Chapter 7

Quasi Regular Semi Local
Functions in Ideal Bitopological Spaces

7.1 Introduction


In section 2 of this chapter, we introduce quasi regular semi local function, \(qrsI\)-open sets and \(qrsI\)-closed sets in ideal bitopological spaces and study some of their properties. The notion of \(qrsI\)-interior is defined and some of its basic properties are studied. Also we introduce the concepts of \(qrsI\)-cluster and \(qrsI\)-neighbourhood in ideal bitopological spaces by using the notions of \(qrsI\)-closed sets and obtain
some of its basic results.

In section 3 of this chapter, a new class of maps called \( qrsI \)-continuous maps in ideal bitopological spaces are introduced and investigated. During this process, some of their properties are obtained. It is found that every \( qI \)-continuous mapping is \( qrsI \)-continuous but not conversely. Also, we have introduced the concept of \( qrsI \)-open map and \( qrsI \)-closed map in ideal bitopological spaces and some of their properties.

Throughout this chapter \((X, \tau_1, \tau_2, I)\) or simply \(X\) denote nonempty ideal bitopological spaces on which no separation axioms are assumed, unless otherwise mentioned and fixed integers \(i, j \in \{1, 2\}\).

### 7.2 Quasi Regular Semi Local Functions

In this section, we introduce and investigate the concept of quasi regular semi local function, \( qrsI \)-open sets and \( qrsI \)-closed sets in ideal bitopological spaces.

**Definition 7.2.1** Given an ideal bitopological space \((X, \tau_1, \tau_2, I)\) the **quasi regular semi local function** of \(A\) with respect to \(\tau_1, \tau_2\) and \(I\) denoted by \(A_{qrs}^*(\tau_1, \tau_2, I)\) is defined as follows:

\[
A_{qrs}^*(\tau_1, \tau_2, I) = \{x \in X | U \cap A \notin I, \forall \text{ quasi regular semiopen set } U \text{ containing } x\}.
\]

When there is no ambiguity \(A_{qrs}^*\) shall be written for \(A_{qrs}^*(\tau_1, \tau_2, I)\).

Recall that, If \(\wp(X)\) is the set of all subsets of \(X\), in a topological space \((X, \tau)\) a set operator \((.)^* : \wp(X) \rightarrow \wp(X)\) called the local function [28] of \(A\) with respect to \(\tau\) and \(I\) and is defined as follows:
\[ A^*(\tau, I) = \{ x \in X | U \cap A \notin I, \forall \ U \in \tau(x) \}, \text{ where } \tau(x) = \{ U \in \tau | x \in U \}. \]

**Theorem 7.2.2** Let \((X, \tau_1, \tau_2, I)\) be an ideal bitopological space and \(A \subseteq X\) then:

(a) \(A_{qrs}^* \subseteq A_q \subseteq A^*(\tau_1, I)\) and \(A_{qrs}^* \subseteq A_q \subseteq A^*(\tau_2, I)\).

(b) \(A_{qrs}^* \subseteq A_{rs}(\tau_1, I)\) and \(A_{qrs}^* \subseteq A_{rs}(\tau_2, I)\).

(c) \(A_{qrs}^*(\tau_1, \tau_2, \{\phi\}) = qrscl(A)\).

(d) \(A_{qrs}^*(\tau_1, \tau_2, \varphi(X)) = \phi\).

(e) If \(A \in I\), then \(A_{qrs}^* = \phi\).

(f) Neither \(A \subseteq A_{qrs}^*\) nor \(A_{qrs}^* \subseteq A\).

**Proof.** Follows from the Definition 7.2.1. \(\blacksquare\)

**Theorem 7.2.3** Let \((X, \tau_1, \tau_2, I)\) be an ideal bitopological space and \(A, B\) be subsets of \(X\) then:

(a) If \(A \subseteq B\), then \(A_{qrs}^* \subseteq B_{qrs}^*\).

