CHAPTER 5
REPLACEMENT PROBLEMS FOR DIFFERENT TYPES OF VARYING MAINTENANCE COST FUNCTIONS

5.1 INTRODUCTION

In engineering and chemical industries there are different types of machines and equipments used for production and they are generally costly. A sizable amount of investment is necessary to make provision for such equipments. These equipments when installed are under constant usage and hence they need to be replaced after certain period when their efficiency goes on deteriorating as per passage of time. It may be noted that their maintenance being expensive, day by day, gives rise to a varying nature of maintenance cost function. Similarly, when the equipment is to be sold after a passage of time, the resale value for the equipment also varies substantially this leads to consider a varying form of the scrap value function. Both these approaches together can be incorporated for the policy-making decisions.

In this chapter, three different types of varying maintenance cost functions pertaining to the discrete cases are considered. The scrap value function is also of varying nature. Three models regarded as models (A), (B) and (C) are developed for all the three relevant cases and they have been illustrated by practical applications. Sensitivity analysis is also carried out for the relevant models that are developed.

5.2 ASSUMPTIONS

This model is derived based on the following assumptions.

(1) The equipment is under constant usage from its installation.
(2) The maintenance cost function consists of a fixed operating cost as well as a variable cost, which is partly linear and partly varying inversely with time. Thus maintenance cost function is of the form

\[ A(t) = a_0 + a_1 t + a_2 / t \]

\[ t = 1, 2, 3, \ldots n. \]

\[ a_0 > a_1 > a_2 > 0 \]
This cost function increases appreciably as per passage of time and it exhibits a linear trend during successive finite and discrete time period.

3) Salvage value function is a linear function of time \( t \) and it is specified by

\[
S(t) = \alpha - \beta t
\]

\( t = 1, 2, 3, \ldots n \)

\( \alpha > \beta > 0 \)

This function is a real, discrete and non-increasing function of time \( t \).

4) The value of money remains the same as per the passage of time.

5.A.1 DERIVATION OF THE MODEL A

A costly machine part is installed at some capital cost \( C \). The annual maintenance cost for year \( t \) is specified by \( A(t) \) and the salvage value for some period \( t \) is given by \( S(t) \). When this equipment is used continuously for some period \( n \), it needs replacement due to deterioration as per the passage of time.

Under above assumptions,

\[
A(t) = a_0 + a_1 t + \frac{a_2}{t}
\]

\( t = 1, 2, 3, \ldots n \)

\( a_0 > a_1 > a_2 > 0 \) (5.A.1)

Where \( a_0 \) is fixed cost. \( a_1 \) and \( a_2 \) are the varying costs.

Then the cumulative maintenance cost during the usage for some period \( n \) is

\[
\sum_{t=1}^{n} A(t) = \sum_{t=1}^{n} \left( a_0 + a_1 t + \frac{a_2}{t} \right)
\]

\[
= a_0 n + a_1 \sum_{t=1}^{n} t + a_2 \sum_{t=1}^{n} \frac{1}{t}
\]

(5.A.2)

Hence average total cost for the equipment is given by

\[
ATC_n = \frac{(C - S) + \int_{1}^{n} A(t) dt}{n}
\]

(5.A.3)

So that,

\[
ATC_n = \frac{C - \alpha}{n} + \left( a_0 + \beta \right) + \frac{a_1 (n+1)}{2} + a_2 H_n
\]

(5.A.4)

Where, \( H_n \) = Harmonic mean

\[
= \frac{1}{n} \sum_{t=1}^{n} \frac{1}{t}
\]
The optimum replacement policy decision rule indicates to find such a value of \( n \) for which \( \text{ATC}_n \) is minimum.

