CHAPTER 3

DYNAMIC RESPONSE OF 2 DOF QUARTER CAR PASSIVE SUSPENSION SYSTEM (QC-PSS) AND 2 DOF QUARTER CAR ELECTROHYDRAULIC ACTIVE SUSPENSION SYSTEM (QC-EH-ASS)

3.1 INTRODUCTION

In this chapter, the dynamic response analysis of a 2 DOF Quarter Car Electro-Hydraulic Active Suspension System (EH-ASS) has been carried out in the spirit and approach of Lin and Kaddissi [50]. This Analysis is further used in the validation of results of Simulation analysis in a state-of-the-art MATLAB SIMULINK R 2010 environment which is reported in chapter 4.

The analysis of vehicle suspension system has been performed using different suspension models. In recent research projects on suspension system, a quarter car model is being preferred over other many DOF models. It has immerged as the most employed and useful vehicle suspension system analysis tool, because of its simplicity and yet it is capable of capturing many critical characteristics of the full car model. In this research work, therefore, it is felt appropriate, to use a 2DOF Quarter Car Model for theoretical analysis of the Road Vehicle Suspension System which is later fabricated and analysed in the institute’s laboratory.
Figure 3.1 describes the scheme of a 2DOF Quarter Car Electro-Hydraulic Active Suspension System Model which is taken up for its dynamic response analysis. Here the sprung mass $m_s$ is one fourth of the mass of car body and attached to unsprung mass through a passive spring and damper alongwith an active element in the form of a hydraulic actuator and electro-hydraulic servo valve. The hydraulic actuator force is viewed as control input. The vehicle tyre is modeled as a simple spring neglecting tyre damping. We can add a damper with damping coefficient $C_u$ in parallel to spring with spring stiffness $K_t$, to model the damping in tires. However, the value of $C_u$ for tires, compared to $C_s$, is very small, and hence, $C_u$ is not considered in modeling of 2 DOF Quarter Car Suspension System. In this case,

$M_s = $ Sprung Mass (mass of car body)

$M_u = $ Unsprung Mass (mass of chassis and wheel)

$K_s = $ Spring Stiffness

$K_t = $ Tire Stiffness

Fig. 3.1 A Typical 2DOF Quarter Car Electro-Hydraulic Active Suspension System Model
3.3 EQUATIONS OF MOTION

Here the scheme as shown in fig. 3.1 is considered for development of the mathematical models for the following:

1. Case 1
2. Case 2

3.3.1 CASE 1 (WITHOUT ACTUATION: PASSIVE SUSPENSION)

Assuming that there is no active element (i.e. actuator in between the sprung and unsprung mass), the system will depict the conventional 2DOF Quarter Car Passive Suspension model.

The governing differential equations of motion for this system are [15],

\[
\begin{align*}
M_s \ddot{x}_s + K_s (x_s - x_{us}) + C_s (\dot{x}_s - \dot{x}_{us}) &= 0 \\
M_{us} \ddot{x}_{us} + K_s (x_{us} - x_s) + C_s (\dot{x}_{us} - r) &= 0
\end{align*}
\]

\[
\begin{equation}
\begin{align*}
M_s \ddot{x}_s + K_s (x_s - x_{us}) + C_s (\dot{x}_s - \dot{x}_{us}) &= 0 \\
M_{us} \ddot{x}_{us} + K_s (x_{us} - x_s) + C_s (\dot{x}_{us} - r) &= 0
\end{align*}
\end{equation}
\] (1)
Taking state variables as
\[ x_1 = x_s, x_2 = x_s', x_3 = x_{uls}, x_4 = x_{uls}' \]
we obtain
\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{1}{M_s}[K_s(x_1 - x_3) + C_s (x_2 - x_4)] \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{1}{M_{uls}}[K_s(x_1 - x_3) + C_s (x_2 - x_4) - K_f (x_3 - r)] 
\end{align*} \] (2)

Frequency domain analysis is used as it is assumed that the passive suspension system is linear.

Here, \( r \) (Road disturbance) is input and displacements \( (x_s) \) and \( (x_{uls}) \) are the responses.

