CHAPTER 5

Stability of Eigenvalues for Predator-Prey Relationships
5.1 Introduction

One of the most well-known predator-prey models is named after the two scientists, Alfred Lotka (1880-1949) and Vito Volterra (1860-1946), who were among the first to study and apply this model. Lotka was born in Poland to American missionaries and moved to United States in 1902. Many of his contributions can be found in the book Elements of Physics Biology (1925). Volterra, an Italian mathematician, published work in partial differential equations, integral equations, and functional analysis.

Predator and prey models have been a interesting part of scientific studies for many years. This type of biological study can be modeled using differential equations. There are many ways to model behavior for predator versus prey.

The predator-prey model is a representation of the interaction between two species of animals that live in the same environment, and the quantity of each group of animals depends on two things: the birth or death rate and the successful encounters [21]. This model is based on important assumptions which restrict actual conditions in the system [22] (For getting more information about this model, see [23] and [24]):

For the purpose of describing the interaction between predators and prey, we decided to imagine the relationship between Sharks and fish. In this model, ‘F’ will always represent the population of the Fish and ‘S’ will represent the population of the Sharks. The population growth of the Fish is dependent on Sharks and of natural causes. The population growth of the Sharks is dependent on the number of Sharks that die from natural causes.

5.2 Formulation of Problem

There are many instances in nature where one species of animal feeds on another species of animal, which in turn feeds on other things. The first species is called the
predator and the second is called the prey. Theoretically, the predator can destroy all
the prey so that the latter become extinct. However, if this happens the predator will
also become extinct since, as we assume, it depends on the prey for its existence.
What actually happens in nature is that a cycle develops where at some time the prey
may be abundant and the predators few. Because of the abundance of prey, the
predator population grows and reduces the population of prey. This results in a
reduction of predators and consequent increase of prey and the cycle continues.
An important problem of ecology, the science which studies the interrelationships of
organisms and their environment, is to investigate the question of coexistence of the
two species. To this end, it is natural to seek a mathematical formulation of this
predator-prey problem and to use it to forecast the behaviour of populations of various
species at different times.

5.2.1 Risk and Food Availability
Sharks appear to be a major threat to fish.
Availability of prey helps animals decide where to live.

5.2.2 Predator-Prey Model: Fish & Sharks
We will create a mathematical model which describes the relationship between
predator and prey in the ocean. Where the predators are sharks and the prey are fish.
In order for this model to work we must first make a few assumptions.

5.2.3 Assumptions
Fish only die by being eaten by sharks, and of natural causes.
Shark only die from natural causes.
The interaction between shark and fish can be described by a function.
5.2.4 Differential Equations and how it relates to Predator-Prey

One of the most interesting applications of systems of differential equations is the predator-prey problem. In this project we will consider an environment containing two related populations—a prey population, such as fish, and a predator population, such as sharks. Clearly, it is reasonable to expect that the two populations react in such a way as to influence each other’s size.

The differential equations are very much helpful in many areas of science. But most of interesting real life problems involves more than one unknown function. Therefore, the use of system of differential equations is very useful. Without loss of generality, we will concentrate on systems of two differential equations.

5.3 The Lotka – Volterra Model

<table>
<thead>
<tr>
<th>System</th>
<th>Initial Conditions</th>
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| \[
\frac{df}{dt} = aF - bF^2 - cFS
\] | \[ F(0) = F^0 \]    |
| \[
\frac{ds}{dt} = -kS + dSF
\] | \[ S(0) = S^0 \]    |

F : The population of the fish at time t.
S : The population of the sharks at time t.
F^0 : The initial size of the fish population.
S^0 : The initial size of the shark population.
a : Reproduction rate of fish.
b : Death rate of fish.
c : Proportional to the number of fish that a shark can eat.
d : Amount of energy that a fish supplies to the consuming
The Model $\frac{df}{dt} = aF - bF^2 - cFS$;

$F'(t)$, The growth rate of the fish population is influenced, according to the first differential equation, by three different terms.

It is positively influenced by the current fish population size, as shown by the term $aF$, where $a$ is a constant, non-negative real number and $aF$ is the birth rate of the fish.