Since \( \text{ATC}_n \) is a discrete function of period \( n \), for optimality criteria,
we must have
\[
\text{ATC}_{n+1} \geq \text{ATC}_n
\]
as well as
\[
\text{ATC}_{n-1} \geq \text{ATC}_n
\]
Simplification gives to the inequalities in (5.A.5) above yields the following inequality
\[
(n-1)\left[ \frac{a_1}{2} + a_2 \Delta H_{n-1} \right] \leq \frac{(C-a)}{n} \leq \left[ \frac{a_1}{2} + a_2 \Delta H_n \right]
\]
Where
\[
\Delta H_{n-1} = H_n - H_{n-1} \\
\Delta H_n = H_{n+1} - H_n
\]
Which determines the optimum replacement period \( n^* \) and then on substituting it in (5.A.4) above, the minimum value of \( \text{ATC}_n \) can be determined.

5.1.2 Fitting Maintenance Cost Function

Let us write the maintenance cost function as given by
\[
A(t) = a_0 + a_1 t + a_2 t^2 + U
\]
Where \( U \) denotes the disturbance term. Let us write \( 1/t = T \)
Then
\[
A(t) = a_0 + a_1 t + a_2 T + U
\]
Here \( a_0, a_1 \) and \( a_2 \) are the parameters of the maintenance cost function. We can collect data pertaining to this cost function for some period and then the parameters can be estimated by using the least squares principles, i.e. Determine \( \hat{a}_0, \hat{a}_1 \) and \( \hat{a}_2 \) so that the residual sum of squares is minimum

Hence the problem now reduces to minimizing
\[
\sum_{i=1}^{n} \left[ A(t) - \hat{a}_0 - \hat{a}_1 t - \hat{a}_2 t^2 \right]^2
\]
(5.A.7)
The normal equations are as under

\[ \sum_{i=1}^{n} A(t_i) = n \hat{a}_0 + a_1 \sum_{i=1}^{n} t_i + a_2 \sum_{i=1}^{n} T_i \]

\[ \sum_{i=1}^{n} TA(t_i) = \hat{a}_0 \sum_{i=1}^{n} t_i + \hat{a}_1 \sum_{i=1}^{n} t_i^2 + n \hat{a}_2 \]

\[ \sum_{i=1}^{n} TA(t_i) = a_0 \sum_{i=1}^{n} T_i + n a_1 + a_2 \sum_{i=1}^{n} T_i^2 \] (5.A.8)

Solving these equations, \( \hat{a}_0 \), \( \hat{a}_1 \) and \( \hat{a}_2 \) can be estimated. These are OLSE of \( a_0 \), \( a_1 \) and \( a_2 \) respectively. So that estimated maintenance cost function is

\[ \hat{A}(t) = \hat{a}_0 + \hat{a}_1 t + \hat{a}_2 \frac{1}{t} \] (5.A.9)

This can now be used for our decision building approach pertaining to this policy. We can apply statistical tests of significance to judge for the validity of the model.

5.A.3 FITTING OF SALVAGE VALUE FUNCTION

Let us consider salvage value function given by

\[ S(t) = \alpha - \beta t + U \] (5.A.10)

Where \( \alpha \) and \( \beta \) are the unknown parameters \( \alpha > \beta > 0 \) and U denotes the disturbance term. Here also based upon past data, the salvage value function can be estimated by using OLSE of \( \hat{\alpha} \) and \( \hat{\beta} \). Hence estimated salvage value function is

\[ \hat{S}(t) = \hat{\alpha} - \hat{\beta} t \] (5.A.11)

Substituting all estimated parameters in (5.A.4) above, we get the estimated ATCn as

\[ \hat{ATCn} = \frac{C - \hat{\alpha}}{n} + \left( \hat{a}_0 + \hat{\beta} \right) \frac{a_1 (n+1)}{2} + \hat{a}_2 H_n \] (5.A.12)

5.A.4 APPLICATION

For a certain engineering production in a small factory a specific type of engine part is utilized. Its purchase cost is Rs. 995 and it needs maintenance from year to year. Data are collected for the salvage value and the maintenance cost which are as given in table 5.A-1
For the above data, the estimators of parameters of the annual maintenance cost function and salvage value function are obtained and the numerical values are as under.

