CHAPTER 2
THERMAL INSTABILITY OF COUPLE-STRESS FLUID IN THE PRESENCE OF VARIABLE GRAVITY FIELD AND ROTATION

2.1 INTRODUCTION

2.2 NOTATIONS

SECTION I

I.1 FORMULATION OF THE PROBLEM

I.2 BASIC STATE AND PERTURBATION EQUATIONS

I.3 ANALYTICAL DISCUSSION
   (a) Stationary Convection
   (b) Stability of the System and Oscillatory Modes
   (c) The Case of Overstability

SECTION II

II.1 EFFECT OF SOLUTE GRADIENT: FORMULATION OF THE PROBLEM

II.2 BASIC STATE AND PERTURBATION EQUATIONS

II.3 ANALYTICAL DISCUSSION
   (a) Stationary Convection
   (b) Stability of the System and Oscillatory Modes
   (c) The Case of Overstability

2.3 NUMERICAL COMPUTATIONS

2.4 CONCLUSIONS
2.1 INTRODUCTION

The problem of thermal instability of fluids in porous medium is of considerable importance in soil science, groundwater hydrology and astrophysics. The theoretical and experimental results on the onset of thermal instability (Bénard convection) in a fluid layer under varying assumptions of hydrodynamics, has been discussed in details by Chandrasekhar [17]. With the growing importance of non-Newtonian fluids in modern technology and industries, the investigations of such fluids are desirable. The theory of couple-stress fluid is proposed by Stokes [133]. One of the applications of couple-stress fluid is its use to the study of the mechanism of lubrication of synovial joints, at which researchers paid their attention. A human joint is a dynamically loaded bearing which has particular cartilage as the bearing and synovial fluid as the lubricant. Normal synovial fluid is clear or yellowish and is a Non-Newtonian, viscous fluid. Couple-stresses are found to appear in noticeable magnitude in fluids with very large molecules. Walicki and Walicka [153] modeled synovial fluid as couple-stress fluid in human joints because of the long chain by lauronic acid molecules is found as additives in synovial fluid. Sharma and Sharma [116] have studied the thermal instability in a Maxwellian viscoelastic fluid in porous medium. It is found that for stationary convection, Maxwellian fluid behaves like a Newtonian fluid and critical Rayleigh number increases with the increase in magnetic field and rotation.

The problem of thermosolutal instability of a Oldroydian visco-elastic fluid in porous medium has been discussed by Sharma and Bhardwaj [110]. They have found that stable solute gradient and rotation has a stabilizing effect on the system. The problem of thermal instability of an Oldroydian visco-elastic fluid in porous medium is investigated by Sharma and Kumar [98]. They have considered the effect of uniform rotation on the thermal instability. Sharma and Sharma [101] have discussed the problem on couple-stress fluid heated from below in porous medium. The problem of thermal convection in couple-stress fluid in porous medium in hydrodynamics is discussed by Sharma and Thakur [102].
The problem of thermosolutal instability of couple-stress binary Rivlin-Ericksen visco-elastic fluid mixture in porous medium in the presence of magnetic field is discussed by Pundir [72]. The thermosolutal convection in Rivlin-Ericksen fluid in porous medium in the presence of uniform vertical magnetic field and uniform rotation is considered by Sharma, Sunil and Pal [104]. They have found the rotation has a stabilizing effect on the system. Bhatia and Steiner [6] have studied the problem of thermal instability of a Maxwellian fluid and found that the rotation has a destabilizing effect in contrast to the stabilizing effect on Newtonian fluids. Sharma and Rana [97] have studied the problem of thermal instability of a Walters' (Model B') elastic-viscous fluid in a porous medium in the presence of variable gravity field and rotation. They have found the principle of exchange of stabilities is valid under certain conditions.

The problem of thermosolutal instability of Rivlin-Ericksen rotating fluid in porous medium is considered by Sharma, Sunil and Chand [119]. They found the stable solute gradient and rotation stabilizes the system and oscillatory modes, if exist, are introduced due to stable solute gradient, rotation, porosity and visco-elasticity, which were non-existent in their absence. Kumar, Lal and Sharma [46] have discussed the instability of two rotating visco-elastic (Rivlin-Ericksen) superposed fluids with suspended particles in porous medium. The problem of thermosolutal instability of Rivlin-Ericksen rotating fluid in the presence of magnetic field and variable gravity field in porous medium is discussed by Sharma and Rana [124]. It is found that for the case of stationary convection, the stable solute gradient has a stabilizing effect on the system.

Since couple-stress fluid plays a significant role in industrial application. It would be of much interest to examine the stability condition of couple-stress fluid. Since the thermal instability of couple-stress fluid in the presence of variable gravity field and rotation seems to be uninvestigated so far. Hence in this chapter, we shall discuss the thermal instability of couple-stress fluid in the presence of variable gravity field and rotation.
2.2 NOTATIONS

\( d \) Depth of layer, [m]

\( a \) Dimensionless wave number,

\( F \) Couple-stress parameter,

\( g \) Acceleration due to gravity, [m/s\(^2\)]

\( g \) Gravity field, [m/s\(^2\)]

\( k \) Wave number, [1/m]

\( k_x, k_y \) Horizontal wave numbers, [1/m]

\( \eta \) Growth rate, [1/s]

\( p \) Fluid pressure, [pa]

\( T_A \) Taylor number, [-]

\( S \) Solute Rayleigh number, [-]

\( R \) Rayleigh number, [-]

\( T \) Temperature, [K]

\( t \) Time, [s]

\( \Omega(0,0,\Omega) \) Rotation vector having components \((0,0,\Omega)\),

\( u, v, w \) Component of velocity after perturbation,

\( \alpha \) Coefficient of thermal expansion, [1/K]

\( \beta \) Uniform temperature gradient, [K/m]

\( \beta' \) Uniform solute gradient, [K/m]

\( \Theta \) Perturbation in temperature, [K]

\( \Gamma \) Perturbation in concentration, [K]

\( k_T \) Thermal diffusivity, [m\(^2\)/s]

\( k_s \) Solute diffusivity, [m\(^2\)/s]

\( v \) Kinematic viscosity, [m\(^2\)/s]

\( v' \) Kinematic viscoelasticity, [m\(^2\)/s]

\( \rho \) Density, [Kg/m\(^3\)]
\[ \nabla \] Del operator,

\[ \partial, D \] Curly operators and derivative with respect to \( z \) (-d/dz).