Following the usual procedure of L.T. transfer functions theory, we get transmissibilities \( H_{1 P(s)} \) and \( H_{3 P(s)} \) as
\[ \begin{align*}
H_{1 P(s)} &= \frac{X_1(s)}{R(s)} = \frac{\omega_0^2(A_4 s^2 + A_2)}{\Delta_p(s)} \\
H_{3 P(s)} &= \frac{X_3(s)}{R(s)} = \frac{\omega_0^2(s^2 + A_4 s + A_2)}{\Delta_p(s)} 
\end{align*} \] (3)

Here \( X_1(s) \) = Laplace transfer function of \( x_1(t) \)
\( X_3(s) \) = Laplace transfer function of \( x_3(t) \)
\( R(s) \) = Laplace transfer function of \( r(t) \) and
\[ \Delta_p(s) = s^4 + (A_1 + A_3)s^3 + (A_2 + A_4 + \omega_0^2)s^2 + A_1 \omega_0^2 s + A_2 \omega_0^2 \] (4)

with,
\[ A_1 = \frac{C_s}{M_s}, A_2 = \frac{K_s}{M_s}, A_3 = \frac{C_s}{M_{uls}}, A_4 = \frac{K_s}{M_{uls}} \quad \text{and} \quad \omega_0 = \sqrt{\frac{K_s}{M_{uls}}} \quad \text{(natural frequency of } X_{uls}) \]

These functions are obtained by the simulation carried out with the MATLAB SIMULINK 2010 environment using values taken from Table 3.1.
TABLE 3.1: Suspension Parameters and Constants used for Transfer Functions in Case-I

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Characteristics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sprung mass (M_s)</td>
<td>8 kg</td>
</tr>
<tr>
<td>Unsprung mass (M_us)</td>
<td>1.5 kg</td>
</tr>
<tr>
<td>Suspension spring stiffness (K_s)</td>
<td>13900 N/m</td>
</tr>
<tr>
<td>Suspension Damping coefficient (C_s)</td>
<td>518 Ns/m</td>
</tr>
<tr>
<td>Tire spring stiffness (K_t)</td>
<td>85000 N/m</td>
</tr>
<tr>
<td>Natural Frequency of X_{ILS} (\omega_0)</td>
<td>76 rad/sec</td>
</tr>
<tr>
<td>Constant A_1 = \frac{C_s}{M_s}</td>
<td>33.45</td>
</tr>
<tr>
<td>Constant A_2 = \frac{K_s}{M_s}</td>
<td>101.4</td>
</tr>
<tr>
<td>Constant A_3 = \frac{C_s}{M_{us}}</td>
<td>35.23</td>
</tr>
<tr>
<td>Constant A_4 = \frac{K_s}{M_{us}}</td>
<td>540.81</td>
</tr>
</tbody>
</table>

Upon substituting above mentioned suspension characteristics of road vehicle of a small car with mass ratio of 5.00, the value of the transmissibility is determined as 0.45 which is nearly equal to the value of 0.477 obtained from simulation of the X_1(t) and X_3(t) in result fig. 4.5 and fig. 4.6.
3.3.2 CASE 2 (WITH ACTUATION: ACTIVE SUSPENSION)

The schematic of the case is shown in fig. 3.3

Figure 3.3: 2 DOF Quarter Car Electro-Hydraulic Active Suspension System Model

Equations of motion:

\[
\begin{align*}
M_s (\ddot{x}_s) + K_s (x_s - x_{us}) + C_s (\dot{x}_s - \dot{x}_{us}) \cdot u_a &= 0 \\
M_{us} \ddot{x}_{us} + K_s (x_{us} - x_s) + C_s (\dot{x}_{us} - \dot{x}_s) + K_f (x_{us} - r) + u_a &= 0
\end{align*}
\]

Using state variables,

\[x_1 = x_s, x_2 = \dot{x}_s, x_3 = x_{us}, x_4 = \dot{x}_{us}\]

Reformulating (5) into state space representation as,

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{M_s} [K_s (x_1 - x_3) + C_s (x_2 - x_4) \cdot u_a] \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{1}{M_{us}} [K_s (x_1 - x_3) + C_s (x_2 - x_4) - K_f (x_3 - r) \cdot u_a]
\end{align*}
\]

Here, \(u_a\) is the control force used as input signal.

Here, before going into the details of the further mathematical modeling, it will be appropriate to discuss the potentials of available control strategies and relevance of selection of a proper control methodology for the active suspension system model undertaken for the analysis. Thus, following section deals with the characteristics and brief comparison of merits and limitations of following control strategies.
3.4 COMPARISON OF POTENTIAL CONTROL STRATEGIES WITH RELATIVE MERITS AND LIMITATIONS

In the past years, various control strategies have been proposed by numerous researchers to improve the trade-off between ride comfort and road handling. These control strategies may be grouped into techniques based on linear, nonlinear and intelligent control approaches. In the following, some of these control approaches that have been reported in the literature will be briefly presented.