\[
\hat{a}_0 = 65, \quad \hat{a}_1 = 23, \quad \hat{a}_2 = 13, \quad \hat{\alpha} = 575, \quad \hat{\beta} = 25
\]

So that estimated maintenance cost function and the salvage value function are obtained as under

\[
\hat{A}(t) = 65 + 23t + \frac{13}{t} \quad \text{and} \quad \hat{S}(t) = 575 - 25t
\]

The estimated parameters are statistically significant at 5% level. Also the coefficient of determination \( R^2 \) is found to be highly significant at 5% level for both the fitted models. This establishes the validity of the above cost function model for the relevant data under consideration.

Hence,

\[
\hat{ATC}_n = \frac{C - \hat{\alpha}}{n} + \left( a_0 + \hat{\beta} \right) \cdot \frac{a_1(n+1)}{2} + \hat{a}_2 H_n
\]

Results for \( \hat{ATC}_n \) are now tabulated in Table 5.A-2 for the successive periods.
Original values: \( C = \text{cost price} = \text{Rs. 995} \)

\( a_0 = 65, \quad a_1 = 23, \quad a_2 = 13, \quad \alpha = 575, \quad \beta = 25 \)

**TABLE 5.A-2**

<table>
<thead>
<tr>
<th>Year</th>
<th>ATCn (Rs.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>546.00</td>
</tr>
<tr>
<td>2</td>
<td>344.25</td>
</tr>
<tr>
<td>3</td>
<td>283.93</td>
</tr>
<tr>
<td>4</td>
<td>259.26</td>
</tr>
<tr>
<td>5</td>
<td>248.93</td>
</tr>
<tr>
<td>6</td>
<td><strong>245.81</strong></td>
</tr>
<tr>
<td>7</td>
<td>246.81</td>
</tr>
<tr>
<td>8</td>
<td>250.42</td>
</tr>
<tr>
<td>9</td>
<td>255.75</td>
</tr>
<tr>
<td>10</td>
<td>262.31</td>
</tr>
</tbody>
</table>

From Table 5.A-2 given above, ATCn is minimum for the period \( n=6 \) years.

Hence the equipment needs replacement after using it for 6 years and the minimum ATCn is Rs. 245.81.

The result is also established from the derived inequalities given in (5.A.6) above.

**GRAPHICAL PRESENTATION (For Maintenance, Salvage Value, ATCn)**

![Graphical Presentation](image-url)
5.A.5 SENSITIVITY ANALYSIS

It may be worthwhile to consider the sensitivity analysis for the derived model. This is considered in two ways:

(I) Total sensitivity Analysis

(II) Partial sensitivity Analysis

(I) Total sensitivity Analysis:

We consider here the case when all the parameters change their value simultaneously.

Original values: \( C = \text{Rs} \, 995 \), \( a_0 = 65 \), \( a_1 = 23 \), \( a_2 = 13 \), \( \alpha = 575 \), \( \beta = 25 \)

<table>
<thead>
<tr>
<th>Changes in the parameter values</th>
<th>( n^* ) (year)</th>
<th>ATC( n^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10% Increase</td>
<td>6</td>
<td>253.80</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(3.25%)</td>
</tr>
<tr>
<td>10% Decrease</td>
<td>7</td>
<td>236.35</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(-3.85%)</td>
</tr>
</tbody>
</table>

(Note: Figures in parenthesis give % change in ATC\( n^* \) as compared to the original solution)

From the above table 5.A-3, we conclude that for 10% increase in all the parameters, minimum ATC\( n^* \) increases by about 3.25% and similarly for 10% decrease in all the parameter values, minimum ATC\( n^* \) decreases by 3.85% as compared to original solution.

For 10% increase in the parameter values, the optimum replacement period remains the same and for 10% decrease in the parameter values, the optimum replacement period increases by one year.
(II) Partial Sensitivity Analysis:

Original values: C = Rs 995, a₀ = 65, a₁ = 23, a₂ = 13, α = 575, β = 25

Here we want to consider the case when only one of the parameter changes by some fixed amount (say 10%) while all other remaining parameters are fixed. This case is illustrated in table 5.A-4 below.

