Chapter - I

Introduction

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Preliminaries
1.1 GENERAL INTRODUCTION:

Algebraic systems endowed with a partial or full ordered are met with in several disciplines of mathematics. In recent years interest in the study of partially ordered and fully ordered semigroups, groups, semirings, semimodules, rings and fields has been increasing enormously. The theory of semigroups had essentially two origins. One was an attempt to generalize both group theory and ring theory to the algebraic system consisting of a single associated operation, which from the group theoretical point of view omits the axioms of the existence of the identities and inverses and from the ring theoretical point of view omits the additive structure of the ring. The ring theory and field theory are two important development theories in algebra.

Much of the theory of rings applied to arbitrary semirings. In particular, one can generalize the theory of Algebra over commutative rings directly to the theory of Algebras over commutative semirings. Then a ring is simply algebra over the commutative semiring $\mathbb{Z}$ of integers. Some mathematicians go so far as to say that semirings are really the more fundamental concept, and specializing to rings should be seen in the same light as specializing to say algebra over the complex numbers.
There are many concepts of universal algebras generalizing that of a ring \((R, +, \cdot)\). Among them are those called semirings, which originate from rings, roughly speaking, by cancelling the assumption that \((R, +)\) has to be a group.

The theory of semirings is attracting the attention of several algebraists due to its applications to Computer Science, Optimization theory, Automata theory, Formal language theory and the Mathematical Modeling of Quantum Physics. Especially semirings with different properties have become important in theoretical computer science.

The concept of semiring was first introduced by Van Diver in 1934. However the developments of the theory in semirings have been taking place since 1950.

Semirings abound in the mathematical world around us. A semiring is one of the fundamental structures in mathematics. Indeed the first mathematical structure we encounter – the set of natural numbers is a semiring. Other semirings arise naturally in such diverse areas of mathematics as combinatorics, functional analysis, topology, graph theory, Euclidean geometry, probability theory, commutative, non-commutative ring theory and the mathematical modeling of quantum physics and parallel computation systems.
Semiring theory stands with a foot in each of two mathematical domains. On one hand, semirings are abstract mathematical structures and their study is part of abstract algebra- arising ab initio from the work of Dedekind, Macaulay, Krull and others on the theory of ideals of a commutative ring and then through the more general work of Vandiver- and the tools used to study them is primarily the tools of abstract algebra. On the other, the modern interest in semirings arises primarily from fields of Applied Mathematics such as Optimization theory, the theory of discrete-event dynamical systems, automata theory and formal language theory, as well as from the allied areas of theoretical computer science and theoretical physics and the questions being asked is, for the most part, motivated by applications.

The study of rings, which are special semirings reveals that multiplicative structure are quite independent of their additive structures though their additive structures are abelian groups. However in semirings it is possible to derive the additive structures from their special multiplicative structures and vice-versa. The developments of semirings and ordered semirings in this direction require semigroup techniques. It is well known that if the multiplicative structure of an ordered semiring is a rectangular band, then its additive structure is a
band. P.H.Karvellas [25] has proved that if the multiplicative structure of a semiring is a regular semigroup and if the additive structure is an inverse semigroup, then the additive structure is commutative. J. Hanumanthachari and K. Venuraju [14] studied the additive structure of a semiring in the case of semiring with identity element, is a band (or) commutative (or) a semilattice. S.Mitchell and P. Sinutoke [28] studied the structures of positive rational domains and semifields.

Semirings have been studied by various researchers in an attempt to broaden techniques coming from the semigroup theory or ring theory or in connection with applications.

The theory of rings and the theory of semigroups have considerable impact on the developments of the theory of semirings and ordered semirings. The works of eminent people like S. Bourne [4,5], P. J. Allen [1,2], S.K. Iseki [20,21,22], M.P. Grillet [11,12], H.E. Stone [44], H.J. Weinert [47,48,49] are to be worth mentioned in the theory of semirings, who use ring theory techniques. J.Hanumanthachari [14], T. Vasanthi [46] have contributed to the theory of semirings using semigroup techniques.

During the last three decades, there is considerable impact of semigroup theory and semiring theory on the development of ordered
semirings both in theory and applications, which are akin to ordered rings and ordered semirings. In this direction the works of H.J. Weinert [47,48], M. Satyanarayana [33,34,35,36,37], J. Hanumanthachari, K. Venuraju and H.J. Weinert[15], M. satyanarayana, J. Hanumanthachari and D. Umamaheswara Reddy [38,39], K.P. Shum and C.S. Hoo [42], K.P. Shum and C.Y. Hung [43], Kehayopulu [26,27], G. Therrin [45], H. Jurgensen, H.J. Shyr, Gerand Lallment, U. Zimmermann and Jonathan S. Golan[8] are to be worth mentioning.