(b) \(A_{qrs}^* = qrscl A_{qrs}^* \subseteq qrscl(A)\) and \(A_{qrs}^*\) is a quasi regular semiclosed set in 
\((X, \tau_1, \tau_2)\).

(c) \((A_{qrs}^*)_{qrs}^* \subseteq A_{qrs}^*\).

(d) \((A \cup B)_{qrs}^* = A_{qrs}^* \cup B_{qrs}^*\).

(e) \(A_{qrs}^* \setminus B_{qrs}^* = (A \setminus B)_{qrs}^* \setminus B_{qrs}^* \subseteq (A \setminus B)_{qrs}^*\).

(f) If \(C \in I\), then \((A \setminus C)_{qrs}^* \subseteq A_{qrs}^* = (A \cup C)_{qrs}^*\).
**Proof.** (a) Suppose $A \subseteq B$ and $x \notin B^*_qrs$ then there exists a quasi regular semiopen set $U$ containing $x$ such that $U \cap B \in I$. Since $A \subseteq B$, $U \cap A \in I$ and so $x \notin A^*_qrs$. Hence $A^*_qrs \subseteq B^*_qrs$.

(b) We have $A^*_qrs \subseteq \text{qrscl}(A^*_qrs)$, in general. Let $x \in \text{qrscl}(A^*_qrs)$, then $A^*_qrs \cap U \neq \emptyset$ for every quasi regular semiopen set $U$ containing $x$. Therefore $\exists \ y \in A^*_qrs \cap U$ and quasi regular semiopen set $U$ containing $y$. Since $y \in A^*_qrs$ and $U \cap A \notin I$, therefore $x \in A^*_qrs$. Hence $\text{qrscl}(A^*_qrs) \subseteq A^*_qrs$. Consequently, $A^*_qrs = \text{qrscl}(A^*_qrs)$. Again let $x \in \text{qrscl}(A^*_qrs) = A^*_qrs$. Then $U \cap A \notin I$ for every quasi regular semiopen set containing $x$. Therefore $x \in \text{qrscl}(A)$. This proves $A^*_qrs = \text{qrscl}(A^*_qrs) \subseteq \text{qrscl}(A)$.

(c) Let $x \in (A^*_qrs)^*$, then for every quasi regular semiopen set $U$ containing $x$, $U \cap A^*_qrs \notin I$ and hence $I \neq \emptyset$. Let $y \in A^*_qrs \cap U$. Then $\exists$ a quasi regular semiopen set $U$ containing $y$ and $y \in A^*_qrs$. Hence we have $U \cap A \notin I$ and $x \in A^*_qrs$. Therefore $(A^*_qrs)^* \subseteq A^*_qrs$.

(d) By (a) $A^*_qrs \cup B^*_qrs \subseteq (A \cup B)^*_qrs$. Let $x \in (A \cup B)^*_qrs$ then for every quasi regular semiopen set $U$ containing $x$, $(U \cap A) \cup (U \cap B) = U \cap (A \cup B) \notin I$. This implies $x \in A^*_qrs$ or $x \in B^*_qrs$. Hence, $x \in A^*_qrs \cup B^*_qrs$.

(e) We have $A^*_qrs = (A \setminus B)^*_qrs \cup (A \cap B)^*_qrs$. Thus $A^*_qrs \setminus B^*_qrs = A^*_qrs \cap (X \setminus B)^*_qrs = (A \setminus B)^*_qrs \cup (A \cap B)^*_qrs \cap (X \setminus B)^*_qrs = (A \setminus B)^*_qrs \cap (X \setminus B)^*_qrs \cup (A \cap B)^*_qrs \cap (X \setminus B)^*_qrs = ((A \setminus B)^*_qrs \setminus B^*_qrs) \cup \phi \subseteq (A \setminus B)^*_qrs$.