This chapter has been divided into two sections.

**In section I**, we formulate a mathematical problem of thermal instability of couple-stress fluid in the presence of variable gravity field and rotation.

**In section II**, we formulate a mathematical problem of thermal instability of couple-stress rotating fluid in the presence of variable gravity field and solute gradient.
SECTION I

I.1 FORMULATION OF THE PROBLEM

Consider a static state in which an incompressible couple-stress fluid layer of thickness $d$, is arranged, confined between two infinite horizontal planes situated at $z = 0$ and $z = d$, which is acted upon by a uniform rotation $\Omega(0, 0, \Omega)$ and variable gravity field $g(0, 0, -g)$, $g = \lambda g_0$, $(g_0 > 0)$ is the value of $g$ at $z = 0$ and $\lambda$ can be positive or negative as gravity increases or decreases upwards from its value $g_0$. The fluid layer is heated from below leading to an adverse temperature gradient $\beta = \frac{T_0 - T_1}{d}$, where $T_0$ and $T_1$ are the constant temperatures of the lower and upper boundaries with $T_0 > T_1$.

Let $p, \rho, T, \alpha, \nu$ and $\mathbf{q}(u, v, w)$ denote respectively, pressure, density, temperature, thermal coefficient of expansion, kinematic viscosity and velocity of the fluid. Then the equations of motion, continuity and heat conduction equations of couple-stress fluid (Stokes [133]) are

\[
\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla) \mathbf{q} = -\frac{1}{\rho_0} \nabla p + g \left(1 + \frac{\delta \rho}{\rho_0}\right) + \left(v - \frac{1}{\rho_0} \nabla^2\right) \nabla^2 \mathbf{q} + 2(\mathbf{q} \times \mathbf{\Omega}), \tag{I.1.1}
\]

\[
\nabla \cdot \mathbf{q} = 0, \tag{I.1.2}
\]

\[
\frac{\partial T}{\partial t} + (\mathbf{q} \cdot \nabla) T = k_T \nabla^2 T. \tag{I.1.3}
\]

The equation of state is

\[
\rho = \rho_0 \left[1 - \frac{\alpha}{C_R} (T - T_0)\right], \tag{I.1.4}
\]

where the suffix zero refers to value at the reference level $z = 0$ and $k_T$ is the thermal diffusivity.

I.2 BASIC STATE AND PERTURBATION EQUATIONS

In the undisturbed state, the fluid is at rest. Constant temperatures are maintained in the fluid and uniform rotation acts in the vertical direction (say in $z$-direction), therefore the basic state of which we wish to examine the stability is characterized by
\( \mathbf{q} = (0, 0, 0), \ \Omega = (0, 0, \Omega), T = T_0 - \beta z, \)  
(I.2.1)

where \( \beta \) may be either positive or negative and this basic state is consistent with the equations (I.1.1) to (I.1.4) provided that

\[
\rho = \rho(z), \ \rho = p(z) \text{ and } \rho = \rho_0 \left[ 1 + \alpha \beta z \right].
\]

(I.2.2)

The character of equilibrium is examined by supposing that the system is slightly perturbed so that every physical quantity is assumed to be the sum of a mean and fluctuating component, later designated as prime quantities and assume to be very small in comparison to their equilibrium state values. Here, we assume that the small disturbances are the functions of space and time variables. Hence, the perturbed flow may be represented as

\[
\mathbf{q} = (0, 0, 0) + (u, v, w), \\
T = T(z) + \theta, \\
\rho = \rho(z) + \delta \rho
\]

and  
\[
\rho = \rho(z) + \delta p.
\]

(I.2.3)

Where \( \mathbf{q}(u, v, w), \ \Theta, \ \delta \rho, \ \delta p \) denote respectively the perturbations in fluid velocity \( \mathbf{q}(0, 0, 0), \) temperature \( T, \) density \( \rho \) and pressure \( p. \) Using equation (I.2.1) into governing equations (I.1.1) to (I.1.4) and linearizing them, we have

\[
\frac{\partial u}{\partial t} - \frac{1}{\rho_0} \frac{\partial }{\partial x} \delta p + \left[ v - \frac{\mu_1}{\rho_0} \nabla^2 \right] \nabla^2 \mathbf{u} + 2 \Omega \mathbf{v},
\]

(I.2.4)

\[
\frac{\partial v}{\partial t} - \frac{1}{\rho_0} \frac{\partial }{\partial y} \delta p + \left[ v - \frac{\mu_1}{\rho_0} \nabla^2 \right] \nabla^2 \mathbf{v} - 2 \Omega \mathbf{u},
\]

(I.2.5)

\[
\frac{\partial w}{\partial t} - \frac{1}{\rho_0} \frac{\partial }{\partial z} \delta p - g \delta p + \left[ v - \frac{\mu_1}{\rho_0} \nabla^2 \right] \nabla^2 \mathbf{w},
\]

(I.2.6)

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,
\]

(I.2.7)

\[
\frac{\partial \theta}{\partial t} = \beta w + k_1 \nabla^2 \theta
\]

(I.2.8)

and  
\[
\delta p = -\alpha p \delta \theta.
\]

(I.2.9)
Analyzing the perturbation into normal modes, we assume that the perturbation quantities are of the form
\[ w, \zeta, \theta | = W(z), Z(z), \Theta(z) \exp(i k_x x + i k_y y + n t), \quad \text{\text{(I.2.10)}} \]
where \( k_x \) and \( k_y \) are the wave numbers in \( x \) and \( y \) directions respectively and
\[ k = \sqrt{k_x^2 + k_y^2} \] is the resultant wave number of propagation and \( n \) is the frequency of any arbitrary disturbance which is, in general, a complex constant.

