CHAPTER 3
THERMOSOLUTAL INSTABILITY OF COUPLE-STRESS ROTATING FLUID IN THE PRESENCE OF MAGNETIC FIELD

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3.1 **INTRODUCTION**

The problem of thermal instability of rotating fluid in the presence of magnetic field is of considerable importance in soil science, ground water hydrology and astrophysics. The theoretical and experimental results on the onset of thermal instability (Bénard convection) in a fluid layer under varying assumption of hydrodynamics, has been discussed in details by Chandrasekhar [17]. With the growing importance of non-Newtonian fluids in technology and industries, the investigations of such fluids are desirable. The theory of couple-stress fluid is proposed by Stokes [133]. One of the applications of couple-stress fluid is its use to the study of the mechanism of lubrication of synovial joints, at which researchers paid their attention. A human joint is a dynamically loaded bearing which has auricular cartilage as the bearing and synovial fluid as the lubricant. Normal synovial fluid is clear or yellowish and is a Non-Newtonian, viscous fluid. Couple-stress is found to appear in noticeable magnitude in fluids with very large molecules. Walicki and Walicka [153] modeled synovial fluid as couple-stress fluid in human joints because of the long chain by lauronic acid molecules is found as additives in synovial fluid. Sharma, Sunil and Chand [119] have discussed the problem of thermosolutal instability of Rivlin-Ericksen rotating fluid in porous medium. It is found that for the case of stationary convection, stable solute gradient and rotation have stabilizing effect on the system. Prakash and Kumar [71] have discussed the effect of suspended particles, rotation and variable gravity field on the thermal instability of Rivlin-Ericksen visco-elastic fluid in porous medium and found that for the case of stationary convection, rotation has a stabilizing effect on the system.

The problem of stability of stratified elastico-viscous Walters’ (Model B’) fluid in the presence of horizontal magnetic field and rotation in porous medium has been discussed by Sharma, Sunil and Gupta [126]. They have found that magnetic field stabilizes the system. Kumar and Singh [45] have studied the problem on a visco-elastic fluid heated from below in a porous medium and found that rotation has a stabilizing effect on the system and a Kuvshiniski visco-elastic fluid behaves like a Newtonian fluid.
in the problem. The problem on couple-stress fluid heated from below in porous medium is considered by Sharma and Sharma [101]. Sharma and Thakur [102] have discussed the problem of thermal convection in couple-stress fluid in porous medium in hydrodynamics.

The thermosolutal convection in Rivlin-Ericksen fluid in porous medium in the presence of uniform vertical magnetic field and rotation is considered by Sharma, Sunil and Pal [104]. They have found that rotation has a stabilizing effect on the system. Sharma and Rana [97] have studied the problem of thermal instability of a Walters’ (Model B’) elastic-viscous fluid in a porous medium in the presence of variable gravity field and rotation and found that Principle of Exchange of Stabilities (PES) is valid under certain conditions. Kumar, Mohan and Lal [49] have studied the effect of magnetic field on thermal instability of a rotating Rivlin-Ericksen visco-elastic fluid and found that rotation has a stabilizing effect while magnetic field has both stabilizing and destabilizing effects on the system. The problem of thermosolutal instability of couple-stress binary Rivlin-Ericksen visco-elastic fluid mixture in porous medium in the presence of magnetic field is discussed by Pundir [72]. Kumar, Singh and Lal [50] have studied the thermal instability of Walters’ B’ visco-elastic fluid permeated with suspended particles in hydromagnetics in porous medium and found that magnetic field stabilizes the system. The problem of thermal instability of couple-stress fluid in the presence of variable gravity field and rotation with the effect of stable solute gradient is discussed by Kumar et al. [51] and found that couple-stress fluid has a stabilizing/destabilizing effect under certain conditions and rotation has a stabilizing effect on the system whereas stable solute gradient has a stabilizing effect on the system.

Since the couple-stress fluid plays a significant role in industrial applications, it would be of much interest to examine the stability conditions of couple-stress fluid. Since the effect of magnetic field on thermosolutal instability of couple-stress rotating fluid in the presence of variable gravity field seems to be uninvestigated so far. Hence in this chapter, we shall discuss the thermosolutal instability of couple-stress rotating fluid in the presence of magnetic field.
3.2 **NOTATIONS**

- \( d \) \quad Depth of layer, [m]
- \( \alpha \) \quad Dimensionless wave number, [ - ]
- \( F \) \quad Couple-stress parameter,
- \( g \) \quad Acceleration due to gravity, [m/s\(^2\)]
- \( g \) \quad Gravity field, [m/s\(^2\)]
- \( k \) \quad Wave number, [1/m]
- \( k_x, k_y \) \quad Horizontal wave numbers, [1/m]
- \( \eta \) \quad Growth rate, [1/s]
- \( p \) \quad Fluid pressure, [pa]
- \( Q \) \quad Chandrasekhar number, [-]
- \( T_A \) \quad Taylor number, [-]
- \( S \) \quad Solute Rayleigh number, [-]
- \( R \) \quad Rayleigh number, [-]
- \( T \) \quad Temperature, [K]
- \( t \) \quad Time, [s]
- \( \Omega \) \quad Rotation vector having components (0, 0, \( \Omega \))
- \( H(\mathbf{h}_x, \mathbf{h}_y, \mathbf{h}_z) \) \quad Magnetic field having components (\( \mathbf{h}_x, \mathbf{h}_y, \mathbf{h}_z \))
- \( u, v, w \) \quad Component of velocity after perturbation,
- \( \alpha \) \quad Coefficient of thermal expansion, [1/K]
- \( \beta \) \quad Uniform temperature gradient, [K/m]
- \( \beta' \) \quad Uniform solute gradient, [K/m]
- \( \Theta \) \quad Perturbation in temperature, [K]
- \( \Gamma \) \quad Perturbation in concentration, [K]
- \( k_r \) \quad Thermal diffusivity, [m\(^2\)/s]
- \( k_s \) \quad Solute diffusivity, [m\(^2\)/s]
3.3 FORMULATION OF THE PROBLEM