(f) Since $A \setminus C \subseteq A$, by (1) $(A \setminus C)^*_qrs \subseteq A^*_qrs$. From (d), we get $(A \cup C)^*_qrs = A^*_qrs \cup C^*_qrs = A^*_qrs \cup \phi = A^*_qrs$. Hence, $(A \setminus C)^*_qrs \subseteq A^*_qrs = (A \cup C)^*_qrs$. □

**Theorem 7.2.4** Let $(X, \tau_1, \tau_2)$ be a bitopological space with ideals $I_1$ and $I_2$ on
$X$ and $A$ is a subset of $X$. Then:

(a) If $I_1 \subseteq I_2$, then $A^*_{qrs}(I_2) \subseteq A^*_{qrs}(I_1)$.

(b) $A^*_{qrs}(I_1 \cap I_2) = A^*_{qrs}(I_1) \cup A^*_{qrs}(I_2)$.

**Proof.** (a) Let $I_1 \subseteq I_2$ and $x \in A^*_{qrs}(I_2)$, then $A \cap U \notin I_2$ for every quasi regular semiopen set $U$ containing $x$. From given $A \cap U \notin I_1$ hence $x \in A^*_{qrs}(I_1)$. Therefore, we have $A^*_{qrs}(I_2) \subseteq A^*_{qrs}(I_1)$.

(b) Let $x \in A^*_{qrs}(I_1 \cap I_2)$, then for every quasi regular semiopen set $U$ containing $x$, $A \cap U \notin (I_1 \cap I_2)$, hence $A \cap U \notin I_1$ and $A \cap U \notin I_2$. This shows, $x \in A^*_{qrs}(I_1)$ or $x \in A^*_{qrs}(I_2)$ that is $x \in A^*_{qrs}(I_1) \cup A^*_{qrs}(I_2)$. Thus, $A^*_{qrs}(I_1 \cap I_2) \subseteq A^*_{qrs}(I_1) \cup A^*_{qrs}(I_2)$. But, $A^*_{qrs}(I_1) \cup A^*_{qrs}(I_2) \subseteq A^*_{qrs}(I_1 \cap I_2)$. Therefore, $A^*_{qrs}(I_1 \cap I_2) = A^*_{qrs}(I_1) \cup A^*_{qrs}(I_2)$.

■

**Definition 7.2.5** In an ideal bitopological space $(X, \tau_1, \tau_2, I)$ the **quasi $^*$-regular semiclosure** $A$ of $X$ denoted by $qrscl^*(A)$ is defined by $qrscl^*(A) = A \cup A^*_{qrs}$.

**Theorem 7.2.6** Let $(X, \tau_1, \tau_2, I)$ be an ideal bitopological space and $A, B$ be the subsets of $X$. Then:

(a) $A \subseteq qrscl^*(A)$.

(b) $qrscl^*(\emptyset) = \emptyset$ and $qrscl^*(X) = X$.

(c) If $A \subseteq B$, then $qrscl^*(A) \subseteq qrscl^*(B)$.

(d) $qrscl^*(A) \cup qrscl^*(B) \subseteq qrscl^*(A \cup B)$.

(e) If $I = \emptyset$, then $qrscl^*(A) = qrscl(A)$.

**Proof.** Follows from Definition 7.2.5. ■
**Definition 7.2.7** A subset $A$ of an ideal bitopological space $(X, \tau_1, \tau_2, I)$ is said to be:

(a) *qrs*-$I$-open if $A \subseteq qrsint(A^*_qrs)$.

(b) *qrs*-$I$-closed if its complement is *qrs*-$I$-open.

The family of all *qrs*-$I$-open (respectively *qrs*-$I$-closed) sets of an ideal bitopological space $(X, \tau_1, \tau_2, I)$ is denoted by $QRSIO(X)$ (respectively $QRSIC(X)$).

The family of all *qrs*-$I$-open sets of $(X, \tau_1, \tau_2, I)$ containing a point $x$ is denoted by $QRSIO(X, x)$.