For the considered form of the perturbations in equation (I.2.10), equations (I.2.4) to (I.2.9) give
\[ n\theta = \beta' W + k_T (D^2 - k^2) \Theta, \quad \text{\text{(I.2.11)}} \]
\[ n(D^2 - k^2)W = -k^2 g \alpha \Theta + \left[ v - \frac{\mu'}{\rho_0} (D^2 - k^2) \right] (D^2 - k^2)^2 W - 2 \partial \partial Z \quad \text{\text{(I.2.12)}} \]
and
\[ nZ = \left[ v - \frac{\mu'}{\rho_0} (D^2 - k^2) \right] (D^2 - k^2)Z + 2 \partial \partial W. \quad \text{\text{(I.2.13)}} \]

We eliminate the physical quantities using the non-dimensional parameter \( a = \frac{nd^2}{v}, p_i = \frac{v}{k_T}, F = \frac{\mu'}{\rho_0 d^2 v} \) and dropping (*) for convenience, equations (I.2.11) to (I.2.13) become
\[ (D^2 - a^2 - E p_i \sigma) \Theta = -\frac{\beta d^2 W}{k_T}, \quad \text{\text{(I.2.14)}} \]
\[ (D^2 - a^2) \left[ \sigma + F(D^2 - a^2)^2 - (D^2 - a^2) \right] W + \frac{\sigma a^2 d^2}{v} \Theta + \frac{2Qd^3}{v} DZ = 0, \quad \text{\text{(I.2.15)}} \]
and
\[ \left[ \sigma + F(D^2 - a^2)^2 - (D^2 - a^2) \right] Z = \frac{2Qd}{v} DW. \quad \text{\text{(I.2.16)}} \]

The perturbation in the temperature is zero at the boundaries because both the boundaries are maintained at constant temperature. The appropriate boundary conditions are
\[ W = 0, Z = 0, \Theta = 0 \] at \( z = 0 \) and \( z = 1 \).
\[ \text{\text{(I.2.17)}} \]

The constitutive equations of couple-stress fluid are
\[ \tau_{ij} = (2\mu - 2\mu' V^2) \rho_{ij} \] and \[ \rho_{ij} = \frac{1}{2} \left( \begin{array}{cc} \epsilon_{ij} & \epsilon_{ij} \\ \epsilon_{ij} & \epsilon_{ij} \end{array} \right) \]
46
and the condition on a free surface are given by

$$\tau_{xz} = (\mu - \mu'\nabla^2) \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = 0 \quad \text{and} \quad \tau_{yz} = (\mu - \mu'\nabla^2) \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = 0. \quad (I.2.18)$$

From the equation of continuity (I.1.2), we conclude that

$$\left[ \frac{\mu - \mu' \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)}{\partial z^2} \right] \frac{\partial^2 w}{\partial z^2} = 0, \quad (I.2.19)$$

which implies that

$$\frac{\partial^2 w}{\partial z^2} = 0, \quad \frac{\partial^4 w}{\partial z^4} = 0 \quad \text{at} \quad z = 0 \text{ and } z = d. \quad (I.2.20)$$

The boundary condition (I.2.20), after using equation (I.2.10) in non-dimensional form is

$$D^4 W = 0, \quad D^4 W = 0 \quad \text{at} \quad z = 0 \text{ and } z = 1. \quad (I.2.21)$$

Now, eliminating $\Theta$ and $Z$ between equations (I.2.14) to (I.2.16), we obtain the stability governing equation

$$\left[ D^2 - a^2 - \sigma p_1 \right] \left[ D^2 - a^2 \right] \left[ \sigma + F(D^2 - a^2)^2 - (D^2 - a^2)^2 \right] W$$

$$- \lambda Ra^2 \left[ \sigma + F(D^2 - a^2)^2 - (D^2 - a^2)^2 \right] W + T_A \left[ D^2 - a^2 - \sigma p_1 \right] D^3 W = 0 \quad (I.2.22)$$

Where $R = \frac{g_0 \rho \alpha d^4}{\nu k_T}$ is the thermal Rayleigh number and $T_A = \frac{2 \Omega d^2}{\nu}$ is Taylor number.

From equation (I.2.21), it is clear that all the even order derivatives of $W$ vanish on the boundaries. Therefore, the proper solution of equation (I.2.22) characterizing the lowest mode is

$$W = W_0 \sin \pi z. \quad (I.2.23)$$

Where $W_0$ is constant. Using equation (I.2.23) in equation (I.2.22), we obtain

$$\left[ D^2 - a^2 - \sigma p_1 \right] \left[ D^2 - a^2 \right] \left[ \sigma + F(D^2 - a^2)^2 - (D^2 - a^2)^2 \right] W_0 \sin \pi z$$

$$- \lambda Ra^2 \left[ \sigma + F(D^2 - a^2)^2 - (D^2 - a^2)^2 \right] W_0 \sin \pi z + T_A \left[ D^2 - a^2 - \sigma p_1 \right] D^3 W_0 \sin \pi z = 0$$

which can be written as

$$R_1 = \frac{(1 + x)}{\lambda x} (1 + x + i \sigma p_1) \left[ i \sigma_1 + F_i (1 + x)^2 + (1 + x) \right] + \frac{T_A (1 + x + i \sigma p_1)}{\lambda x \left[ i \sigma_1 + F_i (1 + x)^2 + (1 + x) \right]} \quad (I.2.24)$$

Where $R_1 = \frac{R}{\pi^4}$, $T_A = \frac{T_A}{\pi^4}$, $i \sigma_1 = \frac{\sigma}{\pi^2}$ and $F_i = \gamma^2 F$.  

47
I.3 ANALYTICAL DISCUSSION

(a) Stationary Convection

At stationary convection, when the instability sets, the marginal state will be characterized by $\varphi = 0$. Thus, putting $\varphi = 0$ in equation (I.2.24), we obtain

$$R_1 = \frac{(1 + x)^3}{\lambda x}[F_i(1 + x) + 1] + \frac{T_{A_i}}{\lambda x[F_i(1 - x) + 1]}.$$  \hspace{1cm} (I.3a.1)

The above relation expresses the modified Rayleigh number $R_1$ as a function of the parameters $F_i$, $T_{A_i}$ and dimensionless wave number $x$. To study the effect of couple-stress and rotation, we examine the nature of $\frac{dR_1}{dF_i}$ and $\frac{dR_1}{dT_{A_i}}$ analytically.