<table>
<thead>
<tr>
<th>Partially Changing Parameter</th>
<th>n* (Year)</th>
<th>ATCn* (Rs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1 a₀</td>
<td>6</td>
<td>252.304</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(2.642%)</td>
</tr>
<tr>
<td>0.9 a₀</td>
<td>6</td>
<td>239.304</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(-2.754%)</td>
</tr>
<tr>
<td>1.1 a₁</td>
<td>6</td>
<td>253.86</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(3.275%)</td>
</tr>
<tr>
<td>0.9 a₁</td>
<td>7</td>
<td>237.60</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(-3.340%)</td>
</tr>
<tr>
<td>1.1 a₂</td>
<td>6</td>
<td>246.334</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.213%)</td>
</tr>
<tr>
<td>0.9 a₂</td>
<td>6</td>
<td>245.274</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(-0.218%)</td>
</tr>
<tr>
<td>1.1 α</td>
<td>6</td>
<td>236.23</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(-3.897%)</td>
</tr>
<tr>
<td>0.9 α</td>
<td>7</td>
<td>255.0243</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(3.749%)</td>
</tr>
<tr>
<td>1.1 β</td>
<td>6</td>
<td>248.31</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.017%)</td>
</tr>
<tr>
<td>0.9 β</td>
<td>6</td>
<td>243.31</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(-1.017%)</td>
</tr>
</tbody>
</table>

( Note: Figures in parenthesis give % change in ATCn * as compared to original solution.)

From the above table it may be concluded that the optimum replacement Period remains the same in almost all the cases except when there is 10% decrease in parameters a₁ and α. The most sensitive parameter is α for which minimum ATCn increases by about 3.75% for 10%
decrease in $\alpha$ alone and it reduces to about 3.90% for 10% increase in $\alpha$ alone as compared to the original solution.

Similarly next sensitive parameter is $a_1$. Here for 10% increase in $a_1$ alone, minimum ATCn increases by about 3.28% and for 10% decrease in $a_1$ alone, minimum ATCn decreases by about 3.34% as compared to the original solution.

It may be noted that $a_2$ is the least sensitive parameter, as it does not appreciably affect the minimum ATCn. The parameters $a_0$ and $\beta$ are other parameters which are more or less sensitive as compared to the original solution.

5.0.1 DERIVATION OF THE MODEL B

For this model with other assumptions as in Model (A), we consider the case of varying maintenance cost function as given by

$$A(t) = a_0 + a_1 t + a_2 \log t$$

$t = 1, 2, 3, \ldots n$. $a_0 > a_1 > a_2 > 0$ (5.B.1)

Where $a_0$ is fixed cost. $a_1$ and $a_2$ are the varying costs.

Then the cumulative maintenance cost during the usage for some period $n$ is

$$\sum_{i=1}^{n} A(t) = \sum_{i=1}^{n} (a_0 + a_1 t + a_2 \log t)$$

$$= a_0 n + a_1 \sum_{i=1}^{n} t + a_2 \sum_{i=1}^{n} \log t$$

(5.B.2)

Hence average total cost for the equipment is given by

$$ATCn = \frac{(C - S) + \int_{1}^{n} A(t) dt}{n}$$

So that, $ATCn = \frac{C - \alpha}{n} + (a_0 + \beta) + \frac{a_1 (n + 1)}{2} + a_2 L_n$ (5.B.3)

Where, $L_n = \frac{1}{n} \sum_{i=1}^{n} \log t$
The optimum replacement policy decision rule indicates to find such a value of \( n \) for which \( \text{ATC}_n \) is minimum.