Consider a static state in which an incompressible, Stokes couple-stress fluid layer of thickness \( d \), is arranged, confined between two infinite horizontal planes situated at \( z = 0 \) and \( z = d \), which is acted upon by a vertical magnetic field \( \mathbf{H}(0, 0, H) \), where \( H \) is a constant, uniform rotation \( \Omega(0, 0, \Omega) \) and variable gravity field \( \mathbf{g}(0, 0, -g) \), \( g = \lambda g_0 \) \((g_0 > 0)\) is the value of \( g \) at \( z = 0 \) and \( \lambda \) can be positive or negative as gravity increase or decrease upwards from its value \( g_0 \). The fluid layer is heated and soluted from below leading to an adverse temperature gradient \( \beta = \frac{T_0 - T_1}{d} \), where \( T_0 \) and \( T_1 \) are the constant temperatures of the lower and upper boundaries with \( T_0 > T_1 \) and \( \beta' = \frac{C_0 - C_1}{d} \), where \( C_0 \) and \( C_1 \) are the constant concentrations of the lower and upper boundaries with \( C_0 > C_1 \).

Let \( \rho, \rho, T, C, \alpha, \alpha', \nu, \mu', k_\nu, k_\rho \) and \( q(u, v, w) \) denote respectively pressure, density, temperature, concentration, thermal coefficient of expansion, an analogous coefficient of expansion, kinematic viscosity, couple-stress viscosity, thermal diffusivity and velocity of the fluid. The equation of motion, continuity and heat conduction of couple-stress fluid (Stokes [133]) are

\[
\begin{align*}
\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \nabla) \mathbf{q} &= -\frac{1}{\rho_0} \nabla p + \mathbf{g} \left( 1 + \frac{\delta \rho}{\rho_0} \right) + \left( \nu - \frac{\mu'}{\rho_0} \right) \nabla^2 \mathbf{q} + 2(\mathbf{q} \times \Omega) \\
&\quad + \frac{\mu_\nu}{4\pi\rho_0} \left( \nabla \times \mathbf{H} \right) \times \mathbf{H}, \quad (3.3.1)
\end{align*}
\]
\[ \nabla \cdot \mathbf{q} = 0, \quad (3.3.2) \]

\[ \frac{\partial T}{\partial t} + (\mathbf{q} \cdot \nabla)T = k_T \nabla^2 T, \quad (3.3.3) \]

\[ \frac{\partial C}{\partial t} + (\mathbf{q} \cdot \nabla)C = k_S \nabla^2 C, \quad (3.3.4) \]

\[ \frac{\partial \mathbf{H}}{\partial t} = (\mathbf{H} \cdot \nabla)q + \eta \nabla^2 \mathbf{H} \quad (3.3.5) \]

and

\[ \nabla \cdot \mathbf{H} = 0. \quad (3.3.6) \]

The equation of state is

\[ \rho = \rho_0 \left( 1 - \alpha(T - T_0) + \alpha'(C - C_0) \right), \quad (3.3.7) \]

where the suffix zero refers to value at the reference level \( z = 0 \).

### 3.4 BASIC STATE AND PERTURBATION EQUATIONS

In the undisturbed state, the fluid is at rest. Constants temperatures and concentrations are maintained in the fluid and uniform rotation acts in the vertical direction (say in \( z \)-direction), therefore the basic state of which we wish to examine the stability is characterized by

\[ \mathbf{q} = (0, 0, 0), \quad \Omega = (0, 0, \Omega), \quad \mathbf{H} = (0, 0, H), \quad T = T_0 - \beta z, \quad C = C_0 - \beta z. \quad (3.4.1) \]

Where \( \beta \) and \( \beta' \) may be either positive or negative and this basic state is consistent with the equations (3.3.1) to (3.3.7) provided that

\[ \rho = \rho(z), \quad p = p(z) \quad \text{and} \quad \rho = \rho_0 \left[ 1 + \alpha \beta z - \alpha' \beta' z \right]. \quad (3.4.2) \]

The character of equilibrium is examined by supposing that the system is slightly perturbed so that every physical quantity is assumed to be the sum of a mean and fluctuating component, later designated as prime quantities and assume to be vary small in comparison to their equilibrium state values. Here, we assume that the small disturbances are the functions of space and time variables. Hence, the perturbed flow may be represented as

\[ \mathbf{q} = (0, 0, 0) + (u, v, w), \]

\[ \mathbf{H} = (0, 0, H) + (h_x, h_y, h_z). \]
\[ T = T(z) + \delta T, \]
\[ C = C(z) + \gamma, \]
\[ \rho = \rho(z) + \delta \rho, \]

and \[ \delta \rho = \rho(z) + \delta \rho. \] (3.4.3)