From equation (I.3a.1), we have

$$\frac{dR_1}{dF_i} = \frac{(1 + x)}{\lambda x}\left[(1 + x)^3 - \frac{T_{A_i}}{[F_i(1 + x) + 1]^2}\right],$$  \hspace{1cm} (I.3a.2)

which shows that couple-stress has a stabilizing/destabilizing effect on the thermal instability of couple-stress fluid under the conditions

$$T_{A_i} < [F_i(1 + x) + 1]^2(1 + x)^3 \quad \text{or} \quad T_{A_i} > [F_i(1 + x) + 1]^2(1 + x)^3.$$  

In the absence of rotation, equation (I.3a.2) becomes

$$\frac{dR_1}{dF_i} = \frac{(1 + x)^4}{\lambda x},$$  \hspace{1cm} (I.3a.3)

which clearly shows that couple-stress has a stabilizing effect on the thermal instability of a couple-stress fluid when gravity increases upwards from its value $g_0$.

From equation (I.3a.1), we have

$$\frac{dR_1}{dT_{A_i}} = \frac{1}{\lambda x[F_i(1 + x) + 1]},$$  \hspace{1cm} (I.3a.4)

which clearly shows that rotation has a stabilizing effect on the thermal instability of a couple-stress fluid when gravity increases upward from its value $g_0$.

(b) Stability of the System and Oscillatory Modes

Multiplying equation (I.2.15) by $W^*$ (conjugate of $W$) and integrate over the range of $z$ and making use of equations (I.2.14) and (I.2.16) together with the boundary
conditions (I.2.17) and (I.2.21), we get

$$\sigma I_1 + I_2 + \sigma^* I_4 + I_5 + \sigma^* I_8 = 0. \quad (I.3b.1)$$

Where, $I_1 = \int (|DW|^2 + a^2 |W|^2) dz$, $I_2 = \int (|D^2W|^2 + 2a^2 |DW|^2 - a^4 |W|^2) dz$,

$$I_3 = \int (|D^3W|^2 + 3a^2 |D^2W|^2 + 3a^4 |DW|^2 - a^6 |W|^2) dz,$$

$$I_5 = \int |DZ|^2 + a^2 |Z|^2 dz,$$

$$I_6 = \int |D^2Z|^2 + 2a^2 |DZ|^2 - a^4 |Z|^2 dz,$$

$$I_7 = \int |D\Theta|^2 + a^2 |\Theta|^2 dz,$$

$$I_8 = \int |\Theta|^2 dz.$$

All the integrals $I_1$ to $I_8$ are positive definite, Putting $\sigma = i\sigma_i$ in equation (I.3b.1), where $\sigma_i$ is real and equating the imaginary part, we get

$$\sigma_i \left[ I_1 - d^2 I_4 + \frac{\lambda g_c a^2 k_r}{\nu \beta} p I_8 \right] = 0. \quad (I.3b.2)$$

In the absence of rotation, equation (I.3b.2) becomes

$$\sigma_i \left[ I_1 + \frac{\lambda g_c a^2 k_r}{\nu \beta} p I_8 \right] = 0. \quad (I.3b.3)$$

It is obvious from equation (I.3b.3) that the terms in the bracket are positive definite. Thus $\sigma_i = 0$ which means that oscillatory modes are not allowed in the system and Principle of Exchange of Stabilities (PES) is satisfied in the absence of rotation in the system. So, we can say that oscillatory modes are introduced due the presence of rotation in the system.

(c) The Case of Overstability

Here, we discuss the case of overstability. When the marginal state is oscillatory, we have $\sigma_i \neq 0$. Since, we wish to examine the Rayleigh number for the onset of instability via a state of pure oscillations, equating the real and imaginary parts of equation (I.2.24), we have

$$R_{\lambda_1} + \cdots + \lambda_n - \sigma \quad + \quad \overline{\sigma} \quad + \quad \overline{T_{\lambda_1} b}. \quad (I.3c.1)$$
and \[ R_i^\lambda x \sigma = 2 \sigma_i (F_i b + 1) b^3 - \sigma_i \rho_1 b + \sigma_i \rho_1 (F_i b + 1)^2 b^3 + \sigma_i \rho_1 T_{\lambda i}. \] (I.3c.2)

Eliminating \( R_i \) between equations (I.3c.1) and (I.3c.2) and using \( b = (1 + x) \), we get
\[ A \sigma_i^2 + B = 0. \] (I.3c.3)

Where \( A = p_1 (F_i b + 1) b^2 + b^2 \)
\[ B = 2(F_i b + 1)^2 b^4 + p_1 (F_i b + 1) - l_1^2 (F_i b + 1)^2 b^4 + T_{\lambda i} \left( \rho_1 (F_i b + 1) - l_1 \right) b. \]

Since \( \sigma_i \) is real for overstability, the value of \( \sigma_i \) is positive. This is clearly impossible from equation (I.3c.3) if \( A > 0, B > 0 \) give the sufficient conditions for the non-existence of overstability which yields
\[ p_1 (F_i b + 1) > 1 \text{ or } k_T < \sqrt{1 + \frac{11'}{\rho_0 d^2 v} (1 + x)}. \]

Hence, \( k_T < \sqrt{1 + \frac{11'}{\rho_0 d^2 v} (1 + x)} \) is the sufficient condition for the non-existence of overstability for the present problem.
SECTION II

II.1 EFFECT OF SOLUTE GRADIENT: FORMULATION OF THE PROBLEM

In this case, consider a static state in which an incompressible couple-stress fluid layer of thickness \( d \), is arranged, confined between two infinite horizontal planes situated at \( z = 0 \) and \( z = d \), which is acted upon by a uniform rotation and variable gravity field. The fluid layer is heated and soluted from below leading to an adverse temperature gradient \( \beta = \frac{T_0 - T_1}{d} \), with \( T_0 > T_1 \) and \( \beta' = \frac{C_0 - C_1}{d} \), where \( C_0 \) and \( C_1 \) are the constant concentrations of the lower and upper boundaries with \( C_0 > C_1 \).

