Chapter 6

Feedback Queueing Model with Correlated Departures

Like the model considered in chapter 3, the model in present chapter also obtains the time dependent and steady state probabilities of the number of units in the system. Here in the present chapter, a feedback queueing system is considered where the departures are correlated too.

Mohan (1955, 1958) introduced the idea in discrete time of dependence of probability of an arrival or of no arrival at a time mark is dependent on the probability of an arrival or of no arrival at the previous time mark. Afterwards a good many research problems are solved in discrete time by Agarwal (1965), Chaudhry (1966), Tuteja (1966), Murari (1969), Sharda (1970) and Sharda & Rana (1985). Later on Mohan and Murari (1972), Sharma (1975) and Singh (1983) handled the same concept in continuous time. These types of queues in discrete and continuous time are named in literature as correlated queues.

Sharda et al. (1986) solved single server queueing system with feedback. However the model solved by them does not assume the dependence of probability of occurrence of an event (arrival or departure) at a time mark on the probability of occurrence or non-occurrence of the corresponding event (arrival or departure) at the previous time mark. But situations do occur in day-to-day life wherein such dependence is observed.

In this chapter a continuous time single server feedback queueing system in which probability of a departure or of no departure at a time mark is dependent on the probability of a departure or of no departure at the previous time mark is considered. Feedback concept is used in the sense that units are given another chance of service, if required. However the units will have to leave the system definitely after getting second service. Transient as well as steady state
solution and the distribution of the interval during which the server is busy are obtained. Some special cases of interest are also derived. Some illustrations are discussed and their numerical solutions are obtained. Thereafter various probabilities are plotted and compared graphically.

The practical situation which corresponds to the above situation can be that of a gymnasium room with some kind of exercising machine, wherein people come for exercising. It is supposed that the departures can occur only at transition points. There may arise two types of situations:

(I) Departure has occurred at previous transition mark.
Now there may be two cases according as
(a) departure occurs
(b) departure does not occur
at the present transition mark.

(II) Departure has not occurred at previous transition mark.
Now there may be two cases according as
(a) departure occurs
(b) departure does not occur
at the present transition mark.

By using the results of the above model, the probability of the number of customers in the gymnasium room can be found.

The feedback queueing system investigated in this chapter is described by the following assumptions in addition to the assumptions (i), (iv), (v) and (vi) given in model A of chapter 2:
(i) All departures take place only at the transition marks \( t_1, t_2, \ldots \) where \( \theta_r = t_r - t_{r-1}, \ r = 1, 2, 3, \ldots \) are identically distributed random variables.

(ii) If at any instance the queue length is \( > 0 \) then the departure probabilities at two successive transition marks are governed by the following transition probability matrix

\[
\begin{pmatrix}
\text{Departure at } t_r & \text{No Departure at } t_r \\
\text{From } t_{r-1} & \\
\text{Departure at } t_{r-1} & \mu_0 & \mu_i \\
\text{No Departure at } t_{r-1} & \mu_i & \mu_0
\end{pmatrix}
\]

(iii) The inter-transition times \( \theta_r \) are exponentially distributed with mean transition time \( \frac{1}{\lambda} \).

(iv) The stochastic processes involved, viz.

a. arrival of units

b. departure of units

c. transition times

are statistically independent of each other.

**Definitions:**

\[ P_{n,i}(t) = \text{Probability that there are } n \text{ units in the system at time } t \text{ and the next unit is to depart for the first time or second time according as } k = 0 \text{ or } 1 \text{ and } i \text{ departure has occurred at the previous transition mark} \]

\[ n \geq 0, \quad i = 0, 1 \]
\( P^{(k)}_n(t) = \) Probability that there are \( n \) units in the system at time \( t \) and the next unit is to depart for the first time or second time according as \( k = 0 \) or \( 1 \) \( \quad n \geq 0 \)

\[
P^{(k)}_n(t) = P^{(k)}_{n,0}(t) + P^{(k)}_{n,1}(t)
\]

\( P_{n,i}(t) = \) Probability that there are \( n \) units in the system at time \( t \) and \( i \) departure has occurred at the previous transition mark \( \quad n \geq 0, \quad i = 0, 1 \)

\[
P_{n,i}(t) = P^{(0)}_{n,i}(t) + P^{(1)}_{n,i}(t)
\]

\( P_n(t) = \) Probability that there are \( n \) units in the system at time \( t \) \( \quad n \geq 0 \)

\[
P_n(t) = P^{(0)}_n(t) + P^{(1)}_n(t)
\]

\[
P_n(t) = P_{n,0}(t) + P_{n,1}(t)
\]

Initially,

\[
P^{(0)}_{0,0}(0) = 1, \quad P^{(0)}_{0,1}(0) = 0
\]

\[
P^{(1)}_{0,0}(t) = P^{(1)}_{0,1}(t) = 0 \quad t \geq 0
\]

The difference differential equations describing the system are

\[
\frac{d}{dt} P^{(0)}_{n,0}(t) = - (\lambda + V) P^{(0)}_{n,0}(t) + \lambda P^{(0)}_{n-1,0}(t) + V \{ \mu_0 P^{(0)}_{n,0}(t) + \mu_1 P^{(0)}_{n,1}(t) \}, \quad n \geq 1 \quad (6.1)
\]

\[
\frac{d}{dt} P^{(1)}_{n,0}(t) = - (\lambda + V) P^{(1)}_{n,0}(t) + \lambda P^{(1)}_{n-1,0}(t) + V \{ \mu_0 P^{(1)}_{n,0}(t) + \mu_1 P^{(1)}_{n,1}(t) \}, \quad n \geq 1 \quad (6.2)
\]

\[
\frac{d}{dt} P^{(0)}_{n,1}(t) = - \lambda P^{(0)}_{n,0}(t) + VP^{(0)}_{n,1}(t) \quad (6.3)
\]
\[
\frac{d}{dt} P_{n,1}^{(0)}(t) = - (\lambda + V)P_{n,1}^{(0)}(t) + \lambda P_{n-1,1}^{(0)}(t) \\
+ V C_1 \left[ \mu_1 \left( q P_{n+1,0}^{(0)}(t) + P_{n+1,0}^{(0)}(t) \right) \right] + \mu_0 \left( q P_{n+1,1}^{(0)}(t) + P_{n+1,1}^{(0)}(t) \right) \\
+ V C_1 \left[ \mu_0 \left( q P_{n+1,0}^{(0)}(t) + P_{n+1,0}^{(0)}(t) \right) \right] (1 - \delta_{n,1}) , \quad n \geq 1 \tag{6.4}
\]

