_1}(y; 0,0,0.75))$, and $B_2$ ≡ “ALD”, $(\mu_{B_2}(y; 0.25,1,1))$.

The fuzzy inference process defines as the following:
Chapter 2

\[ FI: A' \rightarrow \overline{B}_{j_1 \cdots j_6}^i \]  \hspace{1cm} (2.88)

where \( A' \) is an input fuzzy sets in the input space \( X \), and \( \overline{B}_{j_1 \cdots j_6}^i \) the fuzzy sets in the output space \( Y \). Each one of the rules specifies a fuzzy set \( \overline{B}_{j_1 j_2 j_3}^i \subseteq Y \) that is given by the compositional rule of inference:

\[ \overline{B}_{j_1 \cdots j_6}^i = A' \circ \left( \overline{A}_{j_1 \cdots j_6}^i \rightarrow \overline{B}_{j_1 \cdots j_6}^i \right), \]

where \( \overline{A}_{j_1 \cdots j_6}^i = A_{j_1}^i \times A_{j_2}^i \times \ldots \times A_{j_6}^i \) (2.89)

Therefore, from \( \mu_{A'_{j_1 \cdots j_6}}(x_1, \ldots, x_n) = \mu_{A_{j_1}}(x_1) \ast \ldots \ast \mu_{A_{j_6}}(x_n) \), we obtain

\[ \mu_{A'_{j_1 \cdots j_6}}(x_j) = \mu_{A_{j_1 \cdots j_6}}(x_1, \ldots, x_6), (j = 1, 2, \ldots, 6) \]  \hspace{1cm} (2.90)

From \( \left( \mu_B(y) = \sup_{x \in X} \{ \mu_A(x) \oplus \mu_Q(x, y) \} \right) \), the fuzzy sets \( \overline{B}_{j_1 \cdots j_6}^i \) are described by MF:

\[ \mu_{\overline{B}_{j_1 \cdots j_6}^i}(y) = \sup_{x \in X} \left\{ \mu_A(x) \oplus \mu_{A_{j_1 \cdots j_6}^i}(x, y) \right\} \]  \hspace{1cm} (2.91)

Consequently, can be re-express (2.91) as the following:

\[ \mu_{\overline{B}_{j_1 \cdots j_6}^i}(y) = \mu_{A_{j_1 \cdots j_6}^i}(\overline{B}_{j_1 \cdots j_6}^i(x_j), y) = \text{Im} \left( \mu_{A_{j_1 \cdots j_6}^i}(x_j), \mu_{B_{j_1 \cdots j_6}^i}(y) \right), \]  \hspace{1cm} (2.92)

where \( \text{Im}(\cdot) \) is an “implementation”. Since, we used Mamdani’s minimum implication (2.54), therefore, we obtained:

\[ \text{Im} \left( \mu_{A_{j_1 \cdots j_6}^i}(x_j), \mu_{B_{j_1 \cdots j_6}^i}(y) \right) = \min \left\{ \mu_{A_{j_1 \cdots j_6}^i}(x_j), \mu_{B_{j_1 \cdots j_6}^i}(y) \right\} \]  \hspace{1cm} (2.93)

or (2.93) may be written as:

\[ \mu_{\overline{B}_{j_1 \cdots j_6}^i}(y) = \min \left\{ \mu_{A_{j_1 \cdots j_6}^i}(x_j), \mu_{B_{j_1 \cdots j_6}^i}(y) \right\} \]  \hspace{1cm} (2.94)

The aggregation operator has been applied in order to get the fuzzy set \( B' \) that uses the functions ‘\( \text{max} \)’ (\( s \)-norm) or ‘\( \text{min} \)’ (\( t \)-norm) depending on the type of fuzzy implication. The aggregation operator is denoted by:

\[ B' = \bigcup_{i=1}^{I} \overline{B}_{j_1 \cdots j_6}^i, \]  \hspace{1cm} (2.95)

and therefore, the membership function of \( B' \) is computed using the ‘\( \text{max} \)’ function as the following:

\[ \mu_{B'}(y) = \max_{v_i} \{ \mu_{\overline{B}_{j_1 j_2 j_3}^i}(y) \} \]  \hspace{1cm} (2.96)
Step 3:

The defuzzifier performs a mapping as the following:

\[ def = B' \rightarrow f(x), \]  
(2.97)

where \( B' \) is a fuzzy set, \( f(x) \) is a crisp point \( f(x) \in (Y \subset R) \).

