Chapter 6

$(\tau_1, \tau_2) - I_{rw}$ Closed Sets in Ideal Bitopological Spaces

6.1 Introduction

The triple $(X, \tau_1, \tau_2)$, where $X$ is a set, $\tau_1$ and $\tau_2$ are topologies on $X$ is called a bitopological space. Kelly [34] initiated the systematic study of such spaces in 1963. He generalized the topological concepts to bitopological study and published a large number of papers.

R. S. Wali [67] introduced the concepts of regular weakly closed sets and regular weakly open sets in bitopological spaces. Recently many authors are introducing the topological concepts with respect to an ideal $I$. For instance, Palaniappan and Alagar [55] introduced regular generalized closed sets and regular generalized locally closed sets with respect to an ideal. Alagar and Thenmozhi [3] introduced regular generalized star closed sets with respect to an ideal. T. Noiri and N. Rajesh [53] introduced $(i, j) - I_g$ closed sets in bitopological spaces.
In chapter 2, we have introduced and studied the concepts of \( I_{rw} \)-closed sets and \( I_{rw} \)-open sets in ideal topological spaces.

In section 2 of this chapter, \((i, j)\)-\( I_{rw} \)-closed sets in an ideal bitopological space have been introduced and studied. Among many other results it is observed that every \((i, j)\)-\( rw \)-closed set is \((i, j)\)-\( I_{rw} \)-closed set but not conversely.

In section 3 of this chapter, we have introduced \((i, j)\)-\( I_{rw} \)-open sets in an ideal bitopological space and studied some of their properties.

Throughout this chapter \((X, \tau_1, \tau_2, I)\) or simply \(X\) denote nonempty ideal bitopological spaces on which no separation axioms are assumed, unless otherwise mentioned and fixed integers \(i, j \in \{1, 2\}\).

### 6.2 \((i, j)\)-\( I_{rw} \) Closed Sets and Their Basic Properties

In this section, we introduce and investigate the concept of \((i, j)\)-\( I_{rw} \)-closed sets which are introduced in ideal bitopological spaces is analogy with \( I_{rw} \)-closed sets in ideal bitopological spaces. From now on, \(\tau \cdot cl(A)\) denotes the closure of \(A\) relative to a topology \(\tau\).

**Definition 6.2.1** A subset \(A\) of an ideal bitopological space \((X, \tau_1, \tau_2, I)\) is called a \((i, j)\)-regular weakly closed set with respect to an ideal \(I\) (\((i, j)\)-\( I_{rw} \) closed) in \(X\) if and only if \(\tau_j \cdot cl(A) \cap U \equiv I\) whenever \(A \subseteq U\) and \(U\) is \(\tau_i\)-regular semiopen in \(X\), \(i, j = 1, 2\) and \(i \neq j\).

**Example 6.2.2** Let \(X = \{a, b, c, d\}\), \(\tau_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}\), \(\tau_2 = \{\phi, \{a\}\),

95
\{b, c\}, \{a, b, c\}, X\}, \ I = \{\phi, \{a\}, \{d\}, \{a, d\}\}. \ Then \ \phi, X, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \\
\{b, c\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\ are \ (1, 2) - I_{rw} \ closed \ sets \ in \ (X, \tau_1, \tau_2, I).

**Theorem 6.2.3** Let \((X, \tau_1, \tau_2, I)\) be an ideal bitopological space. Every \((i, j)\)-rw closed set is \((i, j) - I_{rw}\) closed in \(X\), \(i, j = 1, 2\) and \(i \neq j\).

**Proof.** Let \(A\) be a \((i, j)\)-rw closed subset of \((X, \tau_1, \tau_2, I)\). Let \(A \subseteq U\) and \(U\) is \(\tau_i\)-regular semiopen in \(X\), \(i, j = 1, 2\) and \(i \neq j\). Then \(\tau_j - cl(A) \subseteq U\). Hence \(\tau_j - cl(A) - U = \phi \in I\). Therefore, \(A\) is \((i, j) - I_{rw}\) closed. \qed

**Remark 6.2.4** The converse of the above Theorem 6.2.3 is not true in general as can be seen from the following example.

**Example 6.2.5** In Example 6.2.2, \(\{a\}\) is \((1, 2) - I_{rw}\) closed, but not \((1, 2) - rw\) closed in \((X, \tau_1, \tau_2, I)\).