\[
\frac{d}{dt} P_{n,1}^{(1)}(t) = - (\lambda + V)P_{n,1}^{(1)}(t) + \lambda P_{n-1,1}^{(1)}(t) \\
+ V C_1 \left[ \mu_1 \left( q P_{n+1,0}^{(1)}(t) + P_{n+1,0}^{(1)}(t) \right) \right] + \mu_0 \left( q P_{n+1,1}^{(1)}(t) + P_{n+1,1}^{(1)}(t) \right) \\
+ V \left( c_2 + c_1 \delta_{n,1} \right) \left[ \mu_0 P_{n,0}^{(1)}(t) + \mu_1 P_{n,1}^{(0)}(t) \right] , \quad n \geq 1 \tag{6.5}
\]

\[
\frac{d}{dt} P_{0,1}^{(0)}(t) = - (\lambda + V)P_{0,1}^{(0)}(t) + V \left[ \mu_1 \left( q P_{1,0}^{(1)}(t) + P_{1,0}^{(1)}(t) \right) \right] + \mu_0 \left( q P_{1,1}^{(1)}(t) + P_{1,1}^{(1)}(t) \right) \tag{6.6}
\]

The steady state difference equations describing the systems are

\[
(\lambda + V)P_{n,0}^{(0)} = \lambda P_{n-1,0}^{(0)} + V \left[ \mu_0 P_{n,0}^{(0)} + \mu_1 P_{n,1}^{(0)} \right] , \quad n \geq 1 \tag{6.7}
\]

\[
\lambda P_{0,0}^{(0)} = VP_{0,1}^{(0)} \tag{6.8}
\]

\[
(\lambda + V)P_{n,0}^{(1)} = \lambda P_{n-1,0}^{(1)} + V \left[ \mu_0 P_{n,0}^{(1)} + \mu_1 P_{n,1}^{(1)} \right] , \quad n \geq 1 \tag{6.9}
\]

\[
(\lambda + V)P_{n,1}^{(0)} = \lambda P_{n-1,1}^{(0)} + V C_1 \left[ \mu_1 \left( q P_{n+1,0}^{(0)} + P_{n+1,0}^{(0)} \right) \right] + \mu_0 \left( q P_{n+1,1}^{(0)} + P_{n+1,1}^{(0)} \right) \\
+ V C_1 \left[ \mu_0 \left( q P_{n+1,0}^{(0)} + P_{n+1,0}^{(0)} \right) \right] (1 - \delta_{n,1}) , \quad n \geq 1 \tag{6.10}
\]

\[
(\lambda + V)P_{n,1}^{(1)} = \lambda P_{n-1,1}^{(1)} + V \left[ \mu_1 \left( q P_{n+1,0}^{(1)} + P_{n+1,0}^{(1)} \right) \right] + \mu_0 \left( q P_{n+1,1}^{(1)} + P_{n+1,1}^{(1)} \right) \\
+ V \left( c_2 + c_1 \delta_{n,1} \right) \left[ \mu_0 P_{n,0}^{(1)} + \mu_1 P_{n,1}^{(0)} \right] , \quad n \geq 1 \tag{6.11}
\]

Taking the Laplace Transformation \( \bar{P}_n(s) = \int_0^\infty e^{-st} P_n(t) dt \); Re \( s > 0 \)

of (6.1)-(6.6)

\[
(\lambda + V + s) \bar{P}_{n,0}^{(0)}(s) = \lambda \bar{P}_{n-1,0}^{(0)}(s) + V \left[ \mu_0 \bar{P}_{n,0}^{(0)}(s) + \mu_1 \bar{P}_{n,1}^{(0)}(s) \right] , \quad n \geq 1 \tag{6.13}
\]
\[(\lambda + s)\bar{P}^{(0)}_{0,0}(s) = 1 + V\bar{P}^{(0)}_{0,1}(s)\]  \hfill (6.14)

\[(\lambda + V + s)\bar{P}^{(0)}_{n,0}(s) = \lambda \bar{P}^{(1)}_{n-1,0}(s) + V]\left[\mu_i\bar{P}^{(0)}_{n,0}(s) + \mu_{\lambda}\bar{P}^{(0)}_{n,1}(s)\right] + n \geq 1 \hfill (6.15)

\[(\lambda + V + s)\bar{P}^{(0)}_{n,1}(s) = \lambda \bar{P}^{(0)}_{n-1,1}(s) + Vc_1[\mu_i\bar{P}^{(0)}_{n,0}(s) + \mu_{\lambda}\bar{P}^{(0)}_{n,1}(s)] + n \geq 1 \hfill (6.16)

\[(\lambda + V + s)\bar{P}^{(0)}_{0,0}(s) = V[\mu_i\bar{P}^{(0)}_{1,0}(s) + \bar{P}^{(0)}_{1,0}(s)] + \mu_0[\bar{P}^{(0)}_{0,1}(s) + \bar{P}^{(0)}_{1,1}(s)]] + n \geq 1 \hfill (6.17)

\[(\lambda + V + s)\bar{P}^{(0)}_{n,1}(s) = \lambda \bar{P}^{(1)}_{n-1,1}(s) + Vc_2[\mu_i\bar{P}^{(0)}_{n,0}(s) + \mu_{\lambda}\bar{P}^{(0)}_{n,1}(s)] + n \geq 1 \hfill (6.18)

**Definitions**

\[P_i^{(k)}(z,t) = \sum_{n=0}^{\infty} P_{n,0}^{(k)}(t)z^n \quad , \quad P_i(z,t) = P_i^{(0)}(z,t) + P_i^{(1)}(z,t) \quad i = 0,1\]

\[P(z,t) = P_0(z,t) + P_1(z,t)\]

and \[\bar{P}_0^{(k)}(z,s) = \int_0^{\infty} e^{-st} P_0^{(k)}(z,t)dt \quad , \quad \bar{P}_i^{(k)}(z,s) = \int_0^{\infty} e^{-st} P_i^{(k)}(z,t)dt\]

\[\bar{P}(z,s) = \int_0^{\infty} e^{-st} P(z,t)dt \quad , \quad \text{all for } k = 0 \quad \text{or} \quad 1 \quad \text{with} \quad |z| \leq 1\]