Where \( \mathbf{q}(u, v, w) \), \( \mathbf{h}(h_x, h_y, h_z) \), \( \Theta \), \( \mathbf{F} \), \( \delta \rho \), \( \delta p \) denote respectively the perturbations in fluid velocity \( \mathbf{q}(0, 0, 0) \), magnetic field \( \mathbf{H} \), temperature \( T \), concentration \( C \), density \( \rho \) and pressure \( P \). Using equation (3.4.3) into governing equations (3.3.1) to (3.3.7) and linearizing them, we have

\[
\frac{\partial \hat{u}}{\partial t} = -\frac{1}{\rho_0} \frac{\partial}{\partial x} \delta \rho + \left[ v - \frac{\mu^\prime}{\rho_0} V^2 \right] \nabla^2 \hat{u} + \frac{\mu H}{4 \pi \rho_0} \left( \frac{\partial h_x}{\partial z} - \frac{\partial h_y}{\partial x} \right) + 2 \hat{\Omega} v, \tag{3.4.4}
\]

\[
\frac{\partial \hat{v}}{\partial t} = -\frac{1}{\rho_0} \frac{\partial}{\partial y} \delta \rho + \left[ v - \frac{\mu^\prime}{\rho_0} V^2 \right] \nabla^2 \hat{v} + \frac{\mu H}{4 \pi \rho_0} \left( \frac{\partial h_y}{\partial z} - \frac{\partial h_z}{\partial y} \right) - 2 \hat{\Omega} u, \tag{3.4.5}
\]

\[
\frac{\partial \hat{w}}{\partial t} = -\frac{1}{\rho_0} \frac{\partial}{\partial z} \delta \rho - g \delta \rho + \left[ v - \frac{\mu^\prime}{\rho_0} V^2 \right] \nabla^2 \hat{w}, \tag{3.4.6}
\]

\[
\frac{\hat{u}}{\hat{c}_x} + \frac{\hat{v}}{\hat{c}_y} + \frac{\hat{w}}{\hat{c}_z} = 0, \tag{3.4.7}
\]

\[
\frac{\hat{\theta}}{\hat{c}_t} = \beta \hat{w} + k_x \nabla^2 \hat{\theta}, \tag{3.4.8}
\]

\[
\frac{\hat{\gamma}}{\hat{c}_t} = \beta' \hat{w} + k_y \nabla^2 \hat{\gamma}, \tag{3.4.9}
\]

\[
\frac{\hat{h}_x}{\hat{c}_t} = H \frac{\hat{u}}{\hat{c}_z} + \eta \nabla^2 \hat{h}_x, \tag{3.4.10}
\]

\[
\frac{\hat{h}_y}{\hat{c}_t} = H \frac{\hat{v}}{\hat{c}_z} + \eta \nabla^2 \hat{h}_y, \tag{3.4.11}
\]

\[
\frac{\hat{h}_z}{\hat{c}_t} = H \frac{\hat{w}}{\hat{c}_z} + \eta \nabla^2 \hat{h}_z, \tag{3.4.12}
\]

\[
\frac{\partial \hat{h}_x}{\partial x} + \frac{\partial \hat{h}_y}{\partial y} + \frac{\partial \hat{h}_z}{\partial z} = 0 \tag{3.4.13}
\]

and \[ \delta p = -\rho \alpha \delta \theta - \alpha \gamma \]. (3.4.14)
Analyzing the perturbations into normal modes, we assume that the perturbation quantities are of the form

\[ w, \theta, \varphi, h, x, z \mid = W(z), \Theta(z), \Gamma(z), K(z), Z(z), X(z) \mid \exp \left( ik_x x + ik_y y + n t \right), \]  (3.4.15)

where \( k_x \) and \( k_y \) are the wave numbers in \( x \) and \( y \) directions respectively and \( k = \sqrt{k_x^2 + k_y^2} \) is the resultant wave number of propagation and \( n \) is the frequency of any arbitrary disturbance which is, in general, a complex constant.

For the considered form of the perturbations in equation (3.4.15), equations (3.4.4) to (3.4.14) give

\[ n(D^2 - k^2)W = -gk^2(\alpha \Theta - \alpha') + \left[ v - \frac{\mu'}{\rho_0} (D^2 - k^2) \right] (D^2 - k^2)^2 W \]
\[ + \frac{\mu'H}{4\pi\rho_0} (D^2 - k^2)DK \cdot 2\Omega DZ, \]  (3.4.16)

\[ nZ = \left[ v - \frac{\mu'}{\rho_0} (D^2 - k^2) \right] (D^2 - k^2)^2 Z + \frac{\mu'H}{4\pi\rho_0} DX \cdot 2\Omega DW, \]  (3.4.17)

\[ nX = \eta(D^2 - k^2)X, \]  (3.4.18)

\[ n\Theta = \beta W + k_r(D^2 - k^2)\Theta, \]  (3.4.19)

\[ n\Gamma = \beta W + k_s(D^2 - k^2)\Gamma \]  (3.4.20)

and \( nK = \eta(D^2 - k^2)K \).  \( \)  (3.4.21)