Since \( \text{ATC}_n \) is a discrete function of period \( n \), for optimality criteria, we must have

\[
\text{ATC}_{n+1} \geq \text{ATC}_n \quad \text{as well as} \quad \text{ATC}_{n-1} \geq \text{ATC}_n
\]

Simplification gives to the inequalities in (5.B.4) above yields the following inequality

\[
(n-1) \left[ \frac{a_1}{2} + a_2 \Delta L_{n-1} \right] \leq \frac{(C-\alpha)}{n} \leq (n+1) \left[ \frac{a_1}{2} + a_2 \Delta L_n \right]
\]

(5.B.5)

Where

\[
\Delta L_{n-1} = L_n - L_{n-1} \\
\Delta L_n = L_{n+1} - L_n
\]

Which determines the optimum replacement period \( n^* \) and then on substituting it in (5.B.3) above, the minimum value of \( \text{ATC}_n \) can be determined.

While dealing with the above form of maintenance cost function and the salvage value function, the usual least squares approach (as indicated in the case of Model (A) above), can be applied for fitting these functions.

5.B.2 APPLICATION

Let us consider an application of this model for a specific production unit in chemical industry. A machine part used here costs Rs. 900 and data are collected for the relevant maintenance and salvage value functions from time to time, which are as given in table 5.B-1.
For the above data, the estimators of parameters of the annual maintenance cost function and salvage value function are obtained and the numerical values are as under.

\[ \hat{a}_0 = 85, \quad \hat{a}_1 = 40, \quad \hat{a}_2 = 23, \quad \hat{\alpha} = 570, \quad \hat{\beta} = 30 \]

So that estimated maintenance cost function and the salvage value function are obtained as under

\[ \hat{A}(t) = 85 + 40t + 23 \log t \quad \text{and} \quad \hat{S}(t) = 570 - 30t \]

The estimated parameters are statistically significant at 5% level. Also the coefficient of determination \( R^2 \) is also found to be highly significant at 5% level for both these functions. This suggests the validity of the above cost function model for the relevant data under consideration, we have

\[ \hat{A}TC_n = \frac{C - \hat{\alpha}}{n} + \left( \hat{a}_0 + \hat{\beta} \right) + \frac{\hat{a}_1 (n+1)}{2} + \hat{a}_2 \log n \]

Results for \( \hat{A}TC_n \) are now tabulated in Table 5.B-2 for the successive periods.
From Table 5.B-2 given above, ATCₙ is minimum for the period n=4 years. Hence the equipment needs replacement after using it for 4 years and the minimum ATCₙ is Rs. 308.08.

The result is also established from the derived inequalities given in (5.B.5) above.

**TABLE 5.B-2**

<table>
<thead>
<tr>
<th>Year</th>
<th>ATCₙ (Rs.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>485.00</td>
</tr>
<tr>
<td>2</td>
<td>343.46</td>
</tr>
<tr>
<td>3</td>
<td>312.12</td>
</tr>
<tr>
<td>4</td>
<td><strong>308.08</strong></td>
</tr>
<tr>
<td>5</td>
<td>314.80</td>
</tr>
<tr>
<td>6</td>
<td>326.78</td>
</tr>
<tr>
<td>7</td>
<td>341.70</td>
</tr>
<tr>
<td>8</td>
<td>358.40</td>
</tr>
<tr>
<td>9</td>
<td>376.28</td>
</tr>
<tr>
<td>10</td>
<td>415.61</td>
</tr>
</tbody>
</table>

**GRAPHICAL PRESENTATION (For Maintenance, Salvage Value, ATCₙ)**

- **Maintenance Cost (Rs.)**
- **Salvage Value (Rs.)**
- **ATCₙ (Rs.)**
5.B.3 SENSITIVITY ANALYSIS

(I) Total sensitivity Analysis:

Original values: C = Rs 900, \( a_0 = 85 \), \( a_1 = 40 \), \( a_2 = 23 \), \( \alpha = 570 \), \( \beta = 30 \)

<table>
<thead>
<tr>
<th>Changes in the parameter value</th>
<th>( n^*(\text{year}) )</th>
<th>( \text{ATCn}^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10% Increase</td>
<td>3</td>
<td>313.3318 (1.70%)</td>
</tr>
<tr>
<td>10% Decrease</td>
<td>4</td>
<td>299.7735 (-2.70%)</td>
</tr>
</tbody>
</table>

(Note: Figures in parenthesis give % change in \( \text{ATCn}^* \) as compared to the original solution)

From the above table, we conclude that for 10% increase in all the parameters, minimum \( \text{ATCn} \) increases by about 1.70% and similarly for 10% decrease in all the parameter values, minimum \( \text{ATCn} \) decreases by 2.70% as compared to original solution.