3.4.1 STUDY OF VARIOUS CONTROL STRATEGIES

The most popular linear control strategy that has been used by researchers in the design of the active suspension system is based on the optimal control concept (Hrovat, 1997). Amongst the optimal control concepts used are the Linear Quadratic Regulator (LQR) approach, the Linear Quadratic Gaussian (LQG) approach and the Loop Transfer Recovery (LTR) approach. These methods are based on the minimization of a linear quadratic cost function where the performance measure is a function of the states and inputs to the system.

Application of the LQR method to the active suspension system has been proposed by Hrovat (1988), Tseng and Hrovat (1990) and Esmailzadeh and Taghirad (1996). Hrovat (1988) has studied the effects of the unsprung mass on the active suspension system. The carpet plots were introduced to give a clear global view of the effect of various parameters on the system performances. The carpet plots are the plots of the root mean square (r.m.s) values of the sprung mass acceleration and unsprung mass acceleration versus the suspension travel. The r.m.s. values of all parameters are obtained from a series of simulations on different weights of the performance index.

Since it is desirable to measure all the system states in the actual implementations, observers are usually used. Ulsoy et al.
(1994) used Kalman filter to reconstruct the states and to address the problem of the sensors noise and road disturbances. In the study, the robustness margin of the LQG controllers with respect to the parameter uncertainties and actuator dynamics were investigated. The results showed that the LQG controllers, using some measurements, such as the suspension stroke, should be avoided since the controllers did not produce the satisfactory robustness. LQG with loop transfer recovery (LQG/LTR) was studied by Ray (1993) as a solution to increase the robustness of the LQG controllers. However, when applying LTR, uncertain system parameters must be identified, and the magnitude of uncertainty expected should be known or estimated; otherwise, LQR/LTR system may exhibit robustness qualities that are not better than the original LQG system.

Most researchers that utilized the linear control approach did not consider the dynamics of the actuators in their study. Thus, the control strategies that have been developed did not represent the actual system which is highly nonlinear due to the hydraulic actuator properties and the presence of uncertainties in the system. Furthermore, when applying the linear control theory to the system, it may not give an acceptable performance due to the presence of uncertainties and nonlinearities in the system.

Suspension systems are intrinsically nonlinear and uncertain and are subjected to a variety of road profiles and suspension dynamics. The nonlinearities of a road profile are due to the roughness and smoothness of the road surfaces while the suspension dynamics are affected by the actuator nonlinearities. Yamashita et al. (1994) presented a control law for a full car model using the actual characteristics of hydraulic actuators based on the H-infinity control theory. The proposed controller has been implemented in an experimental vehicle, and evaluated for robust performance in a four-wheel shaker
and during actual driving. The results showed that the system is robust even when the closed-loop system is perturbed by limited uncertainty. Instead of using the state feedback, Hayakawa et al. (1999) utilized the robust H-infinity output feedback control to a full car active suspension model. In the study, the linear dynamical model of a full car model is intrinsically decoupled into two parts to make the implementation of the output feedback control simpler and realizable.

The combination of the H-infinity and adaptive nonlinear control technique on active suspension system has been reported by Fukao et al. (1999). The study divided the active suspension structure into two parts. The car’s body part utilized the H-infinity control design and the actuator part used the adaptive nonlinear control design technique. On the ride quality, there exists a range of frequency where passengers strongly feel the body acceleration caused by the disturbance from road surface. Therefore, the H-infinity controller through frequency shaping performs improvement of the frequency property. The nonlinearities and the uncertainties of the actuator are overcome by the adaptive nonlinear controller based on the backstepping technique.

Alleyne and Hedrick (1995) presented a nonlinear adaptive control to active suspension system. The study introduced a standard parameter adaptation scheme based on Lyapunov analysis to reduce the error in the model. Then a modified adaptation scheme that enables the identification of parameters whose values change with regions of the state space is developed. The adaptation algorithms are coupled with the nonlinear control law that produces a nonlinear adaptive controller. The performance of the proposed controller is evaluated by observing the ability of the electro-hydraulic actuator to track a desired force. The results showed that the
controller improved the performance of the active suspension system as compared to the nonlinear control law alone.