**Steady State Solution of the Problem**

Using \(Ef(x) = f(x+1)\) equations (6.7),(6.9),(6.10) and (6.12) give (for \(n \geq 2\))

\[- \{ (\lambda + \mu_i)E - \lambda \} P_{n,0}^{(0)} + \mu_i\bar{V}P_{n,1}^{(0)} = 0 \]  \hfill (6.19)
\[-(\lambda + \mu_i)(E - \lambda) P_{n,0}^{(i)} + \mu_i VEP_{n,1}^{(i)} = 0 \] (6.20)

\[
(c_i \mu_i V E^2 + c_i \mu_i V p E) P_{n,0}^{(i)} + c_i \mu_i V E^2 P_{n,0}^{(i)} + (c_i \mu_i V q E + c_i \mu_i V p E - (\lambda + V) E + \lambda) P_{n,1}^{(i)} + c_i \mu_0 V E^2 P_{n,1}^{(i)} = 0
\] (6.21)

\[
(c_2 \mu_i V E^2 + c_2 \mu_i V p E) P_{n,0}^{(i)} + c_2 \mu_i V E^2 P_{n,0}^{(i)} + (c_2 \mu_i V q E + c_2 \mu_i V p E) P_{n,1}^{(i)} + (c_2 \mu_0 V E^2 - (\lambda + V) E + \lambda) P_{n,1}^{(i)} = 0
\] (6.22)

To have a solution of above system of equations, we must have

\[
(E - 1)^k \{(\lambda + \mu_i V) E - \lambda\} \left[ (c_2 + c_i q) (\mu_i \lambda + \lambda V) V E^2 - \lambda (\lambda + \mu_i V + V - \mu_0 V c_i p) E + \lambda^2 \right] = 0
\] (6.23)

The values of \( P_{n,0}^{(i)} \), \( P_{n,1}^{(i)} \), \( P_{n,0}^{(i)} \), and \( P_{n,1}^{(i)} \) are given by

\[
P_{n,0}^{(i)} = \sum_{i=0}^{4} a_i' z_i^n \quad \text{and} \quad P_{n,0}^{(i)} = \sum_{i=0}^{4} b_i' z_i^n
\]

\[
P_{n,1}^{(i)} = \sum_{i=0}^{4} c_i' z_i^n \quad \text{and} \quad P_{n,1}^{(i)} = \sum_{i=0}^{4} d_i' z_i^n \quad , \quad n \geq 2
\]

where \( a_i', b_i', c_i', d_i' (i=0,1,2,3,4) \) are arbitrary constants to be evaluated and \( z_0, z_1, z_2, z_3 & z_4 \) are the roots of (6.23) with \( z_0 = 1 \); \( z_1 = \frac{\lambda}{(\lambda + \mu_i V)} \); \( z_2 = \frac{\lambda}{(\lambda + V)} \)

The other two roots \( z_3 \) and \( z_4 \) depend on the parameters \( \lambda, \mu_i, V, c_1 \), \( p \) and are evaluated from equation

\[
\left[ (c_2 + c_i q) (\mu_i \lambda + \lambda V) V E^2 - \lambda (\lambda + \mu_i V + V - \mu_0 V c_i p) E + \lambda^2 \right] = 0 \quad \text{after putting}
\]

the values of \( \lambda, \mu_i, V, c_1 \), \( p \) in this quadratic equation. If any of \( z_3 \) \& \( z_4 \) is \( \geq 1 \), take

\( a_i' = b_i' = c_i' = d_i' = 0 \) for that \( i \) (i=3, 4). Roots \( z_1 \) and \( z_2 \) are always <1 whatever the values of parameters. To have convergence of the solution any root \( \geq 1 \) must be rejected. Thus rejecting \( z_0 \), we have
\[ P_{n,0}^{(0)} = \sum_{i=1}^{4} a_i z_i^n \quad \text{and} \quad P_{n,1}^{(1)} = \sum_{i=1}^{4} b_i z_i^n \]
\[ P_{n,0}^{(0)} = \sum_{i=1}^{4} c_i z_i^n \quad \text{and} \quad P_{n,1}^{(1)} = \sum_{i=1}^{4} d_i z_i^n, \quad n \geq 2 \]

From equations (6.7), (6.9), (6.10) & (6.12) putting n=2 and equation (6.11), we get the probabilities \( P_{i,0}^{(0)} , \ P_{i,0}^{(1)} , \ P_{i,1}^{(0)} \) and \( P_{i,1}^{(1)} \) respectively in terms of unknown constants \( a_i , \ b_i , \ c_i , \ d_i \ (i = 1,2,3,4) \). Substituting above values in equation (6.8), in equations \{ (6.7), (6.9), (6.10) & (6.12) \} for \( n=1,3,4 \) and in equations \{ (6.7), (6.9) & (6.10) \} for \( n=5 \), we get sixteen relations between sixteen unknown constants \( a_i , \ b_i , \ c_i , \ d_i \ (i = 1,2,3,4) \) and probability \( P_{0,0}^{(0)} \).

From these relations, sixteen unknown constants \( a_i , \ b_i , \ c_i , \ d_i \ (i = 1,2,3,4) \) can be determined in terms of \( P_{0,0}^{(0)} \). Further \( P_{0,0}^{(0)} \) can be obtained by the relation
\[ \sum_{n=0}^{\infty} P_{n,0}^{(0)} + \sum_{n=1}^{\infty} P_{n,0}^{(1)} + \sum_{n=0}^{\infty} P_{n,1}^{(0)} + \sum_{n=1}^{\infty} P_{n,1}^{(1)} = 1 \]

Hence by using the values of \( a_i , \ b_i , \ c_i , \ d_i \ (i = 1,2,3,4) \) and \( P_{0,0}^{(0)} \), the probabilities \( P_{i,0}^{(0)} , \ P_{i,0}^{(1)} , \ P_{i,1}^{(0)} , \ P_{i,1}^{(1)} \) are completely known for various values of \( n \).