For 10% increase in the parameter values, the optimum replacement period decreases by one year and for 10% decrease in the parameter values, the optimum replacement period remains the same.
(II) Partial Sensitivity Analysis:

Original values: C = Rs 900, \( a_0 = 85 \), \( a_1 = 40 \), \( a_2 = 23 \), \( \alpha = 570 \), \( \beta = 30 \)

Table 5.B-4

<table>
<thead>
<tr>
<th>Partially Changing Parameter</th>
<th>( n^* ) (Year)</th>
<th>ATC(n^* ) (Rs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1 ( a_0 )</td>
<td>4</td>
<td>316.58 (2.76%)</td>
</tr>
<tr>
<td>0.9 ( a_0 )</td>
<td>4</td>
<td>299.58 (-2.76%)</td>
</tr>
<tr>
<td>1.1 ( a_1 )</td>
<td>4</td>
<td>318.082 (3.25%)</td>
</tr>
<tr>
<td>0.9 ( a_1 )</td>
<td>4</td>
<td>298.08 (-3.25%)</td>
</tr>
<tr>
<td>1.1 ( a_2 )</td>
<td>4</td>
<td>309.14 (0.35%)</td>
</tr>
<tr>
<td>0.9 ( a_2 )</td>
<td>4</td>
<td>307.0235 (-0.35%)</td>
</tr>
<tr>
<td>1.1 ( \alpha )</td>
<td>3</td>
<td>293.12 (-4.86%)</td>
</tr>
<tr>
<td>0.9 ( \alpha )</td>
<td>4</td>
<td>322.332 (4.63%)</td>
</tr>
<tr>
<td>1.1 ( \beta )</td>
<td>4</td>
<td>311.082 (0.974%)</td>
</tr>
<tr>
<td>0.9 ( \beta )</td>
<td>4</td>
<td>305.082 (-0.974%)</td>
</tr>
</tbody>
</table>

( Note: Figures in parenthesis give % change in ATC\(n^* \) as compared to original solution.)

From the above table 5.B-4 it may be concluded that the optimum replacement period remains the same in almost all the cases except when there is 10% increase in parameter \( \alpha \).

The most sensitive parameter is \( \alpha \) for which minimum ATC\(n \) increases by about 4.63% for 10% decrease in \( \alpha \) alone and it reduces to about 4.86% for 10% increase in \( \alpha \) alone as compared to the original solution.
Similarly next sensitive parameter is $a_1$. Here for 10% increase in $a_1$ alone, minimum ATCn increases by about 3.25% and for 10% decrease in $a_1$ alone, minimum ATCn decreases by about 3.25% as compared to the original solution.

It may be noted that $a_2$ is the least sensitive parameter, as it does not appreciably affect the minimum ATCn. The parameters $a_0$ and $\beta$ are other parameters which are more or less sensitive as compared to the original solution.

### 5.C.1 DERIVATION OF THE MODEL C

For this model with other assumptions as in Model (A), we consider the case of varying maintenance cost function as given by

$$A(t) = a_0 + a_1 t + a_2 e^{\beta t} \quad t = 1, 2, 3, \ldots, n \quad a_0 > a_1 > a_2 > 0$$

(5.C.1)

Where $a_0$ is fixed cost, $a_1$ and $a_2$ are the varying costs.