**Remark 6.2.6** \((1, 2) - I_{rw}\) closed sets and \((1, 2) - I_{ry}\) closed sets, \((1, 2) - I_g\) closed sets are independent in general as can be seen from the following example.

**Example 6.2.7** In Example 6.2.2, \((1, 2) - I_{rw}\) closed sets are \(\phi, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\), \((1, 2) - I_g\) closed sets are \(P(x) - \{\{b\}, \{d\}\}, \{a, b\}, \{a, d\}\) and \((1, 2) - I_{rg*}\) closed sets are \(P(x) - \{\{b\}, \{d\}, \{a, d\}\}. \) Clearly these sets are independent.

**Theorem 6.2.8** Let \(A\) be a subset of an ideal bitopological space \((X, \tau_1, \tau_2, I)\). If \(A\) is \((i, j) - I_{rw}\) closed then \(\tau_j - cl(A) - A\) does not contain \(\tau_i\)-regular semiclosed sets such that \(F \notin I\), \(i, j = 1, 2\) and \(i \neq j\).
Proof. Suppose that \( A \) is \((i,j)\)-\(I_{rw}\) closed, \( i, j = 1, 2 \) and \( i \neq j \). Let \( F \) be an \( \tau_i \)-regular semiclosed set such that \( F \subseteq \tau_j \cdot cl(A) - A \). Since \( F \subseteq \tau_j \cdot cl(A) - A \), we have \( F \subseteq [\tau_j \cdot cl(A)] \cap A^C \). Consequently \( F \subseteq A^C \) and \( F \subseteq \tau_j \cdot cl(A) \). Since \( F \subseteq A^C \), we have \( A \subseteq F^C \). Since \( F \) is \( \tau_i \)-regular semiclosed sets, we have \( F^C \) is \( \tau_i \)-regular semiopen. Since \( A \) is \((i,j)\)-\(I_{rw}\) closed, we have \( \tau_j \cdot cl(A) - F^C = \tau_j \cdot cl(A) \cap F = F \in I \). Thus, \( \tau_j \cdot cl(A) - A \) does not contain \( \tau_i \)-regular semiclosed sets such that \( F \notin I \). 

**Theorem 6.2.9** If \( A \) and \( B \) are \((i,j)\)-\(I_{rw}\) closed sets then \( A \cup B \) is \((i,j)\)-\(I_{rw}\) closed, \( i, j = 1, 2 \) and \( i \neq j \).

Proof. Suppose that \( A \) and \( B \) are \((i,j)\)-\(I_{rw}\) closed sets, \( i, j = 1, 2 \) and \( i \neq j \). We shall show that \( A \cup B \) is \((i,j)\)-\(I_{rw}\) closed. Let \( A \cup B \subseteq U \) and \( U \) is \( \tau_i \)-regular semiopen. Since \( A \cup B \subseteq U \), we have \( A \subseteq U \) and \( B \subseteq U \). Since \( A \subseteq U \) and \( U \) is \( \tau_i \)-regular semiopen, we have \( \tau_j \cdot cl(A) - U \in I \) \{since \( A \) is \((i,j)\)-\(I_{rw}\) closed\}. Since \( B \subseteq U \) and \( U \) is \( \tau_i \)-regular semiopen, we have \( \tau_j \cdot cl(B) - U \in I \) \{since \( B \) is \((i,j)\)-\(I_{rw}\) closed\}. Therefore, \( \tau_j \cdot cl(A \cup B) - U = \{\tau_j \cdot cl(A) - U\} \cup \{\tau_j \cdot cl(B) - U\} \in I \). Hence \( A \cup B \) is \((i,j)\)-\(I_{rw}\) closed.

**Remark 6.2.10** The intersection of two \((i,j)\)-\(I_{rw}\) closed sets is not an \((i,j)\)-\(I_{rw}\) closed set in general as can be seen from the following example.

**Example 6.2.11** In Example 6.2.2, \( A = \{a, b\} \), \( B = \{b, c\} \) are \((1,2)\)-\(I_{rw}\) closed sets, but \( A \cap B = \{b\} \) is not an \((1,2)\)-\(I_{rw}\) closed set in \( X \).

**Lemma 6.2.12** Let \( A \) be an \( \tau_i \)-open set in \((X,\tau_1,\tau_2)\) and let \( U \) be \( \tau_i \)-regular
semiopen in $A$. Then $U = A \cap W$ for some $\tau_i$-regular semiopen set $W$ in $X$, $i, j = 1, 2$ and $i \neq j$.