**Remark 7.2.8** Every *qI*-open set is *qrs*-$I$-open but the converse is not true as shown in the following example.

**Example 7.2.9** Let $X = \{a, b, c\}$ and $\tau_1 = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$, $\tau_2 = \{X, \emptyset, \{c\}, \{a\}\}$, be the topologies on $X$ and $I = \{\emptyset, \{a\}\}$ be an ideal on $X$. Then the set $A = \{b, c\}$ is *qrs*-$I$-open but not *qI*-open in $(X, \tau_1, \tau_2, I)$.

**Remark 7.2.10** Every *qrs*-$I$-open set is quasi regular semiopen but the converse is not true as shown in the following example.

**Example 7.2.11** In Example 7.2.9, the set $A = \{a, c\}$ is quasi regular semiopen but not *qrs*-$I$-open in $(X, \tau_1, \tau_2, I)$.

**Remark 7.2.12** For an ideal bitopological space $(X, \tau_1, \tau_2, I)$ we have the following:

(a) $X$ need not be a *qrs*-$I$-open set.
(b) If $I = \leftrightarrow (X)$, then only the empty set is $qrsI$-open.

(c) If $I = \phi$, $qrsI$-openness and quasi regular semiopenness are equivalent.

**Theorem 7.2.13** If $A$ is $qrsI$-open, then $A^*_qrs = (qrsint(A^*_qrs))^{**}_{qrs}$.

**Proof.** Since $A$ is $qrsI$-open, $A \subseteq qrsint(A^*_qrs)$. Therefore, $A^*_qrs \subseteq (qrsint(A^*_qrs))^{**}_{qrs}$. Also we have $qrsint(A^*_qrs) \subseteq A^*_qrs((qrsint(A^*_qrs))^{**}_{qrs}) \subseteq (A^*_qrs)^* \subseteq A^*_qrs$. Hence, $A^*_qrs = (qrsint(A^*_qrs))^{**}_{qrs}$.  

**Theorem 7.2.14** Any union of a family of $qrsI$-open sets in an ideal bitopological space $(X, \tau_1, \tau_2, I)$ is $qrsI$-open in $X$.

**Proof.** Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of $qrsI$-open sets of an ideal bitopological space $(X, \tau_1, \tau_2, I)$. Then $U_\alpha \subseteq qrsint((U_\alpha)^*_{qrs}), \forall \alpha \in \Delta$. It follows that $\cup_{\alpha \in \Delta}U_\alpha \subseteq \cup_{\alpha \in \Delta}(qrsint((U_\alpha)^*_{qrs})) \subseteq qrsint(\cup_{\alpha \in \Delta}(U_\alpha)) \subseteq qrsint(\cup_{\alpha \in \Delta}(U_\alpha)^*_{qrs})$. Hence $\cup_{\alpha \in \Delta}U_\alpha$ is $qrsI$-open set in $X$.

**Definition 7.2.15** Let $A$ be a subset of an ideal bitopological space $(X, \tau_1, \tau_2, I)$ and $x \in X$. Then:

(a) $x$ is called a $qrsI$-**interior point** of $A$ if $\exists V \in QRSIO(X)$ such that $x \in V \subseteq A$.

(b) The set of all $qrsI$-interior points of $A$ denoted by $qrsI\text{int}(A)$ is called the $qrsI$-**interior** of $A$.

The following theorem summarizes the properties of $qrsI$-interior of subsets in ideal bitopological spaces.
Theorem 7.2.16  Let $A, B$ be subsets of an ideal bitopological space $(X, \tau_1, \tau_2, I)$. Then:

(a) $qrsInt(A) = \cup \{T : T \subseteq A \text{ and } A \in QRSIO(X)\}$.

(b) $qrsInt(A)$ is the largest $qrsI$-open subset of $X$ contained in $A$.

(c) $A$ is $qrsI$-open if and only if $A = qrsInt(A)$.

(d) $qrsInt(qrsInt(A)) = qrsInt(A)$.

(e) If $A \subseteq B$, then $qrsInt(A) \subseteq qrsInt(B)$.

(f) $qrsInt(A) \cup qrsInt(B) \subseteq qrsInt(A \cup B)$.