We eliminate the physical quantities using the non-dimensional parameter \( \alpha = kd, \sigma = \frac{ng^2}{v}, \rho_1 = \frac{v}{k_r}, \rho_2 = \frac{v}{\eta}, q = \frac{v}{k_s}, F = \frac{\mu'}{\rho_0 d^2 v} \) and \( D^* = dD \) and dropping (*) for convenience, equations (3.4.16) to (3.4.21) become

\[ \left[ \sigma + F(J^2 - a^2)^2 - (J^2 - a^2) \right] (J^2 - a^2)W + \frac{\lambda_g \alpha'^2 d^2}{v} \Theta + \frac{g\sigma a^2 \gamma^2}{v} \]
\[ + \frac{\mu'Hd}{4\pi\gamma v} (D^2 - a^2)DK + \frac{2a'}{v} DZ = 0, \]  (3.4.22)
\[ \left[ \sigma + F(D^2 - a^2)^2 - (D^2 - a^2) \right] Z = \frac{Hd}{4\pi \rho \nu} DX + \frac{2}{\nu} DW, \]  
(3.4.23)

\[ (D^2 - a^2 - \sigma p_2) X = -\frac{HdDZ}{\eta}, \]  
(3.4.24)

\[ (D^2 - a^2 - \sigma p_1) = -\frac{\beta d W}{k_r}, \]  
(3.4.25)

\[ (D^2 - a^2 - \sigma q) = -\frac{\beta d q W}{k_s}, \]  
(3.4.26)

and \[ (D^2 - a^2 - \sigma p_2) K = -\frac{HdDW}{\eta}. \]  
(3.4.27)

The perturbation in the temperature is zero at the boundaries because both the boundaries are maintained at constant temperature and constant concentrations. The appropriate boundary conditions are
\[ \eta = 0, \Theta = 0, \Gamma = 0 \text{ at } z = 0 \text{ and } z = 1. \]  
(3.4.28)

The constitutive equations of couple-stress fluid are
\[ \tau_{ij} = (2\mu_1 - 2\mu_1^1\nabla^2) e_{ij} \text{ and } e_{ij} = \frac{1}{2} \left( \frac{\partial q_i}{\partial r_j} - \frac{\partial q_j}{\partial r_i} \right), \]  
and the conditions on a free surface are given by
\[ \tau_{xz} = (\mu - \mu_1^1 \nabla^2) \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = 0 \text{ and } \tau_{yz} = (\mu - \mu_1^1 \nabla^2) \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = 0. \]  
(3.4.29)

Using the equation of continuity (3.4.7), we conclude that
\[ \left[ \mu - \mu_1^1 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \right] \frac{\partial^2 W}{\partial z^2} = 0, \]  
(3.4.30)

which gives
\[ \frac{\partial^2 W}{\partial z^2} = 0, \quad \frac{\partial^4 W}{\partial z^4} = 0 \text{ at } z = 0 \text{ and } z = d. \]  
(3.4.31)

The boundary condition (3.4.31), using equation (3.4.15), in non-dimensional form are
\[ D^2 W = 0, \quad D^4 W = 0 \text{ at } z = 0 \text{ and } z = 1. \]  
(3.4.32)
Multiplying equation (3.4.23) by \((J^2 - a^2 - \sigma p_2)\) and using equation (3.4.24), we get
\[
\left[ \sigma + F(D^2 - a^2)^2 - (D^2 - a^2)^2 \right] \left[ j^2 - a^2 - \sigma p_1 \right] j^2 - a^2 - \sigma p_2  \\
= \frac{2}{V} \frac{d}{dt} (D^2 - a^2 - \sigma p_2) D W . \tag{3.4.33}
\]

On eliminate \(J, Z, K, \chi\) between the equations (3.4.22), (3.4.25) to (3.4.27) and (3.4.33), we obtain the stability governing equation
\[
\left[ \sigma + F(J^2 - a^2)^2 - (J^2 - a^2)^2 \right] \left[ j^2 - a^2 - \sigma p_1 \right] j^2 - a^2 - \sigma p_2  \\
+ T_a \frac{D^2 - a^2 - \sigma q}{\sigma + F(D^2 - a^2)^2 - (D^2 - a^2)^2} \frac{D^2 - a^2 - \sigma p_2}{(D^2 - a^2 - \sigma p_2)^2} \frac{D^2 W}{(D^2 - a^2 - \sigma p_2) + QD^2} = 0. \tag{3.4.34}
\]

Where, \(R = \frac{g_0 \alpha \beta d^4}{\gamma k} \) is the thermal Rayleigh number, \(T_a = \left( \frac{2}{V} \frac{d^2}{dt} \right)^2 \) is the Taylor number, \(Q = \frac{H^2 d^2}{4 \pi \rho_0 v_1} \) is the Chandrasekhar number and \(S = \frac{g_0 \alpha \beta' d^4}{\gamma k} \) is the solute Rayleigh number.