**Lemma 6.2.13** If $A$ is $\tau_i \tau_j$-open and $U$ is $\tau_i$-regular semiopen in $X$ then $U \cap A$ is $\tau_i$-regular semiopen in $A$, $i, j = 1, 2$ and $i \neq j$.

**Lemma 6.2.14** If $A$ is $\tau_i \tau_j$-open in $(X, \tau_1, \tau_2)$, then $\tau_j \cdot cl_A(B) \subseteq A \cap \tau_j \cdot cl(B)$ for any subset $B$ of $A$, $i, j = 1, 2$ and $i \neq j$.

**Theorem 6.2.15** Let $I$ be an ideal in $X$. Let $B \subseteq A$ where $A$ is $\tau_i$-regular semiopen, $\tau_j$-regular semiopen and $(i, j) \cdot I_{rw}$ closed. Then $B$ is $(i, j) \cdot I_{rw}$ closed relative to $A$ with respect to an ideal $I_A = \{ F \subseteq A | F \in I \}$ if $B$ is $(i, j) \cdot I_{rw}$ closed in $X$, $i, j = 1, 2$ and $i \neq j$.

**Proof.** Suppose that $B$ is $(i, j) \cdot I_{rw}$ closed in $X$, $i, j = 1, 2$ and $i \neq j$. We shall show that $B$ is $(i, j) \cdot I_{rw}$ closed relative to $A$. Let $B \subseteq U$ and $U$ is $\tau_i$-regular semiopen in $A$. Since $A$ is $\tau_i$-open in $X$ and $U$ is $\tau_i$-regular semiopen in $A$, we have $U = A \cap W$ for some $\tau_i$-regular semiopen set $W$ in $X$ { By Lemma 6.2.12}. Since $A$ is $\tau_i \tau_j$-open in $X$ and $W$ is $\tau_i$-regular semiopen in $X$, we have $U = A \cap W$ is $\tau_i$-regular semiopen set in $X$ {by Lemma 6.2.13}. Hence $B \subseteq U$ and $U$ is $\tau_i$-regular semiopen set in $X$. Since $B$ is $(i, j) \cdot I_{rw}$ closed in $X$, we have $\tau_j \cdot cl(B) \subseteq U \in I$. Therefore, $\tau_j \cdot cl(B) \cup U^C \subseteq I$. Consequently, $\tau_j \cdot cl(B) \cap A \cap U^C \subseteq I_A$. Since $A$ is $\tau_i \tau_j$-open in $X$, we have $\tau_j \cdot cl(B) \cap A = \tau_j \cdot cl_A(B)$ { by Lemma 6.2.14 }. Hence $\tau_j \cdot cl_A(B) \subseteq U \in I_A$. Therefore, $B$ is $(i, j) \cdot I_{rw}$ closed relative to $A$. 

**Theorem 6.2.16** Let $A$ and $B$ be subsets such that $A \subseteq B \subseteq \tau_j \cdot cl(A)$. If $A$ is $(i, j) \cdot I_{rw}$ closed, then $B$ is $(i, j) \cdot I_{rw}$ closed, $i, j = 1, 2$ and $i \neq j$. 

98
Proof. Let $A$ and $B$ be subsets such that $A \subseteq B \subseteq \tau_j - \text{cl}(A)$. Suppose that $A$ is $(i,j) - I_{rw}$ closed, $i, j = 1, 2$ and $i \neq j$. Let $B \subseteq U$ and $U$ is $\tau_i$-regular semiopen in $X$. Since $A \subseteq B$ and $B \subseteq U$, we have $A \subseteq U$. Hence $A \subseteq U$ and $U$ is $\tau_i$-regular semiopen in $X$. Since $A$ is $(i,j) - I_{rw}$ closed, we have $\tau_j - \text{cl}(A) - U \in I$. Since $B \subseteq \tau_j - \text{cl}(A)$, we have $\tau_j - \text{cl}(B) \subseteq \tau_j - \text{cl}(A)$. Hence $\tau_j - \text{cl}(B) - U \subseteq \tau_j - \text{cl}(A) - U \in I$. Therefore, $B$ is $(i,j) - I_{rw}$ closed.

Theorem 6.2.17 Suppose that $\tau_j - \text{RSO}(X, \tau_1, \tau_2) \subseteq \tau_j - \text{RSC}(X, \tau_1, \tau_2)$, then every subset of $X$ is $(i,j) - I_{rw}$-closed, $i, j = 1, 2$ and $i \neq j$.