Using the techniques of (ordered) semigroups, M. satyanarayana [37] examines in his paper whether the multiplicative structure of semirings determines the order structure as well as the additive structure of the semirings. In other paper, M. Satyanarayana [36] deals with the problem determine when the multiplicative semigroup is o.Archimedean if its additive semigroup is o. Archimedean and conversely in totally ordered semirings.

In a survey it is observed that the theory of semirings and ordered semirings find wide applications in linear and combinatorial optimization problems such as path problems transportation and assignment problems, matching problems and Eigen value problems.
The aim of the author in this thesis is to contribute some new results in the theory of “Structure of Semirings”.
1.2. A BRIEF SURVEY OF THE RESULTS ON SEMIRINGS AND ORDERED SEMIRINGS

Homomorphism theorems for semirings have been discussed by S.Bourne [5] and P.J.Allen [1]. M.P.Grillet [11] has considered subdivision rings of semirings and in [12] he has studied the structure of semirings in which additive semigroup is completely simple. The theory of ideals and quasi-ideals for semirings has been studied by K.Iseki [21, 22]. Radicals and semiradicals in semirings have been considered by S.Bourne [4], K.Iseki and Y.Miyanaga [23] and Bourne and Zassen-haus [6]. Henriksen [18] has generalized Jacobson’s theorem where as Allen [2] has extended the Hilbertbasis theorem to semirings. H.J.Weinert [47] has shown that each semiring is isomorphic to a subsemiring of certain semi-near-ring of partial transformations.

M.Satyanarayana [34] has studied some multiplicative conditions under which the additive structure of a semiring is a band (or) commutative, which again prompted K.Venuraju and J. Hanumanthachari [14] to study the question when the additive structure of semirings; in the case of semiring with identity element is band (or) commutative (or) a semilattice. Sidney S. Mitchell and
Kyungpook Porntip Sinutoke [28] studied the structure of positive rational domains and semifields.

In the recent papers on ordered semirings, the works of M. Satyanarayana [35], J. Hanumanthachari, K. Venuraju and H.J. Weinert [15], M. Satyanarayana, J. Hanumanthachari and D. Umamaheswareddy [38, 39] are to be mentioned. M. Satyanarayana [35] has studied how far the properties of multiplicative structure are reflected in the additive structure and vice-versa. J. Hanumanthachari, K. Venuraju and H.J. Weinert [15] have studied weak partially ordered semirings (w.p.o.s.r) in which additive structures are idempotents. M. Satyanarayana, J. Hanumanthachari and D. Umamaheswara Reddy [38] have studied additive structure of totally ordered semirings (S, +, •) in which the multiplicative structures (S, •) are o-Archimedean and p.t.o. They proved that (S, +) is a band in which the addition is one of the four types namely commutative minimum addition, commutative maximum addition, left zero addition and the right zero addition or (S, +) is o-Archimedean and positively ordered in the strict sense in the case (S, •) is non-cancellative, (S, +) is a nilsemigroup. They also studied in [39], the multiplicative structure of totally ordered semirings (S, +, •) in which (S, +) is o-Archimedean and r.n.t.o. In [16], D. Umamaheswara
Reddy and J.Hanumanthachari studied the properties of maximal and minimal elements in the additive and multiplicative structures of certain classes of totally ordered (t.o.) semirings. Also the inter-relation between cancellation laws and zero-divisors in a totally ordered semiring (t.o.s.r) is discussed. The main aim of this thesis is to study the additive and multiplicative structure of semirings. The results obtained by the author on Structure of Semirings are presented in section 1.3.

For future study, the author is interested to study the applications of semirings and ordered semirings to computer science.
1.3 SUMMARY OF THE RESULTS OBTAINED BY THE AUTHOR:

Chapter one deals with a general introduction and brief survey on the developments of semirings and ordered semirings which also includes the major results of the author which are going to be proved in the subsequent chapters.

In chapter two, we study the properties of semirings and ordered semirings satisfying the identity \( ab + a = a \), for all \( a, b \) in \( S \). We also characterize zerosumfree semirings. The main results of chapter 2 are:

1.3.1: Let \( (S, +, \cdot) \) be a semiring satisfying the identity \( ab + a = a \), for all \( a, b \) in \( S \). If \( S \) contains the multiplicative identity \( 1 \), then \( (S, +) \) is a band.

1.3.2: Let \( (S, +, \cdot) \) be a semiring satisfying the identity \( ab + a = a \), for all \( a, b \) in \( S \). Let \( S \) contain the multiplicative identity \( 1 \) and \( (S, +) \) be commutative. Then \( (S, \cdot) \) is commutative if \( (S, +) \) is not a rectangular band.