From equation (3.4.32), it is clear that all the even order derivatives of \(W\) vanish on the boundaries. Therefore, the proper solution of equation (3.4.34) characterizing the lowest mode is
\[
W = W_0 \sin \pi z. \tag{3.4.35}
\]

Where \(W_0\) is constant using equation (3.4.35), equation (3.4.34) gives
\[
R_i = \frac{(1 + x)}{\lambda x} \left[ i \sigma_i + F_i (1 + x)^2 + (1 + x) \right] 1 + x + i \sigma_i p_i \left| 1 + x + i \sigma_i q \right| S_i \frac{(1 + x + i \sigma_i p_i)}{(1 + x + i \sigma_i q)}  \\
+ \frac{Q_i (1 + x + i \sigma_i p_i)}{\lambda x (1 + x + i \sigma_i p_2)} + \frac{1 + x + i \sigma_i p_2}{\lambda x} \left[ i \sigma_i + F_i (1 + x)^2 + (1 + x) \right] 1 + x + i \sigma_i p_2 + Q_i . \tag{3.4.36}
\]

Where, \(R_i = \frac{R}{\pi}, S_i = \frac{S}{\pi}, i \sigma_i = \frac{\sigma}{\pi}, F_i = \pi^2 F, T_a = \frac{T_a}{\pi}\) and \(x = \frac{a^2}{\gamma c^2} \).
3.5 ANALYTICAL DISCUSSION

(a) Stationary Convection

At stationary convection, when the instability sets, the marginal state will be characterized by \( \alpha = 0 \). Thus, putting \( \alpha = 0 \) in equation (3.4.36), we get

\[
R_1 = \frac{(1 + x)}{\lambda x} \left[ F_1(1 + x) + \frac{T_1(1 + x)}{F_1(1 + x) + 1} (1 + x)^2 + Q_1 \right] - S_1, \tag{3.5a.1}
\]

which express the Rayleigh number \( R_1 \) as a function of the parameters \( S_1, F_1, T_1, Q_1 \) and dimensionless wave number \( \lambda \). To study the effect of stable solute gradient, couple-stress, rotation and magnetic field, we study the behavior of \( \frac{dR_1}{dS_1}, \frac{dR_1}{dF_1}, \frac{dR_1}{dT_1} \) and \( \frac{dR_1}{dQ_1} \) analytically.

From equation (3.5a.1), we have

\[
\frac{dR_1}{dS_1} = 1, \quad \text{(which is positive)}, \tag{3.5a.2}
\]

which clearly shows that stable solute gradient has a stabilizing effect on thermosolutal instability of couple-stress rotating fluid in the presence of magnetic field as can be seen graphically from Fig. 3.1.

From equation (3.5a.1), we have

\[
\frac{dR_1}{dF_1} = \frac{(1 + x)^4}{\lambda x} \left[ 1 - \frac{T_1(1 + x)}{\left[ F_1(1 + x) + 1 \right] (1 + x)^2 + Q_1} \right], \tag{3.5a.3}
\]

which shows that couple-stress has a stabilizing/destabilizing effect on thermosolutal instability of couple-stress rotating fluid in the presence of magnetic field under the conditions

\[
T_1(1 + x) < \left[ F_1(1 + x) + 1 \right] (1 + x)^2 + Q_1. \]

But for the permissible values of various parameters, the said effect is stabilizing only if

\[
T_1(1 + x) < \left[ F_1(1 + x) + 1 \right] (1 + x)^2 + Q_1 \]

as can be seen graphically from Fig. 3.2, Fig. 3.3 and Fig. 3.4.
In the absence of rotation ($T_A = 0$), equation (3.5a.3) becomes

$$\frac{dR_i}{dF_i} = \frac{(1+x)^4}{\lambda x},$$

(3.5a.4)

which shows that couple-stress has a stabilizing effect on thermosolutal instability of couple-stress rotating fluid in the absence of rotation when gravity increases upward from its value $g_0$.

In the absence of magnetic field, equation (3.5a.3) becomes

$$\frac{dR_i}{dF_i} = \frac{1}{\lambda x} \left[ (1+x)^4 - \frac{T_A (1+x)}{[F_i (1+x) + 1]^2} \right],$$

(3.5a.5)

which shows that the couple-stress has a stabilizing/destabilizing effect on thermosolutal instability of couple-stress rotating fluid in the presence of magnetic field under the conditions $T_A < (1+x)^3 [F_i (1+x) + 1]^2$.

But for the permissible values of various parameters, the said effect is stabilizing only if

$$T_A < (1+x)^3 [F_i (1+x) + 1]^2.$$

Again from equation (3.5a.1), we have

$$\frac{dR_i}{dT_A} = \frac{(1+x)^2}{\lambda x \left[ F_i (1+x) + 1 \right] (1+x)^2 + Q_i},$$

(3.5a.6)

which clearly shows that rotation has a stabilizing effect on thermosolutal instability of couple-stress rotating fluid in the presence of magnetic field when gravity increases upwards from its value $g_0$ as can be seen graphically from Fig. 3.5 and Fig. 3.6.

Again from equation (3.5a.1), we have

$$\frac{dR_i}{dQ_i} = \frac{(1+x)}{\lambda x} \left[ 1 - \frac{T_A (1+x)}{\left[ 1 - T_A (1+x) \right]} \right],$$

(3.5a.7)

which shows that magnetic field has stabilizing/destabilizing effect on thermosolutal instability of couple-stress rotating fluid in the presence of magnetic field under the conditions.
when gravity increases upwards from value $g_o$ as can be seen graphically from Fig. 3.7.