Lin and Kanellapoulos (1997a) presented a nonlinear backstepping design for the control of quarter car active suspension model. The intentional introduction of nonlinearity into the control objective allows the controller to react differently in different operating regimes. They improved further their works on nonlinear control design for active suspension system by augmenting such controller with the road adaptive algorithm as reported in Lin and Kanellakopoulos (1997b).

The road adaptive approach is also reported in Fialho and Balas (2002). In the study, combination of the linear parameter varying (LPV) control with a nonlinear backstepping technique that forms the road adaptive active suspension system is proposed. Two level of adaptation is considered with the lower level control to shape the nonlinear characteristic of the vehicle suspension and the higher level design involves adaptive switching between the different nonlinear characteristic based on the road condition.

Chantranuwathana and Peng (1999), D'Amato and Viassolo (2000) decomposed the active suspension control design into two loops. The main loop calculated the desired actuation force. The inner loop controls the nonlinear hydraulic actuators to achieve tracking of the desired actuation force. The results showed that the proposed controller performed better.

Recently, intelligent based techniques such as fuzzy logic, neural network and genetic algorithm have been applied to the active suspension system. Ting et al. (1995) presented a sliding mode fuzzy control technique for a quarter car model active suspension system. In this study, the controller is organized into two levels. At the basic level, the conventional fuzzy control rule sets and inference mechanism are constructed to generate
a fuzzy control scheme. At the supervising level, the control performance is evaluated to modify system parameters. The controller input consists of the input from the sliding mode controller and fuzzy controller. The results showed that the fuzzy SMC attained superior performance in body acceleration and road handling ability but worst in the suspension travel as compared to the conventional sliding mode scheme.

Yoshimura et al. (1999) presented an active suspension system for passenger cars, using linear and fuzzy logic control technique. The studied utilize vertical acceleration of the vehicle body as the principle source of control, and the fuzzy logic control scheme as the complementary control of the active suspension system for passenger cars. The fuzzy control rules are determined by minimizing the mean squares of the time responses of the vehicle body under certain constraints on the acceptable relative displacements between vehicle body and suspension parts and tire deflections.

3.4.2 RELATIVE MERITS AND LIMITATIONS OF POTENTIAL CONTROL STRATEGIES

Following is the brief comparative statement of the relative Merits and Limitations of the potential control strategies based on previous study.

<table>
<thead>
<tr>
<th>Sr. No.</th>
<th>Control Strategy</th>
<th>Merits</th>
<th>Limitations</th>
</tr>
</thead>
<tbody>
<tr>
<td>01</td>
<td>The linear control strategies based on the optimal control theory (LQR, LQG, LTR and H-infinity)</td>
<td>Capable of minimizing a defined performance index</td>
<td>Do not have the capability to adapt to significant system parameter changes and variations in the road profiles</td>
</tr>
<tr>
<td>02</td>
<td>Sliding Mode Control (SMC)</td>
<td>Capable of improving the tradeoff between ride comfort and road handling characteristics. Highly robust to the uncertainties.</td>
<td>Conventional sliding surface has been used.</td>
</tr>
<tr>
<td>Sr. No.</td>
<td>Control Strategy</td>
<td>Merits</td>
<td>Limitations</td>
</tr>
<tr>
<td>--------</td>
<td>------------------</td>
<td>----------------------------------------------------------------------</td>
<td>----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>03</td>
<td>Intelligent based techniques (fuzzy logic, neural network and genetic algorithm)</td>
<td>Can attain superior performance in body acceleration and road handling ability</td>
<td>Worst in the suspension travel control as compared to the conventional sliding mode scheme. These techniques have a potential problem on stability</td>
</tr>
<tr>
<td>04</td>
<td>Nonlinear adaptive control</td>
<td>The controller improved the performance of the active suspension system as compared to the nonlinear control law alone.</td>
<td>Lyapunov analysis adaptation is required</td>
</tr>
<tr>
<td>05</td>
<td>Outer-loop controller</td>
<td>Able to carry out the commanded force accurately</td>
<td>simulations of these outer-loop controllers were frequently done without considering actuator dynamics</td>
</tr>
<tr>
<td>06</td>
<td>Nonlinear backstepping Control</td>
<td>The intentional introduction of nonlinearity into the control objective allows the controller to react differently in different operating regimes.</td>
<td>Higher nonlinear dynamics is involved.</td>
</tr>
</tbody>
</table>

In the above review, various active suspension system models with either quarter or half car models have been used in the design of the controllers. The quarter car model with linear force input has been used by Hac (1987), Hrovat (1997 and 1998), Tseng and Hrovat (1990), Sunwoo et al. (1991), Ray (1993), Ting et al. (1995), Kim and Ro (1998), Huang and Chao (2000) and Yoshimura et al. (2001) in their study. Modeling of the active suspension system as a linear force input is the most simple but it does not give an accurate model of the system because the actuator’s dynamics have been ignored in the design. Thus the controller developed and the result presented may have problem when applying to the active suspension system in the real world.