The center of the area is a final system output that is defined by the following formula:

\[
f(x) = \frac{\sum_{j_1=1}^{5} \cdots \sum_{j_6=1}^{5} y_{j_1-j_6} \left( \min \left( \max \left( \min \left( \frac{x - a_{j_1-j_6}}{b_{j_1-j_6} - a_{j_1-j_6}}, \frac{c_{j_1-j_6} - x}{c_{j_1-j_6} - b_{j_1-j_6}} \right), 0 \right) \right) \right)}{\sum_{j_1=1}^{5} \cdots \sum_{j_6=1}^{5} \left( \min \left( \max \left( \min \left( \frac{x - a_{j_1-j_6}}{b_{j_1-j_6} - a_{j_1-j_6}}, \frac{c_{j_1-j_6} - x}{c_{j_1-j_6} - b_{j_1-j_6}} \right), 0 \right) \right) \right)},
\]
(2.98)

The model of fuzzy system \( f(x) \) has been built from the \( I \) rules of (2.87a) using Mamdani's minimum implication (2.54) with the MIS (2.55), the SF (2.56), and the CAD (2.57) (see Figure 2.4). We have created two programs; the first program depends on the Mamdani model, while the second program depends on the ST model. The programs of FIS have applied, that is given by (2.102), and obtained a good result of target. Therefore, we have applied the measure for accuracy of all data. The difference between the actual and target outputs is given by the formula as the following:

\[ Error = |Rv - FISv|, \]  
(2.99)
where $R_v$ is actual output values and $FIS_v$ is the output target values. The value of accuracy is a very small, where the average error of Simple LD=0.2047 and the average error of Acute LD=0.1158, when FIS is used the Mamdani fuzzy rule model. While, when the FIS is used the ST fuzzy rule model, then the average error of Simple LD=0.1463 and the average error of Acute LD=0.0061. Table 2.2 represents all the results. The ‘surfview’ tool is the surface viewer that helps view the input-output surface of the FIS. This conception is very helpful to understand how the system is going to behave for the entire range of values in the inputs space. Figure 2.5 shows the output surface for the different inputs.

![Output Surface](image)

**Figure 2.5:** The output surface for the different inputs

### 2.7.3 Discussion and Results for the Model of the Adaptive Fuzzy System Using Neural Network

This Section specifies the structure of the fuzzy system to be modeled. Here, we choose the fuzzy system with a MIS, a SF, the CAD, and a Triangular MF that given by the model (2.102). Model (2.102) has not been modeled, because the free parameters $\gamma_{j1}, a_{j1}, b_{j1}, c_{j1}$ are not specified. These parameters should be determined in order to represent the $f(x)$. The Model of the adaptive fuzzy system using neural network may be express as the following:

**Step 1:** Determine the initial parameters $\gamma_{j1}, a_{j1}, b_{j1}, c_{j1}$ according to the linguistic rules from experts, such as when $j = j_h = 1, (h = 1, ..., 6)$, then $a_1(0) = 0, b_1(0) = 0, c_1(0) = 0.18$, similarly for $\forall j_h$, and $j$ and for each input.
Chapter 2

Step 2: From a given inputs-output pair \((x_{10}^p, \ldots, x_{60}^p; y_0^p), \forall p \) (observe), compute the outputs of layers as the following:

(i) The node of the first output \(o_1^{j1}\) is represented by:

\[
o_1^{j1} = \mu_{A1}(x_1), (j_1 = 1, \ldots, 5); o_1^{j2} = \mu_{A1}(x_1), (j_2 = 5, \ldots, 10); o_1^{j3} = \mu_{A1}(x_1), (j_3 = 10, \ldots, 15); o_1^{j4} = \mu_{A1}(x_1), (j_4 = 15, \ldots, 20); o_1^{j5} = \mu_{A1}(x_1), (j_5 = 20, \ldots, 25) \text{ and } o_1^{j6} = \mu_{A1}(x_1), (j_6 = 25, \ldots, 30),
\]

where \(o_1^{j1}\) is the degree of MFs of a fuzzy set.

(ii) To compute the second output, we should use the minimum function of all MFs as:

\[
o_2^{j1⋯j6} = \min \left( \mu_{A1}^{j1⋯j6}(x_1), \mu_{A2}^{j1⋯j6}(x_2), \ldots, \mu_{A6}^{j1⋯j6}(x_6) \right).
\]  

Since \(\mu_{A1}^{j1⋯j6}\) is a Triangular MF, therefore, we obtain:

\[
o_2^{j1⋯j6} = \min \left( \max \left( \min \left( \mu_{A1}^{j1⋯j6}(x_1), \mu_{A2}^{j1⋯j6}(x_2), \ldots, \mu_{A6}^{j1⋯j6}(x_6) \right), 0 \right) \right) \quad (2.100)
\]

(iii) For all rules, we have calculated the \(j_1 \ldots j_6\) node as:

\[
o_3^{j1⋯j6} = \left( \sum_{j_1=1}^{5} \sum_{j_6=1}^{5} \min \left( \max \left( \min \left( \mu_{A1}^{j1⋯j6}(x_1), \mu_{A2}^{j1⋯j6}(x_2), \ldots, \mu_{A6}^{j1⋯j6}(x_6) \right), 0 \right) \right) \quad (2.102)
\]