But for the permissible values of various parameters, the said effect is stabilizing only if

$$T_A (1 + x) < \left[ \left\{ F_i (1 + x) + l_1 (1 + x)^2 + Q_i \right\}^2 \right].$$

In the absence of rotation ($T_A = 0$), equation (3.5a.7) becomes

$$\frac{dR}{dQ} = \frac{(1 + x)}{\lambda x}, \tag{3.5a.8}$$

which clearly shows that magnetic field has a stabilizing effect on thermosolutal instability of couple-stress rotating fluid in the presence of magnetic field when gravity increases upward from its value $g_o$.

(b) Stability of the System and Oscillatory Modes

Multiplying equation (3.4.22) by $W^*$ and integrate over the range at $z$ and making use of equations (3.4.23) to (3.4.27) with the boundary condition (3.4.28), we get

$$\sigma I_1 + I_2 + F I_3 - \frac{\gamma g_0 \alpha k T}{\sqrt{\mu}} \left[ I_4 + \sigma^* p_i I_5 \right] + \frac{\gamma g_0 \alpha k T^2}{\sqrt{\mu}} \left[ I_6 + \sigma^* q I_7 \right]$$

$$+ \frac{\mu \eta}{2} \left[ I_8 - \sigma^* p_2 I_9 \right] + d^2 \left[ \sigma^* I_{10} + F I_{11} + I_{12} \right] + \frac{\mu \eta d^2}{4 \pi \rho \nu} \left[ I_{13} + \sigma \rho I_{14} \right] = 0. \tag{3.5b.1}$$

Where $I_1 = \int |D W|^2 + a^2 |W|^2 dz$, $I_2 = \int |D^2 W|^2 + 2a^2 |DW|^2 + a^2 |W|^2 dz$,

$$I_3 = \int |D^3 W|^2 + 3a^2 |D^2 W|^2 + 3a^2 |DW|^2 + a^2 |W|^2 dz,$$

$$I_4 = \int |D^2|^2 + a^2 |D^2|, I_5 = \int |D^2| ^2 + a^2 |D^2| dz,$$

$$I_6 = \int |D^2|^2 + a^2 |D^2|, I_7 = \int |D^2| dz,$$

$$I_8 = \int |D^2 K|^2 + a^4 |K|^2 + 2a^2 |D K|^2 dz$$

$$I_9 = \int |D K|^2 + a^2 |K|^2 dz,$$

$$I_{10} = \int |Z|^2 dz, I_{11} = \int |D^2 Z|^2 + 2a^2 |DZ|^2 + a^4 |Z|^2 dz,$$

$$I_{12} = \int \left( |DZ|^2 + a^2 |Z|^2 \right) dz,$$

$$I_{13} = \int |DX|^2 + a^2 |X|^2 dz$$

and $I_{14} = \int |X|^2 dz$.  

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Where $\sigma^*$ is the complex conjugate of $\sigma$. All the integrals $I_1$ to $I_{14}$ are positive definite. Putting $\sigma = \sigma_r + i\sigma_i$ in equation (3.5b.1) and equating the real and imaginary parts, we obtain

$$\sigma_r \left[ I_1 - \frac{\tilde{\gamma} g_0 \alpha k^2 a^2}{\nu \beta} p_1 I_5 + \frac{\tilde{\gamma} g_0 \alpha' k^2 a^2}{\nu' \beta'} q l_7 + \frac{\mu v \eta}{4 \pi \rho_b \nu} p_2 I_9 + d^2 I_{10} + \frac{\mu_i d^2 \eta}{4 \pi \rho_b \nu} p_2 I_{14} \right]$$

$$= \frac{\tilde{\gamma} g_0 \alpha k^2 a^2}{\nu \beta} I_5 - F l_5 - \frac{\tilde{\gamma} g_0 \alpha' k^2 a^2}{\nu' \beta'} I_6 - \frac{\mu v \eta}{4 \pi \rho_b \nu} I_8 - d^2 F l_{11} - d^2 I_{12} - \frac{\mu_i d^2 \eta}{4 \pi \rho_b \nu} I_9 \tag{3.5b.2}$$

and $\sigma_i \left[ I_1 + \frac{\tilde{\gamma} g_0 \alpha k^2 a^2}{\nu \beta} p_1 I_5 + \frac{\tilde{\gamma} g_0 \alpha' k^2 a^2}{\nu' \beta'} q l_7 - \frac{\mu v \eta}{4 \pi \rho_b \nu} p_2 I_9 - d^2 I_{10} + \frac{\mu_i d^2 \eta}{4 \pi \rho_b \nu} p_2 I_{14} \right] = 0.$

(3.5b.3)

It may be inferred from equation (3.5b.2) that $\sigma_r$ may be positive or negative which means that system may be stable or unstable.