1.3.3: Let \( (S, +, \cdot) \) be a semiring satisfying the identity \( ab + a = a \), for all \( a, b \) in \( S \). Then the following are true.

(i) If \( (S, \cdot) \) is a band, then \( (S, +) \) is a band.
(ii) Converse is true if \((S, +)\) is right cancellative.

1.3.4: Let \((S, +, \cdot)\) be a semiring satisfying the identity \(ab + a = a\), for all \(a, b \in S\). Let \((S, +)\) be commutative and \((S, \cdot)\) is a rectangular band. Then the following are true.

(i) \(ab = a\) and \(ba = b\).

(ii) \((S, +)\) is a band.

1.3.5: Let \((S, +, \cdot)\) be a zerosumfree semiring with additive identity 0. Then \(ab + a = a\), for all \(a, b \in S\) if and only if \(ab = 0\).

1.3.6: Let \((S, +, \cdot)\) be a zero square semiring, where 0 is the additive identity. If \(S\) satisfies the identity \(ab + a = a\), for all \(a, b \in S\), then \(aba = 0\) and \(bab = 0\).

1.3.7: Let \((S, +, \cdot)\) be a PRD semiring satisfying the identity \(ab + a = a\), for all \(a, b \in S\). Then the following are true

(i) \(a + ab = b + ab\), for all \(a, b \in S\)

(ii) \(a + a^2 = a^2\) and \(a^2 + a = a\), for all ‘a’ in \(S\)

1.3.8: Let \((S, +, \cdot)\) be a semiring satisfying the identity \(ab + a = a\), for all \(a, b \in S\). If \((S, \cdot)\) is left regular semigroup, then \((S, +)\) is E – inversive semigroup.
1.3.9: Let \((S, +, \cdot)\) be a semiring satisfying the identity \(ab + a = a\), for all \(a, b \in S\). If \(S\) contains multiplicative identity which is also an additive identity, then \((S, \cdot)\) is quasi separative.

1.3.10: Let \((S, +, \cdot)\) be a totally ordered semiring satisfying the identity \(ab + a = a\), for all \(a, b \in S\). If \((S, +)\) is p.t.o (n.t.o.) and \((S, \cdot)\) is commutative, then \((S, \cdot)\) is n.t.o. (p.t.o.).

Chapter three deals with semirings and ordered semirings satisfying the identity \(a + ab + a = a\), for all \(a, b \in S\). We characterize zero square semirings. It is proved that in a zero square semiring \((S, +, \cdot)\), where 0 is the additive identity if \(S\) satisfies the identity \(a + ab + a = a\), for all \(a, b \in S\), then \((S, +)\) is a band. The principal results of this chapter are:

1.3.11: Let \((S, +, \cdot)\) be a semiring. If \(S\) contains the multiplicative identity which is also an additive identity, then \((S, \cdot)\) is left singular if and only if \(S\) satisfies the condition \(a + ab + a = a\), for all \(a, b \in S\).

1.3.12: Let \((S, +, \cdot)\) be a semiring satisfying the identity \(a + ab + a = a\), for all \(a, b \in S\). If \(S\) contains the multiplicative identity which is also an additive identity then \((S, +)\) is a rectangular band.
1.3.13: Let \((S, +, \cdot)\) be a semiring satisfying the identity \(a + ab + a = a\), for all \(a, b \in S\). If \(S\) contains the multiplicative identity which is also an additive identity, then \((S, +, \cdot)\) is a monosemiring.

1.3.14: Let \((S, +, \cdot)\) be a semiring satisfying the identity \(a + ab + a = a\), for all \(a, b \in S\). If \(S\) contains the multiplicative identity which is also an additive identity, then

(i) \((S, +)\) is a band

(ii) \((S, \cdot)\) is a band

1.3.15: Let \((S, +, \cdot)\) be a zerosumfree semiring containing the multiplicative identity which is also an additive identity. Then \(a + ab + a = a\), for all \(a, b \in S\) if and only if \(ab=0\).

1.3.16: Let \((S, +, \cdot)\) be a semiring satisfying the identity \(a + ab + a = a\), for all \(a, b \in S\). If \(S\) contains the multiplicative identity which is also an additive identity, then \(S\) is multiplicatively subidempotent.

1.3.17: Let \((S, +, \cdot)\) be a zero square semiring, where 0 is the additive identity. If \(S\) satisfies the identity \(a + ab + a = a\), for all \(a, b \in S\), then \(aba = 0\) and \(bab = 0\).
1.3.18: Let \((S, +, \cdot)\) be a zero square semiring, where 0 is the additive identity. If \(S\) satisfies the identity \(a + ab + a = a\), for all \(a, b\) in \(S\), then \((S, +)\) is a band.