(g) $qrsInt(A \cap B) \subseteq qrsInt(A) \cap qrsInt(B)$.

Proof. (a) Let $x \in \cup \{T : T \subseteq A \text{ and } A \in QRSIO(X)\}$. Then, there exists $T \in QRSIO(X, x)$ such that $x \in T \subseteq A$ and hence $x \in qrsInt(A)$. This shows that $\cup \{T : T \subseteq A \text{ and } A \in QRSIO(X)\} \subseteq qrsInt(A)$. For the reverse inclusion, let $x \in qrsInt(A)$, then there exists $T \in QRSIO(X, x)$, such that $x \in T \subseteq A$ and we obtain $x \in \cup \{T : T \subseteq A \text{ and } A \in QRSIO(X)\}$. This shows that $qrsInt(A) \subseteq \cup \{T : T \subseteq A \text{ and } A \in QRSIO(X)\}$. Therefore $\cup \{T : T \subseteq A \text{ and } A \in QRSIO(X)\} = qrsInt(A)$.

The proof of properties (b)-(e) are obvious.

(f) Clearly $qrsInt(A) \subseteq qrsInt(A \cup B)$ and $qrsInt(B) \subseteq qrsInt(A \cup B)$. Thus $qrsInt(A) \cup qrsInt(B) \subseteq qrsInt(A \cup B)$.

(g) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by (e) we have $qrsInt(A \cap B) \subseteq qrsInt(A)$ and $qrsInt(A \cap B) \subseteq qrsInt(B)$. Then $qrsInt(A \cap B) \subseteq qrsInt(A) \cap qrsInt(B)$.

Definition 7.2.17  Let $A$ be a subset of an ideal bitopological space $(X, \tau_1, \tau_2, I)$
and \( x \in X \). Then:

(a) \( X \) is called a \textit{qrsI-cluster point} of \( A \), if \( V \cap A \neq \emptyset \), for every \( V \in QRSI(X, x) \).

(b) The set of all \textit{qrsI-cluster points} of \( A \) denoted by \( qrsIcl(A) \) is called the \textit{qrsI-closure} of \( A \).

The following theorem summarizes the properties of \textit{qrsI-closure} of subsets in ideal bitopological spaces.

\textbf{Theorem 7.2.18} Let \( A \) and \( B \) be subsets of an ideal bitopological space \((X, \tau_1, \tau_2, I)\).

Then:

(a) \( qrsIcl(A) = \cap\{ F : A \subseteq F \text{ and } F \in QRSIC(X) \} \).

(b) \( qrsIcl(A) \) is the smallest \textit{qrsI-closed} subset of \( X \) containing \( A \).

(c) \( A \) is \textit{qrsI-closed} if and only if \( A = qrsIcl(A) \).

(d) \( qrsIcl(qrsInt(A)) = qrsIcl(A) \).

(e) If \( A \subseteq B \), then \( qrsIcl(A) \subseteq qrsIcl(B) \).

(f) \( qrsIcl(A) \cup qrsIcl(B) = qrsIcl(A \cup B) \).

(g) \( qrsIcl(A \cap B) \subseteq qrsIcl(A) \cap qrsIcl(B) \).

\textbf{Proof.} (a) Suppose \( x \notin qrsIcl(A) \). Then, there exists \( F \in QRSIO(X) \) such that \( F \cap A = \emptyset \). Since \( X \setminus F \) is \textit{qI-closed} set containing \( A \) and \( x \notin X \setminus F \), we obtain \( x \notin \cap\{ F : A \subseteq F \text{ and } F \in QRSIC(X) \} \). Then there exists \( F \in QRSI(X) \) such that \( A \subseteq F \) and \( x \notin F \). Since \( X \setminus F \) is \textit{qrsI-closed} set containing \( x \), we get \( (X \setminus F) \cap A = \emptyset \). This shows that \( x \notin qrsIcl(A) \). Therefore \( qrsIcl(A) = \cap\{ F : A \subseteq F \text{ and } F \in QRSIC(X) \} \).