In the absence of stable solute gradient, magnetic field and rotation, equation (3.5b.3) becomes

$$\sigma_i \left[ I_1 + \frac{\tilde{\gamma} g_0 \alpha k^2 a^2}{\nu \beta} p_1 I_5 + \frac{\tilde{\gamma} g_0 \alpha' k^2 a^2}{\nu' \beta'} q l_7 \right] = 0. \tag{3.5b.4}$$

From equation (3.5b.4), it is obvious that all the terms in the bracket are positive definite. Thus $\sigma_i = 0$ which means that oscillatory modes are not allowed in the system and Principle of Exchange of Stabilities (PES) is satisfied in the absence of stable solute gradient, magnetic field and rotation in the system. So, we can say the oscillatory modes are introduced due to the presence of stable solute gradient, magnetic field and rotation whereas in their absence Principle of Exchange of Stabilities (PES) is satisfied.

(c) Discussion

It is obvious from equation (3.5b.3) that if quantity in bracket is not zero, then $\sigma_i$ is zero and if $\sigma_i$ is arbitrary, then this quantity is zero. Thus, if $\sigma_i \neq 0$. Then in the absence of stable solute gradient, from equation (3.5b.3), we have
Using equation (3.5b.5), equation (3.5b.3) becomes

\[
I_2 + FI_3 + d^2FI_9 + d^2I_{10} + \frac{\mu_r \eta}{4\pi \rho_0 \nu} I_6 + \frac{\mu_r d^2 \eta}{4\pi \rho_0 \nu} I_{11} + 2\sigma_\gamma \left[ I_1 + \frac{\mu_r d^2 \eta}{4\pi \rho_0 \nu} p_2 I_{12} \right] = \frac{\lambda g_0 \alpha k_r}{\nu \beta} I_4
\]

Using Rayleigh-Ritz inequality, equation (3.5b.3) gives

\[
\frac{(\pi^2 + a^2)^3}{a^2} \int_0^1 |W|^2 \, dz + \frac{\pi^2 + a^2}{a^2} \left\{ FI_3 + d^2FI_9 + d^2I_{10} + \frac{\mu_r \eta}{4\pi \rho_0 \nu} I_6 + \frac{\mu_r d^2 \eta}{4\pi \rho_0 \nu} I_{11} \\
+ 2\sigma_\gamma \left[ I_1 + \frac{\mu_r d^2 \eta}{4\pi \rho_0 \nu} p_2 I_{12} \right] \right\} \leq \frac{\lambda g_0 \alpha k_r}{\nu \beta} \int_0^1 |W|^2 \, dz.
\]

Since minimum value of \((\pi^2 + a^2)/a^2\) with respect to \(a^2\) is \(\frac{27\pi^4}{4}\), then equation (3.5b.7) gives

\[
\left[ \frac{27\pi^4}{4} - \frac{\lambda g_0 \alpha k_r}{\nu \beta} \right] \int_0^1 |W|^2 \, dz + \left[ \frac{(\pi^2 + a^2)}{a^2} \left\{ FI_3 + d^2FI_9 + d^2I_{10} + \frac{\mu_r \eta}{4\pi \rho_0 \nu} I_6 \\
+ \frac{\mu_r d^2 \eta}{4\pi \rho_0 \nu} I_{11} + 2\sigma_\gamma \left[ I_1 + \frac{\mu_r d^2 \eta}{4\pi \rho_0 \nu} p_2 I_{12} \right] \right\} \right] \leq 0.
\]

Now, if \(\sigma_\gamma \geq 0\), then from equation (3.5b.8), we have

\[
\frac{\lambda g_0 \alpha k_r}{\nu \beta} > \frac{27\pi^4}{4}
\]

and for \(\sigma_\gamma < 0\), we have

\[
\frac{\lambda g_0 \alpha k_r}{\nu \beta} \leq \frac{27\pi^4}{4}.
\]

Therefore, the system is stable if \(\frac{\lambda g_0 \alpha k_r}{\nu \beta} \leq \frac{27\pi^4}{4}\) and the system is unstable if \(\frac{\lambda g_0 \alpha k_r}{\nu \beta} > \frac{27\pi^4}{4}\).

### 3.6 NUMERICAL COMPUTATIONS

Now, the critical thermal Rayleigh number for the onset of instability is determined numerically using Newton-Raphson method by the condition \(\frac{dR_0}{dx} = 0\).
As a function of \( x \), \( R_t \) is given by equation (3.5a.1) attains its minimum when
\[
\frac{dR_t}{dx} = 0 \quad \text{with} \quad x \text{ determined as a solution of equation by putting } \frac{dR_t}{dx} = 0 \quad \text{in powers of } x.
\]
Equation (3.5a.1) will give the required critical thermal Rayleigh number \( R_t \) for various values of critical wave number \( \lambda \). The numerical values of critical thermal Rayleigh number \( R_t \) and critical wave number \( x \) determined for various values of stable solute gradient \( S_t \), magnetic field \( Q_A \), couple-stress \( F_t \) and rotation \( T_A \).

In Fig. 3.1, critical Rayleigh number \( R_t \) is plotted against stable solute gradient \( S_t \) for fixed value of \( \lambda = 1, F_t = 10, Q_A = 100 \) and \( T_A = 100, 500, 1000 \). The critical Rayleigh number \( R_t \) increases with increase in stable solute gradient \( S_t \) which shows that stable solute gradient has a stabilizing effect on the system.