It is negatively influenced by the natural death rate of the fish, as shown by the term $-bF^2$, where $b$ is a constant, non-negative real number and $bF^2$ is the natural death rate of the fish.

It is also negatively influenced by the death rate of the fish due to consumption by sharks as shown by the term $-cFS$, where $c$ is a constant non-negative real number and $cFS$ is the death rate of the fish due to consumption by sharks.

The Model $\frac{ds}{dt} = -kS + dSF$;

$S'(t)$, the growth rate of the Shark population, is influenced, according to the second differential equation, by two different terms.

It is negatively influenced by the current shark population size as shown by the term $-kS$, where $k$ is a constant non-negative real number and $S$ is the shark population.

It is positively influenced by the shark-fish interactions as shown by the term $dSF$, where $d$ is a constant non-negative real number, $S$ is the shark population and $F$ is the fish population.
5.4 Solution via Equilibrium Points

Once the initial equations are understood, the next step is to find the equilibrium points.

These equilibrium points represent points on the graph of the function which are significant.

These are shown by the following computations.

Let \( X = \frac{dF}{dt} = F(a - bF - cS) \) and \( Y = \frac{ds}{dt} = S(-k + dF) \).

To compute the equilibrium points we solve \( \frac{dF}{dt} = 0 \) and \( \frac{ds}{dt} = 0 \).

\[
\frac{dF}{dt} = 0 \text{ When } F = 0 \text{ or } a - bF - cs = 0
\]

Solution: \( \{F = \frac{(a-cS)}{b}\} \).

\[
\frac{ds}{dt} = 0 \text{ When } S = 0 \text{ or } -k + dF = 0
\]

Solution: \( \{F = \frac{k}{d}\} \).

Now we find all the combinations:

One of our equilibrium points is \((0, 0)\).

For \( F = \frac{(a-cS)}{b} \) When \( S = 0 \), then \( F = \frac{(a-c(0))}{b} = \frac{a}{b} \).

Thus, one of our equilibrium points is \( \left( \frac{a}{b}, 0 \right) \).

For \( F = \frac{(a-cS)}{b} \) and \( F = \frac{k}{d} \), \( \frac{d}{d} \frac{k}{d} = \frac{(a-cS)}{b} \),

Solution: \( \{S = \frac{(-kb + ad)}{dc}\} \).

Thus, one of our equilibrium points is \( \left( \frac{k}{d}, \frac{(-kb + ad)}{dc} \right) \).

Our equilibrium points are \((0, 0), \left( \frac{a}{b}, 0 \right), \text{ and } \left( \frac{k}{d}, \frac{(-kb + ad)}{dc} \right) \).
Now, to study the stability of the equilibrium points we first need to find the Jacobian matrix which is:

\[
J(F, S) = \begin{bmatrix}
\frac{dX}{dF} & \frac{dX}{dS} \\
\frac{dY}{dF} & \frac{dY}{dS}
\end{bmatrix} = \begin{bmatrix}
a - 2bf - cS & -cF \\
dS & -k + dF
\end{bmatrix}
\]

To study the stability of \((0, 0)\):

\[
J(0, 0) = \begin{vmatrix}
a - \lambda & 0 \\
0 & -k - \lambda
\end{vmatrix}, \text{ Solution: } \{\lambda = a\}, \{\lambda = -k\}.
\]

Semi-stable since one eigenvalue is negative and one is positive.

To study the stability of \(\left(\frac{a}{b}, 0\right)\):

\[
J\left(\frac{a}{b}, 0\right) = \begin{vmatrix}
-a - \lambda & -\frac{ca}{b} \\
0 & -k + \frac{ad}{b} - \lambda
\end{vmatrix} = (-a - \lambda)(-k + \frac{ad}{b} - \lambda),
\]

Solution: \(\{\lambda = -a\}, \{\lambda = \frac{-kb + ad}{b}\}\).

Stable if \(\lambda = \frac{-kb + ad}{b} < 0\) \(i.e. ad < kb\).

Semi-stable if \(\lambda = \frac{-kb + ad}{b} > 0\) \(i.e. ad > kb\).