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Statements (b)-(g) have obvious proofs.

**Theorem 7.2.19**  Let \((X, \tau_1, \tau_2, I)\) be an ideal bitopological space and \(A \subseteq X\). Then the following properties hold:

(a) \(qrsIcl(X \setminus A) = X \setminus qrsInt(A)\).

(b) \(qrsInt(X \setminus A) = X \setminus qrsIcl(A)\).

**Proof.** (a) Let \(W\) be a subset of \(X\). Since \(W \subseteq A\) if and only if \(X \setminus A \subseteq X \setminus W\), \(W\) is \(qrsI\) -open if and only if \(X \setminus W\) is \(qrsI\) -closed. Thus, 
\[
qrsIcl(X \setminus A) = \cap \{X \setminus W : W \subseteq A \text{ and } W \in QRSIO(X)\} = X \setminus \cup \{W \subseteq A \text{ and } W \in QRSIO(X)\} = X \setminus qrsInt(A).
\]

(b) Follows from (a).

**Definition 7.2.20**  A subset \(B_x\) of an ideal bitopological space \((X, \tau_1, \tau_2, I)\) is said to be a \(qrsI\) -neighbourhood of a point \(x \in X\) if there exists a \(qrsI\) -open set \(U\) of \(X\) such that \(x \in U \subseteq B_x\).

**Theorem 7.2.21**  A subset of an ideal bitopological space \((X, \tau_1, \tau_2, I)\) is \(qrsI\) -open if and only if it is a \(qrsI\) -neighbourhood of each of its points.

**Proof.** Necessary: Let \(G\) be a \(qrsI\) -open set of \(X\). Then by definition, it is clear that \(G\) is a \(qrsI\) -neighbourhood of each of its points, since \(\forall \; x \in G, x \in G \subseteq G\) and \(G\) is \(qrsI\) -open.

Sufficient: Suppose \(G\) is a \(qrsI\) -neighbourhood of each of its points. Then for each \(x \in G\) there exists \(S_x \in QRSIO(X)\) such that \(S_x \subseteq G\). Therefore \(G = \cup \{S_x : x \in G\}\). Since each \(S_x\) is \(qrsI\) -open and arbitrary union of \(qrsI\) -open sets is \(qrsI\) -open, \(G\) is \(qrsI\) -open in \((X, \tau_1, \tau_2, I)\).
7.3 qrsI-Continuous Mappings

In this section, a new class of maps called qrsI-continuous mappings, qrsI-open maps and qrsI-closed maps in ideal bitopological spaces are introduced and investigated. During this process, some of their properties are obtained. It is found that every qI-continuous mapping is qrsI-continuous but not conversely.

Definition 7.3.1 A mapping $f : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2)$ is called a qrsI-continuous if $f^{-1}(V)$ is a qrsI-open set in $X$ for every quasi open set $V$ of $Y$.

Remark 7.3.2 Every qI-continuous mapping is qrsI-continuous but the converse is not true as shown in the following example.

Example 7.3.3 Let $X = \{a, b, c\}$ and $I = \{\phi, \{b\}\}$ be an ideal on $X$. Let $\tau_1 = \{X, \phi, \{a\}, \{c\}, \{a, c\}\}$, $\tau_2 = \{X, \phi, \{c\}, \{a, b\}\}$, $\sigma_1 = \{X, \phi, \{c\}\}$ and $\sigma_2 = \{\phi, X, \{b, c\}\}$ be topologies on $X$. Then the identity mapping $f : (X, \tau_1, \tau_2, I) \rightarrow (X, \sigma_1, \sigma_2)$ is qrsI-continuous but not qI-continuous.

Theorem 7.3.4 Let $f : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2)$ be a mapping. Then the following statements are equivalent:

(a) $f$ is qrsI-continuous.

(b) $f^{-1}(V)$ is qrsI-closed in $X$ for every quasi closed set $V$ of $Y$.