In order to overcome the problem, Rajamani and Hedrick (1995), Alleyne and Hedrick (1995), Lin and Kanellakopoulos (1997a and 1997b), Fukao et al. (1999), Chantranuwathana and Peng (1999) and Fialho and Balas (2002) have considered the hydraulic actuator dynamics in the design of active suspension system for the quarter car model. All these researchers have utilized the hydraulic actuator dynamics formulated by Merritt (1967).
The active suspension systems for the quarter car models may be modeled as the linear force input or the hydraulically actuated input. The active suspensions systems of the hydraulically actuated input may represent a much more detail of the system dynamics compared to the linear force input. Therefore, the analysis and design of the active suspension systems by using this approach is much closer to the actual systems.

3.4.3 BACKSTEPPING ANALYSIS

From the review of control strategies taken in section 3.4.2, it is seen that for implementing the control strategy for active suspensions, one primary goal is to improve the inherent tradeoff between passenger comfort and road handling. While the use of active components enhances both comfort and handling, this is achieved at the expense of increased suspension travel and fuel consumption. As such, in the present work, a new design methodology which exploits the flexibility of backstepping is introduced to improve the tradeoff between ride quality and suspension travel in the approach of Lin (5) and Kaddissi (40). The resulting closed loop response is fundamentally nonlinear, since it is soft for passenger comfort when the suspension travel is small, but stiffens up very quickly as the suspension approaches its travel limits, to avoid hitting them. This improvement is due to the fact that the resulting nonlinear controller not only handles the inherent nonlinear nature of the hydraulic dynamics, but also introduces additional nonlinearities to make the suspension stiffer near its travel limits. This intentional addition of nonlinearity represents a departure from previous designs which attempt to produce linear closed loop systems.

3.5 BACKSTEPPING CONTROLLER DESIGN STRATEGIES

It is necessary to use a control strategy with the objective of minimization of the forces transmitted to the passengers. A
Backstepping Controller design is incorporated in the equations of motion with following strategies, using the approach of Lin and Kanellakopoulos [15] and Kaddissi et al [50]

### 3.5.1 First Strategy:
Regulated variable: Sprung Mass Acceleration \( (\dot{x}_s) \)

Desired value of hydraulic force is, from (6)

\[
u_a = K_s(x_1 - x_2) + C_s(x_2 - x_4)\]

which gives \( \ddot{x}_2 = 0 \)

Substituting this expression into (6)

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= 0 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -\frac{K_t}{M_{us}}(x_3 - r)
\end{align*}
\]

(7)

With \( \dot{x}_2 \) as regulation variable, road disturbance input will result in sustained wheel oscillations and even diverging car body displacement.

### 3.5.2 Second Strategy:
Regulated variable as sprung mass displacement \( (x_3) \):

If the control strategy is shifted to \( x_1 \), then

1. \( (x_1, x_2) \) subsystem is stabilized and
2. \( (x_3, x_4) \) subsystem represents the zero dynamics.

With this, control force will yield

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= c_1 x_1 - c_2 x_2 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{M_s}{M_{us}}(c_1 x_1 + c_2 x_2) - \frac{K_t}{M_{us}}(x_3 - r)
\end{align*}
\]

(8)

(9)

To regulate \( x_1 \), putting \( x_1 = x_2 = 0 \) into (9)

\[
\begin{align*}
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -\frac{K_t}{M_{us}}(x_3 - r)
\end{align*}
\]

(10)
Thus with \((x_1)\) as controlled variable, the unstable subsystem \((x_1,x_2)\) gets eliminated which was present in (7) with \((\dot{x}_2)\) as a control variable.