**Laplace Transformation of Probability Generating Function of transient-state queue length probabilities**

\[
\mathcal{P}_0^{(0)}(z,s) = \frac{E[F(zE - \mu_0 V[c_2 + c_1(q + pz)]) - \mu_1 z^2 V^2 c_2] + \nu_1 z^2 V^2 [F(zE - V[c_2 + c_1(q + pz)])]_1 + \mu_1 \mathcal{P}_0^{(0)}(s) + \mu_1 \mathcal{P}_1^{(0)}(s)}{EF[zE - \mu_0 V[c_2 + c_1(q + pz)]) - \mu_1 z^2 V^2 [c_2 + c_1(q + pz)]]} \]
\[
\quad \lambda < V; \quad |z| \leq 1 \quad (6.24)
\]
\[
\begin{align*}
\mathcal{P}_0^{(1)}(z, s) &= \mu V^2 c_1 (q + pz) E \left[ \mu_1 - \lambda (1 - z) + s \right] + \mu \bar{P}_0^{(0)}(s) + \mu_0 \bar{P}_0^{(1)}(s) \right] \\
&+ \mu V^2 \left[ zEF - (q + pz) \right] \left[ \mu_1 \bar{P}_1^{(0)}(s) + \mu_0 \bar{P}_1^{(1)}(s) \right] + \mu_2 \left[ \mu_1 \bar{P}_1^{(0)}(s) + \mu_0 \bar{P}_1^{(1)}(s) \right] \\
&- \mu_2 V^2 \left[ e_z + c_1 (q + pz) \right] - \mu_1^2 V^2 \left[ e_z + c_1 (q + pz) \right], \quad \lambda < V; \quad |z| \leq 1 \quad (6.25)
\end{align*}
\]

\[
\begin{align*}
\mathcal{P}_1^{(0)}(z, s) &= Vc_1 (q + pz) E \left[ \mu_1 - \lambda (1 - z) + s \right] + \mu_0 \bar{P}_0^{(0)}(s) + \mu_0 \bar{P}_0^{(1)}(s) \right] \\
&+ V \left[ zEF - V \left( \mu_2 F + \mu_1 \bar{V}^2 \right) \right] \left[ \mu_1 \bar{P}_1^{(0)}(s) + \mu_0 \bar{P}_1^{(1)}(s) \right] + c_2 \left[ \mu_1 \bar{P}_1^{(0)}(s) + \mu_0 \bar{P}_1^{(1)}(s) \right] \\
&- V \left[ zEF - \mu_0 V \left[ e_z + c_1 (q + pz) \right] - \mu_1^2 V^2 \left[ e_z + c_1 (q + pz) \right] \right], \quad \lambda < V; \quad |z| \leq 1 \quad (6.26)
\end{align*}
\]

\[
\begin{align*}
\mathcal{P}_1^{(1)}(z, s) &= Vc_2 (q + pz) E \left[ \mu_1 - \lambda (1 - z) + s \right] + \mu_0 \bar{P}_0^{(0)}(s) + \mu_0 \bar{P}_0^{(1)}(s) \right] \\
&- V \left[ zEF - V \left( q + pz \right) \right] \left[ \mu_2 F + \mu_1 \bar{V}^2 \right] \left[ \mu_1 \bar{P}_1^{(0)}(s) + \mu_0 \bar{P}_1^{(1)}(s) \right] + c_2 \left[ \mu_1 \bar{P}_1^{(0)}(s) + \mu_0 \bar{P}_1^{(1)}(s) \right] \\
&- V \left[ zEF - \mu_0 V \left[ e_z + c_1 (q + pz) \right] - \mu_1^2 V^2 \left[ e_z + c_1 (q + pz) \right] \right], \quad \lambda < V; \quad |z| \leq 1 \quad (6.27)
\end{align*}
\]

\[
\begin{align*}
\mathcal{P}(z, s) &= E \left[ zEF - \mu_0 V \left[ e_z + c_1 (q + pz) \right] + \mu_1 V (q + pz) \right] - \mu_1^2 V^2 \left[ c_1 (q + pz) \right] \\
&- V (1 - z) E \left[ F \left( q + \mu V c_2 p + \mu_1 \bar{V}^2 c_2 p \right) \right] \left[ \mu_1 \bar{P}_0^{(0)}(s) + \mu_0 \bar{P}_0^{(1)}(s) \right] \\
&- V \left[ zEF - \mu_0 V \left[ e_z + c_1 (q + pz) \right] - \mu_1^2 V^2 \left[ e_z + c_1 (q + pz) \right] \right], \quad \lambda < V; \quad |z| \leq 1 \quad (6.28)
\end{align*}
\]

Where \(E = \{- \lambda z + (\lambda + V + s)\}\)

\(F = \{- \lambda z + (\lambda + \mu V + s)\}\)

Let \(D = EF \{ f(z) - g(z) \}\)

where \(f(z) = f_1(z) \ast f_2(z)\)
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\[ f_1(z) = F = \{-\lambda z + (\lambda + \mu_1 V + s)\} \]

\[ f_2(z) = zE - \mu_0 V [c_2 + c_1 (q + pz)] \]

\[ = -\lambda z^2 + [\lambda + \{\mu_1 + \mu_0 (c_2 + c_1 q)\}]V + s]z - \mu_0 (c_2 + c_1 q)V \]

\[ g(z) = \mu_1^2 V^2 \{c_2 + c_1 (q + pz)\} \]

Obviously \( f_2(z) \) have at least one zero inside the unit circle.

Now \( |f(z)| = |f_1(z) \cdot f_2(z)| = |f_1(z)|^* f_2(z) | \]

\[ = \left| -\lambda z + (\lambda + \mu_1 V + s) \right| \]

\[ \cdot \left| -\lambda z^2 + [\lambda + \{\mu_1 + \mu_0 (c_2 + c_1 q)\}]V + s]z - \mu_0 (c_2 + c_1 q)V \right| \]

\[ \geq (\zeta + \mu_1 V)^* (\zeta + \mu_1 V) \quad \text{for} \quad s = \zeta + i\eta, |z| = 1 \]

\[ \geq \mu_1^2 V^2 = |g(z)| \]

Hence \( |f(z)| > |g(z)| \) on \( |z| = 1 \)

Since all the conditions of Rouche’s theorem are satisfied, so denominator \( D \) in \((6.24), (6.25), (6.26), (6.27) \) and \((6.28) \) has at least one zero inside the unit circle. Let it be \( z_0 \).