To study the stability of \(\left(k, \frac{-kb + ad}{dc}\right)\):

\[
J\left(k, \frac{-kb + ad}{dc}\right) = \begin{vmatrix}
a - 2bf - c\frac{ad - kb}{c} - \lambda & -\frac{ck}{d} \\
\frac{ad - kb}{c} & -\lambda
\end{vmatrix} = \begin{vmatrix}
a - 2bf - c\frac{ad - kb}{c} - \lambda & -\frac{ck}{d} \\
\frac{ad - kb}{c} & -\lambda
\end{vmatrix}
\]

\[
= \lambda kb + \lambda^2 dc - k^2 b + kad
\]

Solution: \(\{\lambda = \frac{-kb + \sqrt{k^2 b^2 + 4dk^2 b - 4kad^2}}{2d}\}, \{\lambda = \frac{-kb - \sqrt{k^2 b^2 + 4dk^2 b - 4kad^2}}{2d}\}\).

If we simplify a little more, we get:

\[
\lambda = \frac{-kb + \sqrt{k^2 b^2 + 4dk^2 b - 4kad^2}}{2d} = \frac{-kb + i\sqrt{ -k^2 b^2 - 4dk^2 b + 4kad^2}}{2d}
\]

\[
\lambda = \frac{-kb - \sqrt{k^2 b^2 + 4dk^2 b - 4kad^2}}{2d} = \frac{-kb - i\sqrt{ -k^2 b^2 - 4dk^2 b + 4kad^2}}{2d}
\]

Stable since both of the real parts are negative. The imaginary numbers tells us that it will be periodic.
5.5 Different Case Study for Checking the Variation in Eigen Value

Case 1 ($a \lambda > b k$)

\[ u(x,y) = x(6 - 2x - 4y) \]
\[ v(x, y) = y(-3 + 5x), \]
\[ x(0) = 1 \text{ and } y(0) = .5 \]

INPUT

function ydot = subprog(t,y);
\[
yd(1)=6*y(1)-2*y(1)^2-4*y(1)*y(2);
\]
\[
yd(2)=-3*y(2)+5*y(1)*y(2);
\]
ydot = [yd(1) yd(2)]';

subprog3

x=1
y=.5
[t y]=ode45(@subprog,[0 50],[x y]);
plot(y)
legend('fish',' sharks ')
xlabel('time')
ylabel('fish,sharks')
Stability of Eigenvalues for Predator-Prey Relationships

OUTPUT

\[ x(0) = 2 \quad \text{and} \quad y(0) = 3 \]
x(0) = .5, y(0) = .5

In this case, of course is the critical point for the fish population in a shark-free world, and k/λ for the fish population living with sharks. This case holds true regardless of the initial conditions, as long as F(0) > 0, and S(0) > 0.

Case 2 \((a \lambda < b k)\)

\[ u(x, y) = x(2 - 6x - 4y), \]
\[ v(x, y) = y(-3 + 5x), \]
\[ x(0) = 1 \& y(0) = .5 \]

INPUT

function ydot = subprog(t, y);

\[ yd(1) = 2*y(1) - 6*y(1)^2 - 4*y(1)*y(2); \]
\[ yd(2) = -3*y(2) + 5*y(1)*y(2); \]
\[ ydot = [yd(1) yd(2)]; \]

subprog3

x=1

y=.5
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\[ t \ y = \text{ode45}(@\text{subprog}, [0 \ 50], [x \ y]); \]

plot(y)

legend('fish', 'sharks ')

xlabel('time')

ylabel('fish, sharks')

**OUTPUT**

\[ x(0) = 2 \ \& \ y(0) = 3 \]
x(0) = 0.5 & x(0) = 0.5

In this case, when a/b is less than k/\lambda conclusion, which is supported by simple algebra. When a/b < k/\lambda, then for the shark equation, the critical point becomes a negative number! a/b < k/\lambda \Rightarrow \lambda a < bk results in a negative number. Therefore in this case, the shark population will die out regardless of the initial conditions!

So the solution would converge to the shark-free stable point.