3.5.3 **Third Strategy**: Regulated variable as suspension travel \((x_1-x_3)\)

Regulated variable: \(x_1 - x_3\)

Problem: Zero Dynamics of \((x_3,x_4)\) is oscillatory. To avoid this oscillatory dynamics, reformulating the variable as [12],

\[
z_1 = x_1 - \overline{x}_3
\]

Here,

\[
x_1 = x_s = \text{Sprung Mass displacement}
\]
\[
\overline{x}_3 = \overline{x}_{us} = \text{filtered quantity of unsprung mass displacement}
\]
\[
\overline{x}_3 = \frac{\epsilon}{S+\epsilon} x_3
\]

Here \(\frac{\epsilon}{S+\epsilon}\) is filter

\(\epsilon = \text{positive constant to be selected as per the nature of road}\)

\(S = \text{filter laplace variable}\)

For small values \(\epsilon\), equation (12) becomes low pass filter

1) If road input contains only high frequency components to be rejected.
   i. \(\frac{\epsilon}{S+\epsilon} x_3\) is low pass filter.

   Then use \(z_1 \approx x_1\)

   ii. For constant or slowly changing road structure at very low frequencies and steady state,

   \[z_1 = x_1 - x_3\]

With small \(\epsilon\), focus is on rejection of only high frequency road disturbances which generate large vertical accelerations and cause passenger discomfort. Thus Active suspension system becomes soft. [12].

2) For large \(\epsilon\): High pass filter

In this strategy, high frequency components of road irregularity are allowed to pass through the filter.

\[z_1 \approx x_1 - x_3 \quad (\overline{x}_3 \approx x_3)\]

Thus, Active suspension becomes stiffer, reduces suspension travel by sacrificing significant amount of passenger comfort.[12]
3.6 Backstepping Design Process with third Strategy

By taking \( z_1 \) \([= (x_1 - \bar{x}_3)]\) as regulated variable, now we can proceed to controller design with 2 steps.

1. The derivative of \( Z_1 \),
   \[
   \dot{z}_1 = \dot{x}_1 - \dot{x}_3 \\
   = x_2 + \epsilon(\bar{x}_3 - x_3) \\
   = x_2 + \epsilon(x_1 - z_1 - x_3) \\
   = x_2 + \epsilon(x_1 - x_3) - \epsilon z_1 
   \]  \quad \text{----------- (13)}

   Here we use \( x_2 \) as first virtual control variable whose stabilizing function is
   \[
   \alpha_1 = -c_1 z_1 - \epsilon(x_1 - x_3) 
   \]  \quad \text{----------- (14)}

   where \( c_1 \) is positive design constant

   Error variable, \( z_1 = x_2 - \alpha_1 \) and
   Error equation, \[
   \dot{z}_1 = -(c_1 + \epsilon) z_1 + z_2 
   \]  \quad \text{----------- (15)}

2. The derivative of \( z_2 \),
   \[
   \dot{z}_2 = \dot{x}_2 - \dot{\alpha}_1 \\
   = -\frac{1}{M_s} [K_s(x_1 - x_3) + C_a(x_2 - x_4) - u_a] - \[
   [-c_1(-c_1 z_1 - \epsilon z_1 z_2) - \epsilon(x_2 - x_4)] 
   \]  \quad \text{----------- (16)}

   Here, the actual control input \( u_a \) appears, the Control law is,
   \[
   u_a = M_s [- (c_2 + c_1) z_2 + (c_1^2 - 1 + c_1 \epsilon) z_1 - \epsilon(x_2 - x_4)] \\
   + K_a(x_1 - x_3) + C_a(x_2 - x_4) 
   \]  \quad \text{----------- (17)}

   \( c_2 \) is positive design constant to render derivative of Lyapunov function,

   i.e. \[
   V_a = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2 
   \]  \quad \text{----------- (18)}

   Negative definite,
   \[
   \dot{V} a = -(c_1 + \epsilon) z_1^2 - c_2 z_2^2. 
   \]  \quad \text{----------- (19)}

   Hence the second order systems,
   \[
   \dot{z}_1 = -(c_1 + \epsilon) z_1 + z_2 \\
   \dot{z}_2 = -c_2 z_2 - z_1 
   \]  \quad \text{----------- (20)}
Has a globally exponential stable equilibrium at \((z_1, z_2) = (0.0)\)

This is a second order system. As we started with 5\(^{th}\) order system with active suspension and linear filter \(\ddot{\bar{x}}_3 = \frac{\epsilon}{s+\epsilon} x_3\) the remaining 3 states here are the zero dynamics subsystems of the closed loop systems.