Numerator must also vanish for this zero since \( P_0^{(0)}(z, s), P_0^{(1)}(z, s), P_1^{(0)}(z, s), P_1^{(1)}(z, s) \) and \( \overline{P}(z, s) \) are analytical functions of \( z \). From this we will get one equation in six unknowns \( \overline{P}_0^{(0)}(s), \overline{P}_0^{(1)}(s), \overline{P}_1^{(0)}(s), \overline{P}_1^{(1)}(s) \) and \( \overline{P}_0^{(1)}(s) \) and \( \overline{P}_1^{(1)}(s) \). Eliminating common factor from equations \((6.16) \) and \((6.18) \), we get second equation in above six unknowns for \( n=1 \). The other four equations required for evaluating six unknowns are \((6.14), (6.17) \) and the equations \{(6.13) \& (6.15) \) for \( n=1 \).
Hence the generating functions $\overline{P}_0^{(0)}(z,s)$, $\overline{P}_0^{(1)}(z,s)$, $\overline{P}_1^{(0)}(z,s)$, $\overline{P}_1^{(1)}(z,s)$ and $\overline{P}(z,s)$ are completely known. $\overline{P}_{n,0}^{(0)}(s)$, $\overline{P}_{n,0}^{(1)}(s)$, $\overline{P}_{n,1}^{(0)}(s)$, $\overline{P}_{n,1}^{(1)}(s)$ and $\overline{P}_n(s)$ can be obtained by using the following formulae

$$\overline{P}_{n,0}^{(0)}(s) = \frac{1}{n!} \frac{d^n}{dz^n} \overline{P}_0^{(0)}(z,s) \quad \text{at } z = 0 \quad \text{for } n=0,1,\ldots$$

$$\overline{P}_{n,0}^{(1)}(s) = \frac{1}{n!} \frac{d^n}{dz^n} \overline{P}_0^{(1)}(z,s) \quad \text{at } z = 0 \quad \text{for } n=1,2,\ldots$$

$$\overline{P}_{n,1}^{(0)}(s) = \frac{1}{n!} \frac{d^n}{dz^n} \overline{P}_1^{(0)}(z,s) \quad \text{at } z = 0 \quad \text{for } n=0,1,\ldots$$

$$\overline{P}_{n,1}^{(1)}(s) = \frac{1}{n!} \frac{d^n}{dz^n} \overline{P}_1^{(1)}(z,s) \quad \text{at } z = 0 \quad \text{for } n=1,2,\ldots$$

$$\overline{P}_n(s) = \frac{1}{n!} \frac{d^n}{dz^n} \overline{P}(z,s) \quad \text{at } z = 0 \quad \text{for } n=0,1,\ldots$$

In either case $P_{n,0}^{(0)}(t)$, $P_{n,0}^{(1)}(t)$, $P_{n,1}^{(0)}(t)$, $P_{n,1}^{(1)}(t)$ and $P_n(t)$ can be found by inverting the Laplace transforms $\overline{P}_{n,0}^{(0)}(s)$, $\overline{P}_{n,0}^{(1)}(s)$, $\overline{P}_{n,1}^{(0)}(s)$, $\overline{P}_{n,1}^{(1)}(s)$ and $\overline{P}_n(s)$.

Further $\overline{P}(1, s) = \frac{1}{s}$, as desired.

And

$$\overline{P}(0, s) = \overline{P}_{0,0}^{(0)}(s) + \overline{P}_{0,1}^{(0)}(s) = \overline{P}_0^{(0)}(s) = \overline{P}_0(s)$$

Also $\overline{P}_0^{(0)}(0, s) = \overline{P}_{0,0}^{(0)}(s)$

& $\overline{P}_1^{(0)}(0, s) = \overline{P}_{0,1}^{(0)}(s)$.
Special Cases

(1) When the units are not given the option of rejoining the system after first service i.e. a queueing system without feedback when departures at two successive transition marks are correlated with inter-arrival times and inter-transition times following exponential distribution.

Putting $p=0, q=1$; $c_2 = 0, c_1 = 1$, $P_{0,0}^{(0)}(s) = P_{0,0}(s)$, $P_{0,1}^{(0)}(s) = P_{0,1}(s)$, $P_{1,0}^{(0)}(s) = P_{1,0}(s)$,

$P_{1,1}^{(0)}(s) = P_{1,1}(s)$; $P_{1,0}^{(i)}(s) = P_{1,1}^{(i)}(s) = 0$ in (6.24), (6.26), and (6.28) and on simplification, we get

$$
P_0(z,s) = \frac{EF(zE - \mu_0 V) + VEF(zE - V)\{\mu_1 P_{0,0}(s) + \mu_0 P_{0,1}(s)\}}{EF\{F(zE - \mu_0 V) - \mu_1^2 V^2\}}$$

, $\lambda < V; |z| \leq 1$ (6.29)

$$
P_1(z,s) = \frac{VEF(\mu_1 - \{\lambda(1-z) + s\}\{\mu_1 P_{0,0}(s) + \mu_0 P_{0,1}(s)\})}{EF\{F(zE - \mu_0 V) - \mu_1^2 V^2\}}$$

, $\lambda < V; |z| \leq 1$ (6.30)

$$
P(z,s) = \frac{EF(zE - \mu_0 V + \mu_1 V) - V(1-z)E^2F(\mu_1 P_{0,0}(s) + \mu_0 P_{0,1}(s))}{EF\{F(zE - \mu_0 V) - \mu_1^2 V^2\}}$$

, $\lambda < V; |z| \leq 1$ (6.31)

(2) When departures at two successive transition marks are independent i.e. $\mu_0 = \mu_1 = \frac{\lambda}{2}$, then (6.24) to (6.28) give