(c) for each $x \in X$ and every quasi open set $V$ of $Y$ containing $f(x)$, $\exists$ $W \in QRSIO(X, x)$ such that $f(W) \subseteq V$. 

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(d) for each $x \in X$ and every quasi open set $V$ of $Y$ containing $f(x)$, $f^{-1}$ is a $qrsI$-neighbourhood of $x$.

**Proof.** (a) $\Leftrightarrow$ (b): Obvious from the Definition 7.3.1.

(a) $\Rightarrow$ (c): Let $x \in X$ and $V$ be a quasi open set of $Y$ containing $f(x)$. Since $f$ is $qrsI$-continuous, $f^{-1}(V)$ is a $qrsI$-open set. Putting $W = f^{-1}(V)$, we get $f(W) \subseteq V$.

(c) $\Rightarrow$ (a): Let $A$ be a quasi open set in $Y$. If $f^{-1}(A) = \phi$, then $f^{-1}(A)$ is clearly a $qrsI$-open set. Assume that $f^{-1}(A) \neq \phi$ and $x \in f^{-1}(A)$, then $f(x) \in A \Rightarrow \exists$ a $qrsI$-open set $W$ containing $x$ such that $f(W) \subseteq A$. Thus $W$ is $f^{-1}(A)$. Since $W$ is $qrsI$-open, $x \in W \subseteq qrsint((W)^*_qrs) \subseteq qrsint(f^{-1}(A)^*_qrs)$ and so $f^{-1}(A) \subseteq qrsint(f^{-1}(A)^*_qrs)$. Hence $f^{-1}(A)$ is a $qrsI$-open set and so is $qrsI$-continuous.

(c) $\Rightarrow$ (d): Let $x \in X$ and $V$ be a quasi open set of $Y$ containing $f(x)$ then $\exists$ a $qrsI$-open set $W$ containing $x$ such that $f(W) \subseteq V$. It follows that $W \subseteq f^{-1}(f(W)^*_qrs) \subseteq f^{-1}(V)$. Since $W$ is a $qrsI$-open set, $x \in W \subseteq qrsint((W)^*_qrs) \subseteq qrsint(f^{-1}(V)^*_qrs) \subseteq f^{-1}((V)^*_qrs)$. Hence $f^{-1}((V)^*_qrs)$ is a $qrsI$-neighbourhood of $x$.

(d) $\Rightarrow$ (c): Obvious.

**Definition 7.3.5** A mapping $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2, I)$ is said to be:

(a) **$qrsI$-open** if $f(U)$ is a $qrsI$-open set of $Y$ for every quasi open set $U$ of $X$. 

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(b) **qrsI-closed** if $f(U)$ is a qrsI-closed set of $Y$ for every quasi closed set $U$ of $X$.

**Theorem 7.3.6** Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2, I)$ be a mapping. Then the following statements are equivalent:

(a) $f$ is qrsI-open.

(b) $f(\text{qint}(U)) \subseteq \text{qrsInt}(f(U))$ for each subset $U$ of $X$.

(c) $\text{qint}(f^{-1}(V)) \subseteq f^{-1}(\text{qrsInt}(V))$ for each subset $V$ of $Y$.

**Proof.** (a) $\Rightarrow$ (b): Let $U$ be any subset of $X$. Then $\text{qint}(U)$ is a quasi open set of $X$. Then $f(\text{qint}(U))$ is a qrsI-open set of $Y$. Since $f(\text{qint}(U)) \subseteq f(U)$, $f(\text{qint}(U)) = \text{qrsInt}(f(\text{qint}(U))) \subseteq \text{qrsInt}(f(U))$.

(b) $\Rightarrow$ (c): Let $V$ be any subset of $Y$. Obviously $f^{-1}(V)$ is a subset of $X$. Therefore by (b), $f(\text{qint}(f^{-1}(V))) \subseteq \text{qrsInt}(f(f^{-1}(V))) \subseteq \text{qrsInt}(V)$. Hence, $\text{qint}(f^{-1}(V)) \subseteq f^{-1}(f(\text{qint}(f^{-1}(V)))) \subseteq f^{-1}(\text{qrsInt}(V))$.