In Fig. 3.2, critical Rayleigh number \( R_t \) is plotted against couple-stress parameter \( F_t \) for fixed value of \( \lambda = 1, S_t = 10, Q_A = 100 \) and \( T_A = 100, 500, 1000 \). The critical Rayleigh number \( R_t \) increases with increase in couple-stress parameter \( F_t \) which shows that couple-stress has a stabilizing effect on the system.

In Fig. 3.3, critical Rayleigh number \( R_t \) is plotted against couple-stress parameter \( F_t \) for fixed value of \( \lambda = 1, S_t = 10, Q_A = 10 \) and \( T_A = 6000, 8000, 10000 \). The critical Rayleigh number \( R_t \) increases/decreases with increase in couple-stress parameter \( F_t \) which shows that couple-stress has both stabilizing and destabilizing effects on the system.

In Fig. 3.4, critical Rayleigh number \( R_t \) is plotted against couple-stress parameter \( F_t \) for fixed value of \( \lambda = 1, S_t = 10, T_A = 100 \) and \( Q_A = 150, 250, 350 \). The critical Rayleigh number \( R_t \) increases with increase in couple-stress parameter \( F_t \) which shows that couple-stress has a stabilizing effect on the system.
In Fig. 3.5, critical Rayleigh number $R_i$ is plotted against rotation parameter $T_A$ for fixed value of $\lambda = 1, F_1 = 10, S_i = 10$ and $Q_i = 20,50,80$. The critical Rayleigh number $R_i$ increases with increase in rotation parameter $T_A$ which shows that rotation has stabilizing effects on the system.

In Fig. 3.6, critical Rayleigh number $R_i$ is plotted against rotation parameter $T_A$ for fixed value of $\lambda = 1, F_1 = 10, S_i = 10$ and $Q_i = 100,300,500$. The critical Rayleigh number $R_i$ increases with increase in rotation parameter $T_A$ which shows that rotation has stabilizing effects on the system.

In Fig. 3.7, critical Rayleigh number $R_i$ is plotted against magnetic field parameter $Q_i$ for fixed value of $\lambda = 1, F_1 = 10, S_i = 10$ and $T_A = 100,1000,10000$. The critical Rayleigh number $R_i$ increases with increase in magnetic field parameter $Q_i$ which shows that magnetic field has stabilizing effects on the system.

Fig. 3.1: Variation of critical Rayleigh number $R_i$ with $S_1$ for a fixed $\lambda = 1, F_1 = 10, Q_i = 100$ and $T_A = 100,500,1000$.
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Fig. 3.2: Variation of critical Rayleigh number $R_1$ with $F_1$ for fixed value of

$\lambda = 1, S_1 = 10, Q_1 = 100$ and $T_{A1} = 100, 500, 1000.$

Fig. 3.3: Variation of critical Rayleigh number $R_1$ with $F_1$ for fixed value of

$\lambda = 1, S_1 = 10, Q_1 = 10$ and $T_{A1} = 6000, 8000, 10000.$
Fig. 3.4: Variation of critical Rayleigh number $R_i$ with $F_i$ for fixed value of $\lambda = 1, S_i = 10, T_A = 100$ and $Q = 150, 250, 350$.

Fig. 3.5: Variation of critical Rayleigh number $R_i$ with $T_A$ for fixed value of $\lambda = 1, F_i = 10, S_i = 10$ and $Q = 20, 50, 80$. 
Fig. 3.6: Variation of critical Rayleigh number $R_i$ with $T_A$ for fixed value of

$$\lambda = 1, F_i = 10, S_i = 10 \text{ and } Q_i = 100, 300, 500.$$

Fig. 3.7: Variation of critical Rayleigh number $R_i$ with $Q_i$ for fixed value of

$$\lambda = 1, F_i = 10, S_i = 10 \text{ and } T_A = 100, 1000, 10000.$$
3.7 CONCLUSIONS

In the present chapter, we have investigated the effect of magnetic field on the thermosolutal instability of couple-stress rotating fluid. Dispersion relation governing the effects of stable solute gradient, couple-stress, rotation and magnetic field is derived. The main results from the analysis of this chapter are as follows:

(i) For the case of stationary convection, the stable solute gradient has a stabilizing effect on the system as can be seen from equation (3.5a.2), and graphically from Fig. 3.1.

(ii) Couple-stress has stabilizing/destabilizing effects on the system for the permissible values of various parameters as can be seen from equation (3.5a.3), and graphically, from Fig. 3.2, Fig. 3.3 and Fig. 3.4. In the absence of rotation, couple-stress clearly has a stabilizing effect on the system as can be seen from equation (3.5a.4).

(iii) For the case of stationary convection, the rotation has a stabilizing effect on the system as can be seen from equation (3.5a.6), and graphically, from Fig. 3.5 and Fig. 3.6.

(iv) Magnetic field has stabilizing/destabilizing effect on the system for the permissible values of various parameters when gravity increases upwards from its value $g_0$ as can be seen from equation (3.5a.7), and graphically, from Fig. 3.7. In the absence of rotation, magnetic field has a stabilizing effect on the system as can be seen from equation (3.5a.8).

(v) The oscillatory modes are introduced due to the presence of stable solute gradient, magnetic field and rotation whereas in their absence principle of exchange of stabilities (PES) is satisfied in the system.