$$
P_0^{(0)}(z,s) = \frac{EF[EF(zE - \mu_0 V(c_2 + c_1(q + pz))] - \mu_1^2 V^2 c_2] + VEF[F(zE - V\{\mu_1 c_2 + c_1(q + pz)\})\{\mu_1 P_{0,0}(s) + \mu_0 P_{0,1}(s)\}]}{EF\{F(zE - \mu_0 V) - \mu_1^2 V^2\}}$$
\begin{align*}
\lambda < V; \quad |z| \leq 1 & \quad (6.32) \\
\bar{P}_0^{(1)}(z,s) = & \frac{\mu V^2(c_1 + S + S)}{\mu V^2 \left[ 2EF - (q + pz) \left( \mu_0, P + \mu_1^2 \right) \left( c_0 + S, c_0 + S \right) + c_1 \left( \mu_1 P^{(0)}_0(s) + \mu_0 P^{(0)}_1(s) \right) \right] - \mu V^2 \left[ c_1 + S + S \right] - \mu V^2 \left[ c_1 + S + S \right]} \\
\lambda < V; \quad |z| \leq 1 & \quad (6.33) \\
\bar{F}_0^{(0)}(z,s) = & \frac{Vc_1(q + pz) \left( \mu_0, P + \mu_1^2 \right) \left( c_0 + S, c_0 + S \right) + c_1 \left( \mu_1 P^{(0)}_0(s) + \mu_0 P^{(0)}_1(s) \right) \right] - \mu V^2 \left[ c_1 + S + S \right] - \mu V^2 \left[ c_1 + S + S \right]} \\
\lambda < V; \quad |z| \leq 1 & \quad (6.34) \\
\bar{F}_1^{(0)}(z,s) = & \frac{Vc_1(q + pz) \left( \mu_0, P + \mu_1^2 \right) \left( c_0 + S, c_0 + S \right) + c_1 \left( \mu_1 P^{(0)}_0(s) + \mu_0 P^{(0)}_1(s) \right) \right] - \mu V^2 \left[ c_1 + S + S \right] - \mu V^2 \left[ c_1 + S + S \right]} \\
\lambda < V; \quad |z| \leq 1 & \quad (6.35) \\
\bar{P}(z,s) = & \frac{Vc_1(q + pz) \left( \mu_0, P + \mu_1^2 \right) \left( c_0 + S, c_0 + S \right) + c_1 \left( \mu_1 P^{(0)}_0(s) + \mu_0 P^{(0)}_1(s) \right) \right] - \mu V^2 \left[ c_1 + S + S \right] - \mu V^2 \left[ c_1 + S + S \right]} \\
\lambda < V; \quad |z| \leq 1 & \quad (6.36) \\

\textbf{Busy Period Distribution}

The probability density function for the busy period distribution is given by \( \frac{d}{dt} \left( P^{(0)}_{0,0}(t) + P^{(0)}_{0,1}(t) \right) \) and can be obtained using the equations \((6.1) \& (6.4) \) for \( n \geq 2 \), the equations \((6.2) \& (6.5) \) for \( n \geq 1 \) and the equations:

\begin{equation}
\frac{d}{dt} P^{(0)}_{1,0}(t) = -\left( \lambda + V \right) P^{(0)}_{1,0}(t) + V \left( \mu_1 P^{(0)}_{1,0}(t) + \lambda P^{(0)}_{1,0}(t) \right) \end{equation}
\[
\frac{d}{dt} P_{0,0}(t) = 0 
\]  
(6.38)

\[
\frac{d}{dt} P_{1,1}(t) = -(\lambda + V) P_{1,1}(t) + Vc \left[ \mu_1 \left( q P_{2,0}(t) + P_{2,1}(t) \right) + \mu_0 \left( q P_{3,1}(t) + P_{3,1}(t) \right) \right] 
\]  
(6.39)

\[
\frac{d}{dt} P_{0,1}(t) = V \left[ \mu_1 \left( q P_{1,0}(t) + P_{1,1}(t) \right) + \mu_0 \left( q P_{1,1}(t) + P_{1,1}(t) \right) \right] 
\]  
(6.40)

Initially \( P_{1,1}(0) = 1 \)

Define

\[
G_0^{(k)}(z,t) = \sum_{n=1}^{\infty} P_{n,0}^{(k)}(t) z^n \quad , \quad G_1^{(k)}(z,t) = \sum_{n=1}^{\infty} P_{n,1}^{(k)}(t) z^n 
\]

\[
G_0(z,t) = G_0^{(0)}(z,t) + G_0^{(1)}(z,t) \quad , \quad G_1(z,t) = G_1^{(0)}(z,t) + G_1^{(1)}(z,t) 
\]

\[
G^{(k)}(z,t) = G_0^{(k)}(z,t) + G_1^{(k)}(z,t) \quad , \quad G(z,t) = G^{(0)}(z,t) + G^{(1)}(z,t) 
\]

and \( G(z,t) = G_0(z,t) + G_1(z,t) \) all for \( k = 0 \) or \( 1 \) with \( |z| \leq 1 \)

Laplace Transformation of probability generating function for the busy period distribution

\[
z E \left[ F \left[ z E - \mu_0 V \left( c_2 + c_1 (q + pz) \right) \right] + \mu_1 V (q + pz) \right] - \mu_1^2 V z c_2 p (1 - z) \\
- V z (F + \mu_1 V) \left[ E F (q + c_1 pz) - Vc_1 c_2 p^2 (1 - z) \left( \mu_0 F + \mu_1^2 V \right) \right] \left( \mu_1 P_{3,1}^{(0)}(s) + \mu_0 P_{1,1}^{(0)}(s) \right) \\
\overline{G}(z,s) = \frac{-V z (F + \mu_1 V) \left[ E F (q + c_1 pz) - Vc_1 c_2 p^2 (1 - z) \left( \mu_0 F + \mu_1^2 V \right) \right] \left( \mu_1 \overline{P}_{1,1}^{(0)}(s) + \mu_0 \overline{P}_{3,1}^{(0)}(s) \right)}{E F \left[ z E - \mu_0 V \left( c_2 + c_1 (q + pz) \right) \right] - \mu_1^2 V z \left( c_2 + c_1 (q + pz) \right)} 
\]

, \( \lambda < V \); \( |z| \leq 1 \)  
(6.41)

The denominator of (6.41) is same as that of (6.24), (6.25), (6.26), (6.27) and (6.28).

Numerator of (6.41) must vanish for that one zero of the denominator \( D \) in (6.24), (6.25), (6.26),
(6.27) and (6.28). This equation along with L.T. of equations \{(6.37), (6.2) for n=1, (6.39) & 
(6.5) for n=1\} (eliminating a common factor among last two equations for finding solution) will 
determine the four unknowns $\mathbf{P}^{(0)}_{1,0}(s)$, $\mathbf{P}^{(1)}_{1,0}(s)$, $\mathbf{P}^{(0)}_{1,1}(s)$ and $\mathbf{P}^{(1)}_{1,1}(s)$. Using LAPLACE inverse 
of $\mathbf{P}^{(0)}_{1,0}(s)$, $\mathbf{P}^{(1)}_{1,0}(s)$, $\mathbf{P}^{(0)}_{1,1}(s)$ and $\mathbf{P}^{(1)}_{1,1}(s)$ in (6.38) and (6.40), $\frac{d}{dt}\{P^{(0)}_{0,0}(t) + P^{(0)}_{0,1}(t)\}$ can be 
obtained.
The numerical results are generated using MATLAB programming following the work of Bunday (1986). Various probabilities are listed in Table 6.1 for the case when

\[ \rho' \left( = \frac{\lambda}{V} \right) = 0.3, \mu_0 = 0.3, c_1 = 0.8, q = 0.75 \]
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Various probabilities are plotted Vs time through Figs. 6.1 to 6.8 for the data from Table 6.1. Fig. 6.1 shows plot of probabilities $P_{0,0}^{(0)}$ and $P_{0,1}^{(0)}$ with respect to time (average transition times). It is clear from the graph that probability $P_{0,0}^{(0)}$ decreases rapidly in the starting moments and then becomes almost steady from the initial value one at time $t=0$. Probability $P_{0,1}^{(0)}$ increases in the starting moments, then decreases very slowly and becomes almost steady from the initial value zero at time $t=0$.