(iv) The fourth output depends on a node MF of output \(\bar{z}^{j1⋯j6}\) with an adaptive node \(\bar{y}^{j1⋯j6}\), therefore, we obtain:

\[
o_4^{j1⋯j6} = o_3^{j1⋯j6} f_{j1⋯j6}(x_1, x_2, \ldots, x_6) \quad (2.103)
\]

Since \(f_{j1⋯j6}(x_1, x_2, \ldots, x_6) = a_1^{j1⋯j6} x_1 + a_2^{j1⋯j6} x_2 + \ldots + a_6^{j1⋯j6} x_6 + a_7^{j1⋯j6}, \) then we should determine the initial parameters of \(a_j^{j1⋯j6}, \forall j = 1, \ldots, 7.\).

(v) In order to compute the final output, we must take the summation of all results \(o_4^{j1⋯j6}\) as the following:
\[
o_5^{j_1\ldots j_6} = \sum_{j_1=1}^{5} \sum_{j_6=1}^{5} \frac{o_2^{j_1\ldots j_6}}{\sum_{j_1=1}^{5} \ldots \sum_{j_6}^{5}} \alpha_1^{j_1\ldots j_6} x_1 + \ldots + \alpha_6^{j_1\ldots j_6} x_6 + \alpha_7^{j_1\ldots j_6} \tag{2.104}\]

**Step 3:** In step 2, it is specified the initial parameters of \( \bar{y}^{j_1\ldots j_6}(q) \), \( a_k^{j_1\ldots j_6}(q) \), \( b_k^{j_1\ldots j_6}(q) \), \( c_k^{j_1\ldots j_6}(q) \), when \( q = 0 \). Consequently, we update these parameters at \( q + 1 \), using the least squares model and repeat all the procedure in step 2 for compute \( K \), \( L \) and \( f \). From (2.82) to (2.84), we compute the new parameters as follows:

\[
\theta(2) = \theta(1) + t(2) \begin{bmatrix} 0.462 - (o_2^{j_1\ldots j_6})^T \theta(1) \end{bmatrix}
\]

\[
t(2) = \frac{P(2) o_2^{j_1\ldots j_6}}{\left[ P(2) o_2^{j_1\ldots j_6} (o_2^{j_1\ldots j_6})^T + 1 \right]} \tag{2.105}
\]

\[
P(2) = 1 - \frac{P(1) o_2^{j_1\ldots j_6}}{\left[ P(1) o_2^{j_1\ldots j_6} (o_2^{j_1\ldots j_6})^T + 1 \right]} 1 * (o_2^{j_1\ldots j_6})^T
\]

Similarly, we can update all parameters of \( \theta \) for all training data. Repeat all procedures in step 2 with \( q = q + 1 \) until the last specific number of \( q \) is reached. From the program of the AFS using a neural network that is given by model (2.102), we have obtained better results with the AFS using a neural network than the result of the FIS. We have applied the measure of accuracy for all data, and obtained the average error of Simple LD=0.000943, while the average error of Acute LD=0.0055, (see Table 2). Additionally, obtained the best average testing error of training data (0.00000679) with epoch 100, and the average testing error of checking data (0.03802), see Figure 2.6. Table 3 presents the representation of results of the FIS with Mamdani model, the FIS with ST models, and the ANFIS with their errors through Table 2.3.
2.8 CONCLUSION

This chapter focused on how to use the fuzzy models to solving fuzzy mathematics problems. Here, we have constructed two different models, namely fuzzy inference system, and adaptive neuro-fuzzy inference system. Further, suggested an extension FS and ANFS at $N$-dimensions those depended on the MMI with the MIS, the SF, and the CAD. It is provided the theorem for accuracy of proposed models, as well as, adapted the model of an adaptive FS using a neural network. In addition, we have provided a medical application of the fuzzy mathematics models. We have diagnosed the liver disorder disease that is a high interest to researchers of fuzzy modeling and fuzzy system because the liver is the largest internal member in the human body. Therefore, we have applied the fuzzy mathematical models on the real data to the liver disorder disease. Presented discussion and results for the model of the FS with Mamdani and ST models, and the ANFIS, respectively. We have used the software ‘MATLAB’ in order to
performe the results of different models. Consequently, we have gotten good results for accuracy of these models. The comparsion between the three models the FS with Mamdani model, the ST model, and the ANFIS was presented. We have obtained the best result with the ANFIS. Finally, we have presented the representation of results of the different models with their errors through Table 2.3. In the future work, we can develop these models of fuzzy system to generate many outputs or extending the number of input variables. As well as, we can change the fuzzy inference system with another types, or with different type of fuzzifier or defuzzifier.

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