**Case 3 All constants are equal**

u(x, y) = x(1 - 1x - 1y),

v(x, y) = y(-1 + 1x),

x(0) = 2 & y(0) = 3

**INPUT**

function ydot=subprog(t,y);

yd(1)=1*y(1)-1*y(1)^2-1*y(1)*y(2);

yd(2)=-1*y(2)+1*y(1)*y(2);

ydot=[yd(1) yd(2)];
subprog3

x=1

y=.5

[t y]=ode45(@(subprog,[0 50],[x y]);

plot(y)

legend('fish',' sharks ')

xlabel('time')

ylabel('fish, sharks')

OUTPUT
X(0) = .5 & y(0) = 1.5

In this case, what if all of the constants were the same? A simple glance at the equations tells us that this would be similar to the second case:
We get \( \frac{a}{b} = \frac{k}{\lambda} = 1 \), yet the critical point would be \((1, 0)\) which is on the y-axis (S) and identical to the stable point for the fish population in a shark-free world. So here the sharks die out again.

**Case 4 (**\( \mathbf{b} = \mathbf{0} \)**)

\[
\begin{align*}
u(x, y) &= x(2 - 0x - 1y), \\
v(x, y) &= y(-1 + 1x), \\
x(0) &= 1 & y(0) &= 0.5.
\end{align*}
\]

**INPUT**

function ydot=subprog(t,y);

\[
\begin{align*}
yd(1) &= 2*y(1) - 0*y(1)^2 - 0*y(1)*y(2); \\
yd(2) &= -1*y(2) + 1*y(1)*y(2); \\
ydot &= [yd(1) \quad yd(2)];
\end{align*}
\]

**subprog3**

\[
\begin{align*}
x &= 2 \\
y &= 3 \\
[t \ y] &= \text{ode45}(@\text{subprog},[0 \ 50],[x \ y]); \\
\text{plot}(y) \\
\text{legend}(\text{fish, sharks }) \\
\text{xlabel}(\text{time}) \\
\text{ylabel}(\text{fish, sharks})
\end{align*}
\]

**OUTPUT**
In this case, we make \( b = 0 \) which turns the model into the simplest form of the predator/prey model.

The new equations look like this:

\[
F'(t) = F (a - 0F - cS) = F (a - cS) \quad \text{and} \quad S'(t) = S (-k + \lambda F)
\]
So the critical point becomes \((k/\lambda, a/c)\) and we get an ellipse around the critical point, the shape and size depending on the constants and initial conditions.

So both the fish and shark populations wax and wane in a cyclic pattern with the sharks lagging behind the fish.

**Case 5**

\[ \frac{k}{\lambda} = \frac{a\lambda - bk}{c\lambda} \]

\[ u(x,y) = x(2 - 1x - 1y), \]

\[ v(x, y) = y(-1 + 1x), \]

\[ x(0) = 1 & y(0) = 0.5 \]

**INPUT**

```matlab
function ydot=subprog(t,y);
yd(1)=2*y(1)-1*y(1)^2-1*y(1)*y(2);
yd(2)=-1*y(2)+1*y(1)*y(2);
ydot=[yd(1) yd(2)]';
```

```matlab
x=1
y=.5
[t y]=ode45(@subprog,[0 50],[x y]);
plot(y)
legend('fish',' sharks ')
xlabel('time')
label('fish,sharks')
```
OUTPUT

\[ x(0) = 0.5 \quad \text{and} \quad y(0) = 0.5 \]

Now for the last case, we pose the question:

What happens when \( F(t) = S(t) \), i.e. \( k/\lambda = (a\lambda - bk)/c\lambda \)?
Answer: This is similar to the first case, since $a/b > k/\lambda$, with a simpler spiral as the result. The only significant impact is the location of the critical point.

There are other cases that we have yet to explore here, such as $a = 0, c = 0, k = 0, \lambda = 0$, or a combination of those, but those would render the model meaningless, as $\lambda$ they would cancel the relationship between the fish and the sharks or eliminate the fish’s growth rate or the shark’s death rate.

5.6 Conclusion

We conclude from the above graph, the shark fish and other fish quantity are correlative increased and decrease at some time but after a long time there will be no relative difference between their quantities or we can say that they are relatively stable. So finally we say that in the usual system both are survive no one is drop from the system. If, other fish will die then existence of sharks are not possible in this world and if sharks will die then the other fish will die after some time due to the exotic level of their own community fish.