![Probabilities $P_{0,0}^{(0)}$ and $P_{0,1}^{(0)}$ Vs time](image)

Fig. 6.2 shows relative change in probabilities $P_{1,0}^{(0)}$, $P_{1,0}^{(1)}$, $P_{1,1}^{(0)}$ and $P_{1,1}^{(1)}$. All these probabilities first increase then decrease to some extent and finally attain some steady values for higher values of $t$. Figs. 6.3 and 6.4 show respectively the probability plots of $P_{2,0}^{(0)}$, $P_{2,0}^{(1)}$, $P_{2,1}^{(0)}$, $P_{2,1}^{(1)}$ and $P_{4,0}^{(0)}$, $P_{4,0}^{(1)}$, $P_{4,1}^{(0)}$, $P_{4,1}^{(1)}$ versus time. All these probabilities firstly increase and finally settle down to some steady values.
Chapter 6

Fig. 6.2

Probabilities $P_{1,0}^{(0)}$, $P_{1,0}^{(1)}$, $P_{1,1}^{(0)}$ and $P_{1,1}^{(1)}$ Vs Time

Probabilities $P_{2,0}^{(0)}$, $P_{2,0}^{(1)}$, $P_{2,1}^{(0)}$ and $P_{2,1}^{(1)}$ Vs Time

$\rho' = 0.3, \mu_0 = 0.3, c_1 = 0.8, q = 0.75$
Comparison among four probabilities when no departure has occurred at the previous transition mark and the unit at the head of the queue is to join the server for the first time i.e. among $P_{0,0}^{(0)}$, $P_{0,1}^{(0)}$, $P_{1,0}^{(0)}$ and $P_{4,0}^{(0)}$ is done through Fig. 6.5. It is apparent that probability $P_{n,0}^{(0)}$ decreases as $n$ increases. It is also seen that all the four probabilities increase rapidly in the starting moments, then $P_{1,0}^{(0)}$ decreases to some extent and all show slight variability before finally approaching to some steady values. Fig. 6.6 shows similar type of relationship among probabilities $P_{1,1}^{(0)}$, $P_{2,0}^{(0)}$, $P_{3,0}^{(0)}$ and $P_{4,0}^{(0)}$ as shown by Fig. 6.5 among $P_{1,0}^{(0)}$, $P_{2,0}^{(0)}$, $P_{3,0}^{(0)}$ and $P_{4,0}^{(0)}$ with a difference that decrease in $P_{1,0}^{(0)}$ is not that much pronounced as is there in case of $P_{1,0}^{(0)}$ (Fig. 6.5). It is also observed that as $n$ increases, probabilities increase with less rapidness in their initial times as compared to the probabilities with lower $n$ values.
Chapter 6

Probabilities $P_{1,0}^{(0)}$, $P_{2,0}^{(0)}$, $P_{3,0}^{(0)}$ and $P_{4,0}^{(0)}$ Vs Time

Fig. 6.5

Probabilities $P_{1,0}^{(1)}$, $P_{2,0}^{(1)}$, $P_{3,0}^{(1)}$ and $P_{4,0}^{(1)}$ Vs Time

Fig. 6.6
Fig. 6.7 illustrates the relationship among four probabilities when a departure has occurred at the previous transition mark and the unit at the head of the queue is to join the server for the first time i.e. among $P_{1,1}^{(0)}$, $P_{2,1}^{(0)}$, $P_{3,1}^{(0)}$ and $P_{4,1}^{(0)}$. It shows that probability decreases as n(number of units in the system) increases. It is also seen that $P_{1,1}^{(0)}$ increases rapidly in the starting moments, then decreases to some extent for higher values of $t$(average transition times) and show slight variability before finally approaching to a steady value. Other three probabilities increase rapidly in the initial moments, then increase slowly and finally attain some constant values for higher values of $t$. Fig. 6.8 shows similar type of relationship among probabilities $P_{1,1}^{(1)}$, $P_{2,1}^{(1)}$, $P_{3,1}^{(1)}$ and $P_{4,1}^{(1)}$ when plotted against time as is shown by the Fig. 6.7 among $P_{1,1}^{(0)}$, $P_{2,1}^{(0)}$, $P_{3,1}^{(0)}$ and $P_{4,1}^{(0)}$. 

Fig. 6.7
Probabilities $P_{1,1}^{(1)}$, $P_{2,1}^{(1)}$, $P_{3,1}^{(1)}$ and $P_{4,1}^{(1)}$ Vs Time

$\rho = 0.3, \mu = 0.3, \alpha = 0.8, q = 0.75$

Fig. 6.8
Various probabilities are listed in Table 6.2 for the case when

\[ \rho^*(1 = \frac{\lambda}{\mu} = 0.8, \mu_0 = 0.3, c_1 = 0.8, q = 0.75) \]

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To study the behavior of various probabilities in case when $\rho'$ takes a comparatively high value, data of various probabilities is generated for $\rho' = 0.8$. Figs. 6.9 to 6.16 show graphs for the data of Table 6.2 corresponding to the graphs plotted for the data of Table 6.1.
Chapter 6

Probabilities $P_{0,0}^{(0)}$ and $P_{0,1}^{(0)}$ Vs time

Fig. 6.9

Probabilities $P_{1,0}^{(0)}$, $P_{1,0}^{(1)}$, $P_{1,1}^{(0)}$ and $P_{1,1}^{(1)}$ Vs Time

Fig. 6.10
Chapter 6

Probabilities $P_{2,0}^{(0)}$, $P_{2,0}^{(1)}$, $P_{2,1}^{(0)}$ and $P_{2,1}^{(1)}$ Vs Time