The equations of motion, continuity and heat conduction of couple-stress fluid (Stokes [133]) are

\[
\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla) \mathbf{q} = -\frac{1}{\rho_0} \nabla p + \mathbf{g} \left( 1 + \frac{\rho \Omega^2}{\rho_0} \right) + \left( \mathbf{v} - \frac{\Omega}{\rho_0} \nabla^2 \right) \mathbf{q} + 2(\mathbf{q} \times \mathbf{\Omega}), \quad (II.1.1)
\]

\[
\nabla \cdot \mathbf{q} = 0, \quad (II.1.2)
\]

\[
\frac{\partial T}{\partial t} + (\mathbf{q} \cdot \nabla) T = k_T \nabla^2 T, \quad (II.1.3)
\]

\[
\frac{\partial C}{\partial t} + (\mathbf{q} \cdot \nabla) C = k_s \nabla^2 C. \quad (II.1.4)
\]

The equation of state is

\[
\rho = \rho_0 \left[ 1 - \alpha(T - T_0) + \alpha'(C - C_0) \right], \quad (II.1.5)
\]

where the suffix zero refers to value at the reference level \( z = 0 \) and \( k_T, k_s \) are the thermal diffusivity and solute diffusivity respectively and \( \alpha \) is the thermal coefficient of expansion and \( \alpha' \) is a coefficient analogous to \( \alpha \).

II.2 BASIC STATE AND PERTURBATION EQUATIONS

In the undisturbed state, the fluid is at rest. Constant temperatures and concentrations are maintained in the fluid and uniform rotation acts in the vertical direction (say in \( z \)-direction), therefore the basic state of which we wish to examine the stability is characterized by
\( \mathbf{q} = (0, 0, 0), \ \Omega = (0, 0, \Omega), T = T_0 - \beta z, C = C_0 - \beta' z, \) (II.2.1)

where \( \beta \) and \( \beta' \) may be either positive or negative and this basic state is consistent with the equations (II.1.1) to (II.1.5) provided that

\[ \rho = \rho(z), \ p = p(z) \text{ and } \rho = \rho_0 \left[ 1 - \alpha \beta z + \alpha' \beta' z \right]. \] (II.2.2)

The character of equilibrium is examined by supposing that the system is slightly perturbed so that every physical quantity is assumed to be the sum of a mean and fluctuating component, later designated as prime quantities and assume to be very small in comparison to their equilibrium state values. Here, we assume that the small disturbances are the functions of space and time variables. Hence, the perturbed flow may be represented as

\[ \mathbf{q} = (0, 0, 0) + (u, v, w), \]

\[ T = T(z) + \theta, \]

\[ C = C(z) + \gamma, \]

\[ \rho = \rho(z) + \delta \rho \]

and \[ p = p(z) + \delta p. \] (II.2.3)

Where \( q(u, v, w), \ 0, \ \delta \rho, \ \delta p \) denote respectively the perturbations in fluid velocity \( q(0, 0, 0), \) temperature \( T, \) density \( \rho \) and pressure \( p. \) Using equation (II.2.3) into governing equations (II.1.1) to (II.1.5) and linearizing them, we have

\[ \frac{\partial \tilde{u}}{\partial t} = -\frac{1}{\rho_0} \frac{\partial}{\partial x} \tilde{\rho} + \left[ v - \frac{\mu'}{\rho_0} \right] V^2 u + 2\Omega v, \] (II.2.4)

\[ \frac{\partial \tilde{v}}{\partial t} = -\frac{1}{\rho_0} \frac{\partial}{\partial y} \tilde{\rho} + \left[ v - \frac{\mu'}{\rho_0} \right] V^2 v - 2\Omega u, \] (II.2.5)

\[ \frac{\partial \tilde{w}}{\partial t} = -\frac{1}{\rho_0} \frac{\partial}{\partial z} \tilde{\rho} - g \delta \rho + \left[ v - \frac{\mu'}{\rho_0} \right] V^2 w, \] (II.2.6)

\[ \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial z} = 0, \] (II.2.7)

\[ \frac{\partial \tilde{T}}{\partial t} = \beta w + k_\gamma V^2 \theta, \] (II.2.8)
\[
\frac{\partial \gamma'}{\partial t} = \beta' w + k_x V^2 \gamma
\]  
(II.2.9)

and 

\[
\delta \rho = - \rho_0 (\alpha \theta - \alpha' \gamma).
\]  
(II.2.10)

Analyzing the perturbations into normal modes, we assume that the perturbation quantities are of the form

\[
[w, \zeta, \theta, \gamma] = W(z), Z(z), \Theta(z), \Gamma(z) \exp \left[ ik_x x + ik_y y + i n \gamma \right],
\]  
(II.2.11)

where \( k_x \) and \( k_y \) are the wave numbers in \( x \) and \( y \) directions respectively and \( k = \sqrt{k_x^2 + k_y^2} \) is the resultant wave number of propagation and \( n \) is the frequency of any arbitrary disturbance which is, in general, a complex constant.

For the considered form of the perturbations in equation (II.2.11), equations (II.2.4) to (II.2.10) give

\[
n \Theta = \beta W + k_t (D^2 - k^2) \Theta, \]
(II.2.12)

\[
n \Gamma = \beta' W + k_s (D^2 - k^2) \Gamma, \]
(II.2.13)

\[
n (D^2 - k^2) W = -g k^2 (\alpha \Theta - \alpha' \Gamma) + \left[ v - \frac{\mu'}{\rho_0} (D^2 - k^2) \right] (D^2 - k^2) W - 2 \Omega DZ
\]  
(II.2.14)

and 

\[
n (D^2 - k^2) Z = \left[ v - \frac{\mu'}{\rho_0} (D^2 - k^2) \right] (D^2 - k^2) Z + 2 \Omega DW.
\]  
(II.2.15)

We eliminate the physical quantities using the non-dimensional parameter \( a = k d, \sigma = \frac{nd^2}{v}, p_t = \frac{v}{k_t}, q = \frac{v}{k_s}, F = \frac{\mu'}{\rho_0 d^2 v} \) and \( D^4 = d D \) and dropping (*) for convenience, equations (II.2.12) to (II.2.15) become

\[
(D^2 - a^2 - E p_0 \sigma) \Theta = -\frac{\beta d^2 W}{k_t},
\]  
(II.2.16)

\[
(D^2 - a^2 - \sigma q) \Gamma = -\frac{\beta' d^2 W}{k_s},
\]  
(II.2.17)

\[
(D^2 - a^2) \left[ \frac{\sigma + F (D^2 - a^2)^2}{v} - \left( D^2 - a^2 \right) \right] W + \frac{g \alpha a^2 d^2}{v} \Theta - \frac{g \alpha' a^2 d^2}{v} \Gamma
\]

\[
+ \frac{2 \Omega d^3}{v} DZ = 0
\]  
(II.2.18)
and
\[
\sigma + F(D^2 - a^2)^2 - (D^2 - a^2)^2 - (D^2 - a^2) = \frac{2\Omega x}{V} \cdot \text{DW.} \tag{II.2.19}
\]