Proof. Suppose that $\tau_j - \text{RSO}(X, \tau_1, \tau_2) \subseteq \tau_j - \text{RSC}(X, \tau_1, \tau_2)$, $i, j = 1, 2$ and $i \neq j$. Let $A$ be a subset of $X$. Let $A \subseteq U$ and $U$ is $\tau_i$-regular semiopen in $X$. Since $\tau_j - \text{RSO}(X, \tau_1, \tau_2) \subseteq \tau_j - \text{RSC}(X, \tau_1, \tau_2)$, we have $U$ is $\tau_j$-regular closed in $X$. Then $\tau_j - \text{cl}(U) = U$. Since $A \subseteq U$, we have $\tau_j - \text{cl}(A) \subseteq \tau_j - \text{cl}(U) = U$. Therefore, $\tau_j - \text{cl}(A) \subseteq U$. Consequently, $\tau_j - \text{cl}(A) - U = \phi \in I$. Hence $A$ is $(i,j) - I_{rw}$-closed.

6.3 $(i,j) - I_{rw}$ Open Sets and Their Basic Properties

In this section, we introduce $(i,j) - I_{rw}$ open sets in ideal bitopological spaces and study some of their properties.

Definition 6.3.1 A subset $A$ of an ideal bitopological space $(X, \tau_1, \tau_2, I)$ is called $(i,j)$-regular weakly open with respect to an ideal $I$ ($ (i,j) - I_{rw}$ open) in $X$ if and only if its complement is $(i,j)$-regular weakly closed with respect to an
ideal \((i, j) - I_{rw}\) closed) in \(X\), \(i, j = 1, 2\) and \(i \neq j\).

**Example 6.3.2** In Example 6.2.2, \(\phi, X, \{a\}, \{c\}, \{d\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}\) are \((1, 2) - I_{rw}\) open sets in \((X, \tau_1, \tau_2, I)\).

**Theorem 6.3.3** A subset \(A\) of an ideal bitopological space \((X, \tau_1, \tau_2, I)\) is \((i, j) - I_{rw}\) open if and only if \(F - \tau_j - \text{int}(A) \in I\) whenever \(F \subseteq A\) and \(F\) is \(\tau_i\)-regular semiclosed in \(X\), \(i, j = 1, 2\) and \(i \neq j\).

**Proof.** Suppose that \(A\) is \((i, j) - I_{rw}\) open, \(i, j = 1, 2\) and \(i \neq j\). We shall show that \(F - \tau_j - \text{int}(A) \in I\) whenever \(F \subseteq A\) and \(F\) is \(\tau_i\)-regular semiclosed in \(X\). Let \(A \subseteq F\) and \(F\) is \(\tau_i\)-regular semiclosed in \(X\). Then \(A^C \subseteq F^C\) and \(F^C\) is \(\tau_i\)-regular semiopen in \(X\). Since \(A\) is \((i, j) - I_{rw}\) open, we have \(A^C\) is \((i, j) - I_{rw}\) closed. Hence \(\tau_j - \text{cl}(A^C) = F^C \in I\). Consequently, \([\tau_j - \text{int}(A)]^C \cap F = F - \tau_j - \text{int}(A) \in I\).

Conversely, suppose that \(F - \tau_j - \text{int}(A) \in I\) whenever \(F \subseteq A\) and \(F\) is \(\tau_i\)-regular semiclosed in \(X\). Let \(A^C \subseteq U\) and \(U\) is \(\tau_i\)-regular semiopen in \(X\). Then, \(U^C \subseteq A\) and \(U^C\) is \(\tau_i\)-regular semiclosed in \(X\). By our assumption, we have \(U^C - \tau_j - \text{int}(A) \in I\). Hence \([\tau_i - \text{int}(A)]^C - U = \tau_j - \text{cl}(A^C) - U \in I\). Consequently, \(A^C\) is \((i, j) - I_{rw}\)-closed. Hence \(A\) is \((i, j) - I_{rw}\) open. \(\blacksquare\)

**Remark 6.3.4** \((1, 2) - I_{rw}\) open sets and \((1, 2) - I_{rg}\) open sets, \((1, 2) - I_g\) open sets are independent in general as can be seen from the following example.