Fig. 6.11

Probabilities $P_{4,0}^{(0)}$, $P_{4,0}^{(1)}$, $P_{4,1}^{(0)}$ and $P_{4,1}^{(1)}$ Vs Time

Fig. 6.12
Chapter 6

Fig. 6.13

Probabilities $P_{1,0}^{(0)}$, $P_{2,0}^{(0)}$, $P_{3,0}^{(0)}$, and $P_{4,0}^{(0)}$ Vs Time

Fig. 6.14

Probabilities $P_{1,0}^{(1)}$, $P_{2,0}^{(1)}$, $P_{3,0}^{(1)}$, and $P_{4,0}^{(1)}$ Vs Time

$\rho' = 0.8, \mu_0 = 0.3, c_i = 0.8, q = 0.75$
Chapter 6

Probabilities $P_{1,1}^{(0)}$, $P_{2,1}^{(0)}$, $P_{3,1}^{(0)}$ and $P_{4,1}^{(0)}$ Vs Time

Fig. 6.15

Probabilities $P_{1,1}^{(1)}$, $P_{2,1}^{(1)}$, $P_{3,1}^{(1)}$ and $P_{4,1}^{(1)}$ Vs Time

Fig. 6.16
To study the effect of various parameters on different probabilities of the model, data of various probabilities is generated for different values of \( \rho' \) keeping other parameters constant. The set of values that \( \rho' \) took is \{0.2, 0.3, 0.5, 0.8\}. The other parameters were fixed at \( \mu_0 = 0.3, c_1 = 0.8, q = 0.75 \). The probability \( P_{0,0}^{(0)} \) is plotted against time for different values of \( \rho' \) in Fig. 6.17. From the figure it is concluded that the probability \( P_{0,0}^{(0)} \) decreases with increasing \( \rho' \). So more the \( \rho' \) i.e. more customers are arriving per unit average transition time less is the probability of zero units in the system.

![Graph showing P_{0,0}^{(0)} vs time for different values of \( \rho' \)](image)

Behavior of the probabilities \( P_{2,0}^{(0)}, P_{2,1}^{(0)}, P_{4,0}^{(0)} \) and \( P_{4,1}^{(0)} \) with respect to time(average transition times) and changing \( \rho' \) is apparent from Figs. 6.18 to 6.21.
Chapter 6

Fig. 6.18

$P_{2,0}^{(0)}$ Vs time for different values of $\rho' (= \lambda/V)$

$\mu_0 = 0.3, c_1 = 0.8, q = 0.75$

$P_{2,0}^{(0)}(\rho \leq 0.2)$

$P_{2,0}^{(0)}(\rho \leq 0.3)$

$P_{2,0}^{(0)}(\rho \leq 0.5)$

$P_{2,0}^{(0)}(\rho \leq 0.8)$

Fig. 6.19

$P_{2,1}^{(0)}$ Vs time for different values of $\rho' (= \lambda/V)$

$\mu_0 = 0.3, c_1 = 0.8, q = 0.75$

$P_{2,1}^{(0)}(\rho \leq 0.2)$

$P_{2,1}^{(0)}(\rho \leq 0.3)$

$P_{2,1}^{(0)}(\rho \leq 0.5)$

$P_{2,1}^{(0)}(\rho \leq 0.8)$
$P_{4,0}^{(0)}$ Vs time for different values of $\rho'(=\lambda/V)$

Fig. 6.20

$P_{4,1}^{(0)}$ Vs time for different values of $\rho'(=\lambda/V)$

Fig. 6.21
Effect of change in probability $q$ (probability of leaving the system after getting first service) is studied by generating data of various probabilities for different values of probability $q$ keeping other parameters constant. The set of values that $q$ took is $\{0.45, 0.6, 0.75\}$. The other parameters were fixed at $\rho' = 0.5$, $\mu_0 = 0.3$, $c_1 = 0.8$. In Fig. 6.22, the probability $P_{0,0}^{(0)}$ is plotted against time for different values of $q$. From the figure it is clear that there is no significant change in the value of probability $P_{0,0}^{(0)}$ as $q$ increases. So no significant effect of varying probability $q$ is observed when there is no unit in the system.

![Graph showing $P_{0,0}^{(0)}$ vs time for different values of $q$.](image)

Fig. 6.23 shows probability plot of $P_{1,1}^{(0)}$ with respect to $t$ (average transition times) for a set of $q$ values. It is clear from the graph that in all the three cases probability $P_{1,1}^{(0)}$ increases rapidly in the starting moments then decreases and finally becomes almost steady for higher
values of $t$. It is observed that probability $P_{1,1}^{(0)}$ takes higher values for high value of $q$. This can be explained as $P_{1,1}^{(0)}$ (i.e. zero state) is more probable when $q$ value is high. Probability $P_{1,1}^{(1)}$ is plotted vs time in Fig. 6.24. Here it is observed that probability $P_{1,1}^{(1)}$ takes smaller values for high value of $q$ as $P_{1,1}^{(1)}$ (i.e. one state) is less probable when $q$ takes a high value.

![Graph showing $P_{1,1}^{(0)}$ vs time for different values of probability $q$.](image)

Fig. 6.23
Figs. 6.25 & 6.26 show the probability plots of \( P_{2,1}^{(0)} \) and \( P_{2,1}^{(1)} \). These also show similar kind of relationship as shown by probabilities \( P_{1,1}^{(0)} \) and \( P_{1,1}^{(1)} \) in Figs. 6.23 & 6.24 respectively and also can be explained by same argument.

After analyzing the graphs plotted for different probabilities it is interpreted that all the probabilities tend to attain some steady values for high values of time \( t(\text{average transition times}) \).
Fig. 6.25

$P_{2,1}^{(0)}$ Vs time for different values of probability $q$

- $P_{2,1}^{(0)}(q=0.45)$
- $P_{2,1}^{(0)}(q=0.6)$
- $P_{2,1}^{(0)}(q=0.75)$

$\rho^*=0.5, \mu_b = 0.3, c_i = 0.8$

Fig. 6.26

$P_{2,1}^{(1)}$ Vs time for different values of probability $q$

- $P_{2,1}^{(1)}(q=0.45)$
- $P_{2,1}^{(1)}(q=0.6)$
- $P_{2,1}^{(1)}(q=0.75)$

$\rho^*=0.5, \mu_b = 0.3, c_i = 0.8$