Now, eliminating \(\Theta\), \(\Gamma\) and \(Z\) between equations (II.2.16) to (II.2.19), we obtain the final stability governing equation
\[
\begin{align*}
\left[ D^2 - a^2 - \sigma p_1 \right] & \left[ D^2 - a^2 - \sigma q \right] \left[ D^2 - a^2 \right] \left[ \sigma + F(D^2 - a^2)^2 - (D^2 - a^2) \right] W \\
- \lambda \left[ D^2 - a^2 - \sigma q \right] & \left[ \sigma + F(D^2 - a^2)^2 - (D^2 - a^2) \right] W - \lambda Sa_2 \left[ D^2 - a^2 - \sigma p_1 \right] \\
\left[ \sigma + F(D^2 - a^2)^2 - (D^2 - a^2) \right] W + T_\Lambda \left[ D^2 - a^2 - \sigma p_1 \right] \left[ D^2 - a^2 - \sigma q \right] D^2 W &= 0. \tag{II.2.20}
\end{align*}
\]

Where \( R = \frac{g_0 \alpha \beta d^4}{v k_T} \) is the thermal Rayleigh number, \( S = \frac{g_0 \alpha' \beta' d^4}{v k_s} \) is the solute Rayleigh number and \( T_\Lambda = \frac{2\Omega x^2}{V} \) is the Taylor number.

Taking the case of two free boundaries, the dimensionless boundary conditions are
\[
W = 0, D^2 W = 0, \Theta = 0, \Gamma = 0, DZ = 0 \text{ at } z = 0 \text{ and } z = 1. \tag{II.2.21}
\]

Obviously, all the even order derivatives of \( W \) vanish on the boundaries. Therefore, the proper solution of equations the system of equation (II.2.20) characterizing the lowest mode is
\[
W = W_0 \sin \pi z. \tag{II.2.22}
\]

Where \( W_0 \) is a constant then using equation (II.2.22), equation (II.2.20) gives
\[
\begin{align*}
\left[ D^2 - a^2 - \sigma p_1 \right] & \left[ D^2 - a^2 - \sigma q \right] \left[ D^2 - a^2 \right] \left[ \sigma + F(D^2 - a^2)^2 - (D^2 - a^2) \right] W_0 \sin \pi z \\
- \lambda R & \left[ D^2 - a^2 - \sigma q \right] \left[ \sigma + F(D^2 - a^2)^2 - (D^2 - a^2) \right] W_0 \sin \pi z \\
- \lambda S & \left[ D^2 - a^2 - \sigma p_1 \right] \left[ \sigma + F(D^2 - a^2)^2 - (D^2 - a^2) \right] W_0 \sin \pi z \\
+ T_\Lambda & \left[ D^2 - a^2 - \sigma p_1 \right] \left[ D^2 - a^2 - \sigma q \right] D^2 W_0 \sin \pi z = 0,
\end{align*}
\]

which gives
\[
R_i = \frac{\left(1 + x\right)}{\lambda x} \left[ i\sigma_1 + F_i (1 + x)^2 + (1 + x) \right] + \frac{S_i (1 + x + i\sigma_i p_1)}{(1 + x + i\sigma_i q)} \\
T_\Lambda_i \left(1 + x + i\sigma_i p_1\right) \frac{T_\Lambda \left(1 + x + i\sigma_i p_1\right)}{\lambda x [i\sigma_1 + F_i (1 + x)^2 + (1 + x)]}. \tag{II.2.23}
\]

Where, \( R_i = \frac{R}{\pi^2} \), \( T_\Lambda_i = \frac{T_\Lambda}{\pi^2} \), \( i\sigma_1 = \frac{\sigma}{\pi^2} \), \( x = \frac{a^2}{\pi^2} \) and \( F_i = \frac{\chi^2 F_i}{\pi^2} \).
II.3 ANALYTICAL DISCUSSION

(a) Stationary Convection

At stationary convection, when the instability sets, the marginal state will be characterized by $0$. Thus putting $\varphi = 0$ in equation (II.2.23), we obtain

$$R_i = \frac{1}{\lambda x} \left[ (1+x)^3 [F_i(1+x) + 1] + \frac{T_{\lambda i}}{[F_i(1+x) + 1]} \right] + S_i. \quad (\text{II.3a.1})$$

The above relation expresses modified Rayleigh number $R_i$ as a function of the parameters $S_i, F_i, T_{\lambda i}$ and dimensionless wave number $x$. To study the effect of stable solute gradient, couple-stress and rotation, we study the behavior of $\frac{dR_i}{dS_i}, \frac{dR_i}{dF_i}$ and $\frac{dR_i}{dT_{\lambda i}}$ analytically.

From equation (II.3a.1), we have

$$\frac{dR_i}{dS_i} = 1, \quad \text{(which is positive)} \quad (\text{II.3a.2})$$

which clearly shows that stable solute gradient has a stabilizing effect on the thermosolutal instability of couple-stress rotating fluid as can be seen graphically from Fig. 2.1 and Fig. 2.2.

From equation (II.3a.1), we have

$$\frac{dR_i}{dF_i} = \frac{1}{\lambda x} \left[ (1+x)^4 - \frac{T_{\lambda i}(1+x)}{[F_i(1+x) + 1]^2} \right], \quad (\text{II.3a.3})$$

which shows that the couple-stress has a stabilizing/destabilizing effect on the thermosolutal instability of couple-stress rotating fluid under the conditions

$$T_{\lambda i} < \text{or} > (1+x)^3 [F_i(1+x) + 1]^2$$

as can be seen graphically from Fig. 2.3 and Fig. 2.4.

But for the permissible values of various parameters, the said effect is stabilizing only if

$$T_{\lambda i} < \text{or} > (1+x)^3 [F_i(1+x) + 1]^2.$$
In the absence of rotation, equation (II.3a.3) becomes

$$\frac{dR_{1}}{dT_{a1}} = \frac{(1+x)^{4}}{\lambda x}$$, \hspace{1cm} (II.3a.4)

It is obvious from equation (II.3a.4) that couple-stress clearly has a stabilizing effect on the thermosolutal instability of couple-stress rotating fluid when gravity increases upward from its value \(g_0\).