**Example 6.3.5** In Example 6.2.2, \((1, 2) - I_{rw}\) open sets are \(\phi, \{a\}, \{c\}, \{d\}, \{a, d\}, \{c, d\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\), \((1, 2) - I_g\) open sets are \(P(x) - \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}\) and \((1, 2) - I_{rg}\) open sets are \(P(x) - \{b, c\}, \{a, c, d\}, \{a, b, c\}\). Clearly these sets are independent.
Theorem 6.3.6  Let $A$ and $B$ be subsets such that $\tau_j \cdot \text{int}(A) \subseteq B \subseteq A$. If $A$ is $(i,j) \cdot I_{rw}$ open, then $B$ is $(i,j) \cdot I_{rw}$ open, $i,j = 1,2$ and $i \neq j$.

Proof. Suppose that $A$ and $B$ are subsets such that $\tau_j \cdot \text{int}(A) \subseteq B \subseteq A$. Let $A$ be $(i,j) \cdot I_{rw}$ open, $i,j = 1,2$ and $i \neq j$. Let $F \subseteq B$ and $F$ is $\tau_i$-regular semiclosed in $X$. Since $F \subseteq B$ and $B \subseteq A$, we have $F \subseteq A$. Since $A$ is $(i,j) \cdot I_{rw}$ open, we have $F - \tau_j \cdot \text{int}(A) \in I$. Since $\tau_j \cdot \text{int}(A) \subseteq B$, we have $\tau_j \cdot \text{int}(A) \subseteq \tau_j \cdot \text{int}(B)$. Therefore, $F - \tau_j \cdot \text{int}(B) \subseteq F - \tau_j \cdot \text{int}(A) \in I$. Consequently, $B$ is $(i,j) \cdot I_{rw}$ open. $lacksquare$

Theorem 6.3.7 If a subset $A$ is $(i,j) \cdot I_{rw}$ closed, then $\tau_j \cdot \text{cl}(A) - A$ is $(i,j) \cdot I_{rw}$ open, $i,j = 1,2$ and $i \neq j$.

Proof. Suppose that $A$ is $(i,j) \cdot I_{rw}$ closed, $i,j = 1,2$ and $i \neq j$. Let $F \subseteq \tau_j \cdot \text{cl}(A) - A$ and $F$ is $\tau_i$-regular semiclosed. Since $A$ is $(i,j) \cdot I_{rw}$ closed, we have $\tau_j \cdot \text{cl}(A) - A$ does not contain $\tau_i$-regular semiclosed such that $F \notin I$ {by Theorem 6.2.8}. Hence, $F \in I$. Therefore, $F - \tau_j \cdot \text{int}[\tau_j \cdot \text{cl}(A) - A] \in I$. Consequently, $\tau_j \cdot \text{cl}(A) - A$ is $(i,j) \cdot I_{rw}$ open. $lacksquare$

Theorem 6.3.8 If $A$ and $B$ are $(i,j) \cdot I_{rw}$ open sets then $A \cap B$ is $(i,j) \cdot I_{rw}$ open, $i,j = 1,2$ and $i \neq j$.

Proof. Suppose that $A$ and $B$ are $(i,j) \cdot I_{rw}$ open sets, $i,j = 1,2$ and $i \neq j$. Let $F \subseteq A \cap B$ and $F$ is $\tau_i$-regular semiclosed. Since $F \subseteq A \cap B$, we have $F \subseteq A$ and $F \subseteq B$. Since $F \subseteq A$ and $F$ is $\tau_i$-regular semiclosed, we have $F - \tau_j \cdot \text{int}(A) \in I$ {since $A$ is $(i,j) \cdot I_{rw}$ open}. Since $F \subseteq B$ and $F$ is $\tau_i$-regular

101
semiclosed, we have $F - \tau_j - \text{int}(B) \in I \{ \text{since } B \text{ is } (i, j) - I_{rw} \text{ open} \}$. Therefore, $F - \tau_j - \text{int}(A \cap B) = \{ F - \tau_j - \text{int}(A) \} \cap \{ F - \tau_j - \text{int}(B) \} \in I$. Hence $A \cap B$ is $(i, j) - I_{rw}$ open.

**Remark 6.3.9** The union of two $(i, j) - I_{rw}$ open sets is not an $(i, j) - I_{rw}$ open set in general as can be seen from the following example.

**Example 6.3.10** In Example 6.2.2, $A = \{a\}$, $B = \{c\}$ are $(1, 2) - I_{rw}$ open sets, but $A \cup B = \{a, c\}$ is not an $(1, 2) - I_{rw}$ open set in $X$. 