Then the cumulative maintenance cost during the usage for some period $n$ is

$$\sum_{t=1}^{n} A(t) = \sum_{t=1}^{n} \left( a_0 + a_1 t + a_2 e^{\beta t} \right)$$

$$= a_0 n + a_1 \sum_{t=1}^{n} t + a_2 \sum_{t=1}^{n} e^{\beta t}$$

(5.C.2)

Hence average total cost for the equipment is given by

$$ATCn = \frac{(C - S) + \int_{1}^{n} A(t) dt}{n}$$

So that, $ATCn = \frac{C - \alpha}{n} + (a_0 + \beta) + \frac{a_1(n+1)}{2} + a_2 Q_n$

(5.C.3)

Where, $Q_n = \frac{1}{n} \sum_{t=1}^{n} e^{\beta t}$
The optimum replacement policy decision rule indicates to find such a value of \( n \) for which \( \text{ATC}_n \) is minimum.

Since \( \text{ATC}_n \) is a discrete function of period \( n \), for optimality criteria,

\[ \text{we must have } \quad \text{ATC}_{n+1} \geq \text{ATC}_n \quad \text{as well as } \quad \text{ATC}_{n-1} \geq \text{ATC}_n \quad (5.C.4) \]

Simplification gives to the inequalities in (5.B.4) above yields the following inequality

\[ \left( n - 1 \right) \left[ \frac{a_1}{2} + a_2 \Delta Q_{n-1} \right] \leq \frac{(C - \alpha)}{n} \leq \left( n + 1 \right) \left[ \frac{a_1}{2} + a_2 \Delta Q_n \right] \quad (5.C.5) \]

Where

\[ \Delta Q_{n-1} = Q_n - Q_{n-1} \]
\[ \Delta Q_n = Q_{n+1} - Q_n \]

Which determines the optimum replacement period \( n^* \) and then on substituting it in (5.C.3) above, the minimum value of \( \text{ATC}_n \) can be determined.

While dealing with the above form of maintenance cost function and the salvage value function, the usual least squares approach (as indicated in the case of Model (A) above), can be applied for fitting these functions.

5.C.2 APPLICATION

An engine value used in engineering industry costs Rs 1060 at its market value. Due to constant usage, this part of the machine gets deteriorated and hence it needs to be replaced after some time. Data are collected for the relevant maintenance and salvage value functions from time to time, which are as given in table 5.C-1
For the above data, the estimators of parameters of the annual maintenance cost function and salvage value function are obtained and the numerical values are as under.

\[
\hat{a}_0 = 33, \quad \hat{a}_1 = 13, \quad \hat{a}_2 = 7, \quad \hat{\lambda} = 0.5, \quad \hat{\alpha} = 430, \quad \hat{\beta} = 20
\]

So that estimated maintenance cost function and the salvage value function are obtained as under

\[
\hat{A}(t) = 33 + 13t + 7 e^{-0.5t} \quad \text{and} \quad \hat{S}(t) = 430 - 20t
\]

The estimated parameters are statistically significant at 5% level. Also the coefficient of determination \(R^2\) is also found to be highly significant at 5% level for both these functions. This suggests the validity of the above cost function model for the relevant data under consideration, we have

\[
\hat{ATC}_n = \frac{C - \hat{\alpha}}{n} + (\hat{a}_0 + \hat{\beta}) + \frac{\hat{a}_1(n+1)}{2} + \hat{a}_3 Q_n
\]

Results for \(\hat{ATC}_n\) are now tabulated in Table 5.C-2 for the successive periods.
From Table 5.C-2 given above, ATCn is minimum for the period n = 5 years. Hence the equipment needs replacement after using it for 4 years and the minimum ATCn is Rs. 279.50. The result is also established from the derived inequalities given in (5.C.5) above.