**Zero dynamics.**

To find the zero dynamics, let us set the output identically equal to zero, i.e.,
\[
y = z_1 = x_1 - \bar{x}_3 = 0
\]
\[
\dot{y} = x_2 + \epsilon (\bar{x}_3 - x_3) = 0
\]
\[
\dot{\bar{y}} = -\frac{1}{M_s} [K_s (x_1 - x_3) + c_a (x_2 - x_4) - u_a] + \epsilon [-\epsilon (\bar{x}_3 - x_3) - x_4] = 0
\]

\[\text{(21)}\]

Using the last equation of (6) we substitute
\[
[K_s (x_1 - x_3) + c_a (x_2 - x_4) - u_a] = M_s \epsilon [-\epsilon (\bar{x}_3 - x_3) - x_4] \quad \text{(22)}
\]

Substituting this into \(\dot{x}_4\) - equation of (6) to obtain the zero dynamics as
\[
\ddot{\bar{x}}_3 = -\epsilon (\bar{x}_3 - x_3)
\]
\[
\dot{x}_3 = x_4
\]
\[
\dot{x}_4 = \frac{M_s}{M_{us}} \epsilon [-\epsilon (\bar{x}_3 - x_3) - x_4] - \frac{K_t}{M_{us}} (x_3 - r) \quad \text{(23)}
\]

This can be rewritten into matrix form:
\[
\begin{bmatrix}
\ddot{\bar{x}}_3 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} =
\begin{bmatrix}
-\epsilon & \epsilon & 0 \\
0 & 0 & 1 \\
-\epsilon^2 & \frac{M_s}{M_{us}} & -\epsilon \frac{K_t}{M_{us}}
\end{bmatrix}
\begin{bmatrix}
\bar{x}_3 \\
x_3 \\
x_4
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
\frac{K_t}{M_{us}}
\end{bmatrix} r \quad \text{(24)}
\]

By using Hurwitz criteria,
The zero dynamics are exponentially stable for all \(\epsilon > 0\).

**Frequency domain Analysis:**

1. Rewriting the control law given by (13),
\[
u_a = -M_s (c_2 + c_1) \{x_2 - [-c_1 (x_1 - \bar{x}_3) - \epsilon (x_1 - x_3)]\} + \\
M_s (c_1^2 - 1 + c_1 \epsilon) (x_1 - \bar{x}_3) - \epsilon M_s (x_2 - x_4) + K_s (x_1 - x_3) + c_a (x_2 - x_4)
\]
\[ \begin{align*}
- M_s (c_2 + c_1) x_2 + [K_s - \epsilon M_s (c_2 + c_1)] (x_1 - x_3) + \\
- (c_2 - \epsilon M_s) (x_2 - x_4) + M_s [c_1 (\epsilon - c_2) - 1] (x_1 - \bar{x}_3)
\end{align*} \] 
--------- (25)

Writing this in matrix form as a closed-loop system,

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
-q_1 & -(q_2 + \epsilon) & -q_3 & \epsilon q_2 & \epsilon \\
0 & 0 & -\epsilon & \epsilon & 0 \\
m_r q_1 & m_r (q_2 + \epsilon) & m_r q_3 & -(m_r \epsilon q_2 + \omega_0^2) & -m_r \epsilon
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
\omega_0^2
\end{bmatrix} r
\]
--------- (26)

Here, \( m_r = \frac{M_s}{M_{us}} \)

\( q_1 = c_2 (c_1 + \epsilon) + 1, q_2 = c_2 + c_1, q_3 = c_1 (\epsilon - c_2) - 1 \)

Computation to transfer functions relating the road input \( r \) to car body displacement \( x_1 \) and wheel travel \( x_3 \). For this, calculating these transfer function directly from state-space representation of (26)

Also equivalent and much simple way is to utilize control design results. Results of control design depict that, error variable \( \bar{z}_1 \) converges to zero exponentially i.e. \( x_1 \) approaches \( \bar{x}_3 \) exponentially fast. As this result is independent of frequency content, transfer function from \( \bar{x}_3 \) to \( x_1 \) is equal to 1.