(c) $\Rightarrow$ (a): Let $V$ be any quasi open set of $X$. Then $\text{qint}(V) = V$ and $f(V)$ is a subset of $Y$. So $V = \text{qint}(V) \subseteq \text{qint}(f^{-1}(f(V))) \subseteq f^{-1}(\text{qrsInt}(f(V)))$. Then $f(V) \subseteq f(f^{-1}(\text{qrsInt}(f(V)))) \subseteq \text{qrsInt}(f(V))$ and $\text{qrsInt}(f(V)) \subseteq f(V)$. Hence, $f(V)$ is a qrsI-open set of $Y$ and $f$ is qrsI-open.

**Theorem 7.3.7** Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2, I)$ be a qrsI-open mapping. If $V$ is a subset of $Y$ and $U$ is a quasi closed subset of $X$ containing $f^{-1}(V)$, then there exists a qrsI-closed set $F$ of $Y$ containing $V$ such that $f^{-1}(F) \subseteq U$.

**Proof.** Let $V$ be any subset of $Y$ and $U$ a quasi closed subset of $X$ containing $f^{-1}(V)$ and let $F = Y \setminus (f(X \setminus U))$. Then $f(X \setminus U) \subseteq f(f^{-1}(X \setminus U)) \subseteq (X \setminus U)$ and
$X \setminus U$ is a quasi open set of $X$. Since $f$ is $qrsI$-open, $f(X \setminus U)$ is a $qrsI$-open set of $Y$. Hence $F$ is a quasi closed subset of $Y$ and $f^{-1}(F) = f^{-1}(Y \setminus (f(X \setminus U))) \subseteq U$. ■

**Theorem 7.3.8** A mapping $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, I)$ is $qrsI$-closed if and only if $qrsIcl(f(V)) \subseteq f(qcl(V))$ for each subset $V$ of $X$.

**Proof.** Necessary: Let $f$ be a $qrsI$-closed mapping and $V$ be any subset of $X$. Then $f(V) \subseteq f(qcl(V))$ and $f(qcl(V))$ is a $qrsI$-closed set of $Y$. Thus $qrsIcl(f(V)) \subseteq qrsIcl(f(qcl(V))) = f(qcl(V))$.

Sufficient: Let $V$ be a quasi closed set of $X$. Then by hypothesis $f(V) \subseteq qrsIcl(f(V)) \subseteq f(qrsIcl(V)) = f(V)$. And so, $f(V)$ is a $qrsI$-closed subset of $Y$. Hence, $f$ is $qrsI$-closed. ■

**Theorem 7.3.9** A mapping $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, I)$ is $qrsI$-closed if and only if $f^{-1}(qrsIcl(V)) \subseteq qcl(f^{-1}(V))$ for each subset $V$ of $Y$.

**Proof.** Follows from the Definition 7.3.5. ■

**Theorem 7.3.10** Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, I)$ be a $qrsI$-closed mapping. If $V$ is a subset of $Y$ and $U$ is a quasi open subset of $X$ containing $f^{-1}(V)$, then there exists a $qrsI$-open set $F$ of $Y$ containing $V$ such that $f^{-1}(F) \subseteq U$.

**Proof.** Suppose $f$ is $qrsI$-closed map. Let $V \subseteq Y$ and $U$ is a quasi open set in $X$ such that $f^{-1}(V) \subseteq U$. Now $X - U$ is a quasi closed set in $X$. Since $f$ is $qrsI$-closed map, $f(X - U)$ is $qrsI$-closed set in $Y$. Then $F = Y - f(X - U)$ is a $qrsI$-open set in $Y$. Note that $f^{-1}(V) \subseteq U$ implies $V \subseteq F$ and $f^{-1}(F) = X - f^{-1}(f(X - U)) \subseteq X - (X - U) = U$. That is $f^{-1}(F) \subseteq U$. ■