**GRAPHICAL PRESENTATION (For Maintenance, Salvage Value, ATCn)**

![Graphical presentation](image)
5.C.3 SENSITIVITY ANALYSIS

(I) Total sensitivity Analysis:

Original values: \( C = \text{Rs 1060}, \hat{a}_0 = 33, \hat{a}_1 = 13, \hat{a}_2 = 7, \lambda = 0.5, \alpha = 430, \beta = 20 \)

<table>
<thead>
<tr>
<th>Changes in the parameter value</th>
<th>n*(year)</th>
<th>ATCn*</th>
</tr>
</thead>
<tbody>
<tr>
<td>10% Increase</td>
<td>4</td>
<td>296.4510 (6.065%)</td>
</tr>
<tr>
<td>10% Decrease</td>
<td>6</td>
<td>263.5956 (-5.6898%)</td>
</tr>
</tbody>
</table>

(Note: Figures in parenthesis give % change in ATCn* as compared to the original solution)

From the above table, we conclude that for 10% increase in all the parameters, minimum ATCn increases by about 6.065% and similarly for 10% decrease in all the parameter values, minimum ATCn decreases by 5.6898% as compared to original solution.

For 10% increase in the parameter values, the optimum replacement period decreases by one year and for 10% decrease in the parameter values, the optimum replacement period increases by one year.
(II) Partial Sensitivity Analysis:

Original values: \( C = \text{Rs} \, 1060 \), \( a_0 = 33 \), \( a_1 = 13 \), \( a_2 = 7 \), \( \lambda = 0.5 \), \( \alpha = 430 \), \( \beta = 20 \)

Table 5.04

<table>
<thead>
<tr>
<th>Partially Changing Parameter</th>
<th>( n^* ) (Year)</th>
<th>ATCn* (Rs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1 ( a_0 )</td>
<td>5</td>
<td>282.7985</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.18%)</td>
</tr>
<tr>
<td>0.9 ( a_0 )</td>
<td>5</td>
<td>276.1985</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(-1.18%)</td>
</tr>
<tr>
<td>1.1 ( a_1 )</td>
<td>5</td>
<td>283.3986</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.395%)</td>
</tr>
<tr>
<td>0.9 ( a_1 )</td>
<td>5</td>
<td>275.5985</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(-1.395%)</td>
</tr>
<tr>
<td>1.1 ( a_2 )</td>
<td>5</td>
<td>285.6484</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(2.2003%)</td>
</tr>
<tr>
<td>0.9 ( a_2 )</td>
<td>5</td>
<td>273.3487</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(-2.2003%)</td>
</tr>
<tr>
<td>1.1 ( \alpha )</td>
<td>5</td>
<td>270.8985</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(-3.077%)</td>
</tr>
<tr>
<td>0.9 ( \alpha )</td>
<td>5</td>
<td>288.099</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(3.077%)</td>
</tr>
<tr>
<td>1.1 ( \beta )</td>
<td>5</td>
<td>281.4985</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.716%)</td>
</tr>
<tr>
<td>0.9 ( \beta )</td>
<td>5</td>
<td>277.4985</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(-0.716%)</td>
</tr>
<tr>
<td>1.1 ( \lambda )</td>
<td>5</td>
<td>290.4914</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(3.933%)</td>
</tr>
<tr>
<td>0.9 ( \lambda )</td>
<td>5</td>
<td>270.4573</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(-3.235%)</td>
</tr>
</tbody>
</table>

(Note: Figures in parenthesis give % change in ATCn* as compared to original solution.)

From the above table 5.04 it may be concluded that the optimum replacement period remains the same in almost all the cases.
The most sensitive parameter is $\lambda$ for which minimum ATCn increases by about 3.93% for 10% increase in $\lambda$ alone and it reduces to about 3.24% for 10% decrease in $\lambda$ alone as compared to the original solution.

Similarly, the next sensitive parameter is $\alpha$. Here for 10% increase in $\alpha$ alone, minimum ATCn decreases by about 3.08% and for 10% decrease in $\alpha$ alone, minimum ATCn increases by about 3.08% as compared to the original solution.

It may be noted that $\beta$ is the least sensitive parameter, as it does not appreciably affect the minimum ATCn. The parameters $a_0, a_1$, and $a_2$ are other parameters which are more or less sensitive as compared to the original solution.