Again, from equation (II.3a.1), we have

$$\frac{dR_{1}}{dT_{a1}} = \frac{1}{\lambda x[F_{1}(1+x) + 1]}$$, \hspace{1cm} (II.3a.5)

which clearly shows that rotation has a stabilizing effect on the thermosolutal instability of couple-stress rotating fluid when gravity increases upward from its value \(g_0\) as can be seen graphically from Fig. 2.5.

(b) Stability of the System and Oscillatory Modes

Multiplying equation (II.2.18) by \(W^*(\text{conjugate of } W)\) and integrate over the range of \(z\) and making use of equations (II.2.16), (II.2.17) and (II.2.19) together with the boundary condition (II.2.21), we obtain

$$\sigma I_{1} + I_{2} + F I_{3} + d^{2} \sigma I_{4} + I_{5} + F I_{6} - \frac{\lambda g_{0} \alpha^{2} k}{\beta^{2}} [I_{7} + \sigma I_{10}]$$

$$+ \frac{\lambda g_{0} \alpha^{2} k}{\beta^{2}} [I_{7} + \sigma I_{10}] = 0$$, \hspace{1cm} (II.3b.1)

Where

$$I_{1} = \int (|D W|^2 + a^2 |W|^2) dZ$$,

$$I_{2} = \int (|D^2 W|^2 + 2a^2 |DW|^2 + a^4 |W|^2) dZ$$,

$$I_{3} = \int (|D^3 W|^2 + 3a^2 |D^2 W|^2 + 3a^4 |DW|^2 - a^6 |W|^2) dZ$$,

$$I_{4} = \int |Z|^2 dZ$$, \hspace{1cm} $$I_{5} = \int |DZ|^2 + a^2 |Z|^2 dZ$$,

$$I_{6} = \int |D^2 Z|^2 + 2a^2 |DZ|^2 + a^4 |Z|^2 dZ$$, \hspace{1cm} $$I_{7} = \int |D \Theta|^2 + a^2 |\Theta|^2 dZ$$,

$$I_{8} = \int |\Theta|^2 dZ$$, \hspace{1cm} $$I_{9} = \int |D| dZ$$, \hspace{1cm} $$I_{10} = \int |\Gamma|^2 dZ$$.
All the integrals $I_i$ to $I_{10}$ are positive definite. Putting $\sigma = \sigma_r + i\sigma_i$ in equation (II.3b.5) and equating the real and imaginary parts, we obtain
\[
\sigma_r \left[ I_1 - d^2I_4 - \frac{\dot{\lambda}g_0\alpha a^2k_T}{v\beta} p_1I_8 + \frac{\dot{\lambda}g_0\alpha' a'^2k_s}{v\beta'} qI_{10} \right]
\]
\[
= - \left[ I_2 + FI_3 - d^2(I_5 + FI_6) - \frac{\dot{\lambda}g_0\alpha k_T a^2}{v\beta} I_7 - \frac{\dot{\lambda}g_0\alpha' k_s a'^2}{v\beta'} I_9 \right] \tag{II.3b.2}
\]
and
\[
\sigma_i \left[ I_1 - d^2I_4 + \frac{\dot{\lambda}g_0\alpha a^2k_T}{v\beta} p_1I_8 - \frac{\dot{\lambda}g_0\alpha' a'^2k_s}{v\beta'} qI_{10} \right] = 0. \tag{II.3b.3}
\]

It is obvious from equation (II.3b.2) that $\sigma_r$ may be positive or negative which means that system may be stable or unstable.

In the absence of stable solute gradient and rotation, equation (II.3b.3) becomes
\[
\sigma_i \left[ I_1 - d^2I_4 - \frac{\dot{\lambda}g_0\alpha a^2k_T}{v\beta} p_1I_8 - \frac{\dot{\lambda}g_0\alpha' a'^2k_s}{v\beta'} qI_{10} \right] = 0. \tag{II.3b.4}
\]
It is obvious from equation (II.3b.4) that all the terms in bracket are positive definite. Thus $\sigma_i = 0$ which means that oscillatory modes are not allowed in the system and Principle of Exchange of Stabilities (PES) is satisfied in the system in the absence of stable solute gradient and rotation. So, we can say that oscillatory modes are introduced due to the presence of stable solute gradient and rotation in the system.

(c) **The Case of Overstability**

In this case, we discuss the case of overstability. Since we wish to examine the Rayleigh number for the onset of instability via a state of pure oscillations, equating real and imaginary parts of equation (II.2.23), we get
\[
R_{\lambda, x} \left[ (Fb + 1)b^2 - \sigma_i^2 q \right] = \left[ (Fb + 1)b^2 - \sigma_i^2 \{b^3 - \sigma_i^2 p_1 q b\} \right]
\]
\[- 2\sigma_i^2 (p_i + q)(Fb + 1)b^3 \right] + S_{\lambda, x} \left[ (Fb + 1)b^2 - \sigma_i^2 p_1 \right] + T_{\lambda, i} \left[ b^2 - \sigma_i^2 p_i q \right] \tag{I.3c.1}
\]
and
\[
R_{\lambda, x} \sigma_i b - \sigma_i q(Fb + 1)b^3 - \sigma_i \left\{ \sigma_i (p_i + q)b^2 - \frac{1}{\zeta} (Fb + 1)^2 b^2 - \sigma_i^{-2} \right\} + 2\sigma_i (Fb + 1)b^3 \left\{ b^3 - \sigma_i^2 p_1 q b \right\} \right] + S_{\lambda, x} \left[ \sigma_i b + \sigma_i p_i (Fb + 1)b^3 + 2\sigma_i (Fb + 1)b^3 \right] \tag{I.3c.2}
\]
Where $b = (1 + x)$. Eliminating $R_1$ between equations (I.3c.1) and (I.3c.2), we get