Thus considering the variables,

\[
\begin{align*}
\xi_1 &= M_s x_1 + M_{us} x_3 \\
\xi_2 &= M_s x_2 + M_{us} x_4
\end{align*} \] 
--------- (27)

Whose derivatives are computed from (6) as,

\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= -\omega_0^2 \xi_1 + K_t (m_r x_1 + r)
\end{align*} \] 
--------- (28)

Now we can represent the closed loop system in the equivalent simplified block diagram
Fig. 3.4: Equivalent simplified block diagram of active suspension control with backstepping

From block diagram,

\[ x_3 = \frac{1}{M_{us}} \xi_1 \cdot m_r \cdot x_1 \]
\[ = \frac{1}{M_{us}} \cdot \frac{1}{s^2 + \omega_0^2} \cdot m_r (m_r \ddot{x}_3 + r) - m_r \ddot{x}_3 \]
\[ = \frac{\omega_0^2}{s^2 + \omega_0^2} r - \frac{m_r \omega_0^2}{s^2 + \omega_0^2} \frac{\varepsilon}{s + \varepsilon} x_3 \]
\[ \text{i.e.} \left[ \frac{1 + m_r \varepsilon^2}{(s^2 + \omega_0^2)(s + \varepsilon)} \right] x_3 = \frac{\omega_0^2}{s^2 + \omega_0^2} r \]

Thus, the transfer function of interest are computed as,

\[ H_{1a}(s) = \frac{X_1(s)}{R(s)} = \frac{\omega_0^2(s)}{\Delta a(s)} \]
\[ H_{3a}(s) = \frac{X_3(s)}{R(s)} = \frac{\omega_0^2(s + \varepsilon)}{\Delta a(s)} \]

Where \( X_1(s), X_3(s) \) and \( R(s) \) are the Laplace transforms of \( x_1(t) \), \( x_3(t) \) and \( r(t) \) and

\[ \Delta a(s) = (s^2 + \omega_0^2)(s + \varepsilon) + m_r \varepsilon s^2 \]
\[ = s^3 + \varepsilon (m_r + 1) s^2 + \omega_0^2 s + \omega_0^2 \varepsilon \]
It is observed that, in the above transfer functions of active suspension, the positive design constants $c_1$ and $c_2$ have not appeared though they are used in state equations. This means that these constants do not affect the closed loop transfer functions.

These functions are plotted in the simulation graphs in section 4.3 carried out with the MATLAB SIMULINK 2010 environment using following values which are taken from the physical setup developed for experimental analysis with typical parameters of small car of mass ratio 5.

Table 3.3: Suspension Parameters and Constants used for Transfer Functions in Case-II

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Characteristics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sprung mass ($M_s$)</td>
<td>8 kg</td>
</tr>
<tr>
<td>Unsprung mass ($M_{us}$)</td>
<td>1.5 kg</td>
</tr>
<tr>
<td>Suspension spring stiffness ($K_s$)</td>
<td>13900 N/m</td>
</tr>
<tr>
<td>Suspension Damping coefficient ($C_s$)</td>
<td>518 Ns/m</td>
</tr>
<tr>
<td>Tire spring stiffness ($K_t$)</td>
<td>85000 N/m</td>
</tr>
<tr>
<td>Natural Frequency of $X_{US}$ ($\omega_0$)</td>
<td>76 rad/sec</td>
</tr>
<tr>
<td>Constant $A_1 = \frac{C_s}{M_s}$</td>
<td>33.45</td>
</tr>
<tr>
<td>Constant $A_2 = \frac{K_s}{M_s}$</td>
<td>101.4</td>
</tr>
<tr>
<td>Constant $A_3 = \frac{C_s}{M_{us}}$</td>
<td>35.23</td>
</tr>
<tr>
<td>Constant $A_4 = \frac{K_s}{M_{us}}$</td>
<td>540.81</td>
</tr>
<tr>
<td>Low Pass Filter Constant ($c$)</td>
<td>1.5</td>
</tr>
<tr>
<td>Suspension Mass Ratio ($m_r$)</td>
<td>5.33</td>
</tr>
</tbody>
</table>

With the typical suspension transfer function characteristics used in experimental setup and mentioned in Table 3.3, the equivalent simplified block diagram, of active suspension control with backstepping, is as shown in fig. 3.5 which gives the transfer functions for a typical road vehicle of small car type with mass ratio of 5.33.
These transfer functions can be determined at some invariant point as follows:

1. At natural frequency of unsprung mass, \( \omega_0 \) i.e. at normalized frequency \( \bar{\omega} = 1 \), transmissibility with transfer function in (31),
   \[
   H_{1a} (j \omega_0) = 1/mr = 1/5 = 0.20
   \]
   which is very close to motion transmissibility of 0.24 obtained from the Simulations of figures 4.5 and 4.6.

In the next chapter, the analysis of 2 DOF Quarter Car Passive Suspension System (QC-PSS) and Electro-Hydraulic Active Suspension System (QC-EH-ASS), using the theory outlined in this chapter, is carried out by the simulation technique (Simulink and Simscape).