1.3.19: Let \((S, +, \cdot)\) be a PRD satisfying the identity \(a + ab + a = a\), for all \(a, b\) in \(S\). Then \(a + 1 + a = a\), for all ‘\(a\)’ in \(S\).

1.3.20: Every Boolean semiring in which \(a + ab + a = a\), for all \(a, b\) in \(S\), then \(S = \{a, 2a\} \cup \{b, 2b\} \cup \ldots \ldots\)

1.3.21: Let \((S, +, \cdot)\) be a totally ordered monosemiring and satisfying the identity \(a + ab + a = a\), for all \(a, b\) in \(S\). If \((S, +)\) is p.t.o., then \(a + b = a\).

1.3.22: Let \((S, +, \cdot)\) be a totally ordered semiring and satisfying the identity \(a + ab + a = a\), for all \(a, b\) in \(S\). If \((S, +)\) is p.t.o (n.t.o.), then \((S, \cdot)\) is n.t.o. (p.t.o.).

In chapter four, we discuss viterbi semirings. We also study the properties of viterbi semirings and ordered viterbi semirings. The main results of this chapter are:

1.3.23: If \((S, +, \cdot)\) is a semiring in which \((S, +)\) and \((S, \cdot)\) are left singular semigroups, then \((S, +, \cdot)\) is a viterbi semiring.
1.3.24: Let \((S, +, \cdot)\) be a semiring containing multiplicative identity. If \(S\) contains absorbing element 1 w.r.to `\+', then

(i) \((S, +, \cdot)\) is a viterbi semiring

(ii) The converse is also true if \((S, \cdot)\) is left cancellative and \((S, +)\) is commutative

1.3.25: Every C–semiring is a viterbi semiring.

1.3.26: Let \((S, +, \cdot)\) be a viterbi semiring. If \(S\) satisfies IMP, then the following are true.

(i) \((S, \cdot)\) is a band

(ii) If \((S, \cdot)\) is quasi commutative, then \((S, \cdot)\) is commutative

1.3.27: Let \((S, +, \cdot)\) be a viterbi semiring containing multiplicative identity 1. If \((S, +)\) is right cancellative, then \((S, +)\) is a rectangular band.

1.3.28: Let \((S, +, \cdot)\) be a viterbi semiring. Then \(S\) satisfies the condition \(a^2 = a + a^2\), for all `\(a\)’ in \(S\), if and only if \((S, \cdot)\) is a band.

1.3.29: Let \((S, +, \cdot)\) be a t.o. PRD viterbi semiring. If \((S, +)\) is p.t.o., then 1 is the maximum element.

1.3.30: Let \((S, +, \cdot)\) be a t.o. viterbi semiring. If \((S, +)\) is non–negatively ordered, then \((S, \cdot)\) is non–positively ordered.
In Chapter five, we study the properties of complemented semirings and ordered complemented semirings. Properties of Boolean like semirings are also studied.

**1.3.31:** Let `a` is a complemented element in a semiring. Then \( a^n + b^n = 1 \), for all \( n \geq 1 \).

**1.3.32:** If \((S, +, \cdot)\) is a complemented semiring containing the multiplicative identity 1. If \(S\) contains additive identity zero, then

(i) \((S, \cdot)\) is a band

(ii) \((S, +)\) is commutative

**1.3.33:** Let \((S, +, \cdot)\) be a t.o. complemented semiring. If \((S, +)\) is p.t.o., then \((S, \cdot)\) is n.t.o.

**1.3.34:** Let \((S, +, \cdot)\) be a t.o. complemented semiring. If \((S, \cdot)\) is p.t.o., then \((S, +)\) is a band.

**1.3.35:** Let \((S, +, \cdot)\) be a boolean like semiring. If \(S\) contains the multiplicative identity 1 and \(a + 1 = 1\), then \((S, \cdot)\) is multiplicatively subidempotent.
1.3.36: Let \((S, +, \cdot)\) be a boolean like semiring containing the multiplicative identity which is also an additive identity, then \((S, \cdot)\) is an E - inverse semigroup.

1.3.37: Let \((S, +, \cdot)\) be a boolean like semiring containing the multiplicative identity 1. If \((S, +)\) is left singular, then \((S, \cdot)\) is left regular.

1.3.38: Let \((S, +, \cdot)\) be a boolean like semiring. Then the set \(X\) of all zero square elements is a multiplicative ideal of \(S\).

1.3.39: Let \((S, +, \cdot)\) be a t.o. boolean like semiring. If \((S, \cdot)\) is p.t.o. (n.t.o.), then 0 is the maximum (minimum) element.

The results of chapter 2 are published in “The International Journal of Mathematical Archive”, Vol.35 (2013), pp.149-156.

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