$$\Lambda_0 \sigma_i^4 + B_0 \sigma_i^2 + C_0 = 0. \quad \text{(II.3c.3)}$$

Where $A_0 = q^2 \left\{ p_i (Fb + l) + 1 \right\} b^2$, $B_0 = -S_i \lambda_i (b - l)(p_i - q) + T_{\lambda_i} q^2 - p_i (Fb + l) - I \lambda_i^2 b$,

$$+ q^2 \left\{ p_i (Fb + l) + l \right\} (Fb + l)b^4 + p_i (Fb + l) + I \lambda_i^2 b^4,$$

and $C_0 = S_i \lambda_i (b - l) (p_i - q)(Fb + l) b^3 + T_{\lambda_i} p_i (Fb + l) - I \lambda_i^2 b^3$,

$$+ \left\{ p_i (Fb + l) + l \right\} (Fb + l)^2 b^6.$$

Since $\sigma_i$ is real for overstability, the two values of $\sigma_i^2$ is positive. But this is clearly impossible if

$$p_i > q \quad \text{and} \quad p_i (Fb + l) - I > 0$$

which gives $k_s > k_T$ and $k < \sqrt{\frac{\mu'}{\rho_o d^2 \nu}} (1 + x) + 1 \right\}.$

Hence, $k_s > k_T$ and $k_T < \sqrt{\frac{\mu'}{\rho_o d^2 \nu}} (1 + x) + 1 \right\}$ are the sufficient conditions for the non-existence of overstability.

### 2.3 NUMERICAL COMPUTATIONS

Now, the critical thermal Rayleigh number for the onset of instability is determined numerically using Newton-Raphson method by the condition $\frac{dR_1}{dx} = 0$.

As a function of $x$, $R_1$ is given by equation (II.3a.1) attains its minimum when $\frac{dR_1}{dx} = 0$ with $x$ determined as a solution of equation by putting $\frac{dR_1}{dx} = 0$ in powers of $x$.

Equation (II.3a.1) will give the required critical thermal Rayleigh number $R_1$ for various values of critical wave number $\chi$. The numerical values of critical thermal Rayleigh number $R_1$ and critical wave number $\chi$ determined for various values of stable solute gradient $S_i$, couple-stress $F_i$ and rotation $T_{\lambda_i}$. 
In Fig. 2.1, critical Rayleigh number \( R_i \) is plotted against stable solute gradient \( S_i \) for fixed value of \( \lambda =1, F_1 =10 \) and \( T_A =100, 300, 500 \). The critical Rayleigh number \( R_i \) increases with increase in stable solute gradient \( S_i \) which shows that stable solute gradient has a stabilizing effect on the system.

In Fig. 2.2, critical Rayleigh number \( R_i \) is plotted against stable solute gradient \( S_i \) for fixed value of \( \lambda =1, F_1 =10 \) and \( T_A =1000, 1300, 1600 \). The critical Rayleigh number \( R_i \) increases with increase in stable solute gradient \( S_i \) which shows that stable solute gradient has a stabilizing effect on the system.

In Fig. 2.3, critical Rayleigh number \( R_i \) is plotted against couple-stress parameter \( F_1 \) for fixed value of \( \lambda =1, S_i =10 \) and \( T_A =100, 300, 500 \). The critical Rayleigh number \( R_i \) increases with increase in couple-stress parameter \( F_1 \) which shows that couple-stress has a stabilizing effect on the system.

In Fig. 2.4, critical Rayleigh number \( R_i \) is plotted against couple-stress parameter \( F_1 \) for fixed value of \( \lambda =1, S_i =10 \) and \( T_A =6000, 8000, 10000 \). The critical Rayleigh number \( R_i \) increases/decreases with increase in couple-stress parameter \( F_1 \) which shows that couple-stress has both stabilizing and destabilizing effects on the system.

In Fig. 2.5, critical Rayleigh number \( R_i \) is plotted against rotation parameter \( T_A \) for fixed value of \( \lambda =1, F_1 =10 \) and \( S_i =10, 30, 50 \). The critical Rayleigh number \( R_i \) increases with increase in \( T_A \) which shows that rotation has stabilizing effects on the system.
Fig. 2.1: Variation of critical Rayleigh number $R_1$ with $S_1$ for a fixed $\lambda = 1, F_i = 10, T_A = 100, 300, 500$.

Fig. 2.2: Variation of critical Rayleigh number $R_1$ with $S_1$ for a fixed $\lambda = 1, F_i = 10, T_A = 1000, 1300, 1600$. 
Fig. 2.3: Variation of critical Rayleigh number $R$ with $F$ for a fixed $\lambda = 1, S_1 = 10$ and $T_A = 100, 300, 500$.

Fig. 2.4: Variation of critical Rayleigh number $R$ with $F$ for a fixed $\lambda = 1, S_1 = 10$ and $T_A = 6000, 8000, 10000$. 
Fig. 2.5: Variation of critical Rayleigh number $R_1$ with $T_{A_1}$ for a fixed $\lambda = 1, F_1 = 10$ and $S_1 = 10, 30, 50$

2.4 CONCLUSIONS

With the growing importance of Non-Newtonian fluids in modern technology and industries, the investigations on couple-stress fluid are desirable. In the present chapter, the problem of thermal instability of couple-stress rotating in the presence of a variable gravity field with the effect of stable solute gradient is considered. Dispersion relation governing the effects of stable solute gradient, couple-stress and rotation is derived. The main results from the analysis of this chapter are as follows:

(i) For the case of stationary convection, the stable solute gradient has a stabilizing effect on the system as can be seen from equation (II.3a.2), and graphically, from Fig. 2.1 and Fig. 2.2.

(ii) Couple-stress has stabilizing/destabilizing effects on the system for the permissible values of various parameters as can be seen from equation (II.3a.3), and graphically, from Fig. 2.3 and Fig. 2.4. In the absence of rotation, couple-stress clearly has a stabilizing effect on the system as can be seen from equation (II.3a.4).
(iii) For the case of stationary convection, the rotation has a stabilizing effect on the system as can be seen from equation (II.3a.5), and graphically, from Fig. 2.5.

(iv) The oscillatory modes are introduced due to the presence of stable solute gradient and rotation whereas in their absence principle of exchange of stabilities (PES) is satisfied in the system.

(v) The conditions $k_s > k_T$ and $k_T < v\left\{\frac{1}{1 + x} + 1\right\}$ are the sufficient conditions for the non-existence of overstability.